Bull. Math. Soc. Sci. Math. Roumanie
Tome 50(98) No. 3, 2007, 249–258

A sequence attached to powerful numbers

by
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Abstract

For every powerful number k, let c_k be the least positive integer such that kc_k is a square. We give asymptotic estimations for several series whose terms depend on the sequence $(c_k)_{k\geq 1}$, from which we mention $\sum\limits_{k\leq x} c_k = \frac{3}{\pi^2}\zeta\left(\frac{4}{3}\right)\sqrt[3]{x^2} + O(\sqrt{x}\ln x)$.

Key Words: Prime numbers, squarefree numbers, powerful numbers, sequences, series.
2000 Mathematics Subject Classification: Primary 11N25, Secondary: 11N60.

A positive integer is said to be powerful if it is divisible by the square of each of its prime factors.

In a recent paper [1], we considered for each positive integer m the least positive integer b_m such as mb_m is a square, and we studied the properties of the sequence $(b_m)_m$. In this paper, we will study some properties of the subsequence of $(b_m)_m$ associated to the powerful numbers.

We will use throughout the paper the result of Gegenbauer [2]:

$$Q(x) = \frac{6}{\pi^2}x + O(\sqrt{x}),$$

where $Q(x)$ denotes as usual the number of squarefree positive integers less or equal than x.

Golomb [3] proved that every powerful number can be written in an unique manner as n^2m^3 where m is squarefree.

Let $(f_n)_{n\geq 1}$ be the sequence of the squarefree numbers. The powerful numbers that do not exceed x, can be organised as follows:

$$f_1^3\cdot 1^2, f_1^3\cdot 2^2, \ldots, f_1^3\cdot k_1^2, \text{ where } k_1 = \left[\sqrt{\frac{x}{f_1^3}}\right]$$

$$f_2^3 \cdot 1^2, f_2^3 \cdot 2^2, \ldots, f_2^3 \cdot k_2^2, \text{ where } k_2 = \left[\sqrt{\frac{x}{f_2^3}}\right]$$
$$\cdots\cdots\cdots\cdots$$
$$f_L^3 \cdot 1^2, f_L^3 \cdot 2^2, \ldots, f_L^3 \cdot k_L^2, \text{ where } k_L = \left[\sqrt{\frac{x}{f_L^3}}\right],$$

L denoting the value $Q(\sqrt[3]{x})$.

Let us denote by $K(x)$ the number of powerful numbers that do not exceed x. Since $f_1^3 \cdot 1^2 = 1$, which is not a powerful number, we obtain:

$$\sum_{\substack{f \le \sqrt[3]{x} \\ f \text{ squarefree}}} \left[\sqrt{\frac{x}{f^3}}\right] = K(x) + 1. \tag{1}$$

The same powerful numbers can be counted in a different way:

$$1^2 \cdot f_1^3, 1^2 \cdot f_2^3, \ldots, 1^2 \cdot f_{h_1}^3, \text{ where } h_1 = Q\left(\sqrt[3]{\frac{x}{1^2}}\right);$$

$$2^2 \cdot f_1^3, 2^2 \cdot f_2^3, \ldots, 2^2 \cdot f_{h_2}^3, \text{ where } h_2 = Q\left(\sqrt[3]{\frac{x}{2^2}}\right);$$
$$\cdots\cdots\cdots\cdots$$
$$[\sqrt{x}]^2 \cdot f_1^3, [\sqrt{x}]^2 \cdot f_2^3, \ldots, [\sqrt{x}]^2 \cdot f_{h_{[\sqrt{x}]}}^3, \text{ where } h_{[\sqrt{x}]} = Q\left(\sqrt[3]{\frac{x}{[\sqrt{x}]^2}}\right);$$

we thus obtain

$$\sum_{i \le \sqrt{x}} Q\left(\sqrt[3]{\frac{x}{i^2}}\right) = K(x) + 1. \tag{2}$$

For each powerful number k let c_k be the least positive integer such that kc_k is a square. Since every powerful number k can be written as $n_k^2 m_k^3$, with m_k squarefree, it follows immediately that $c_k = m_k$.

We will now prove some properties of the sequence $(c_k)_{k \ge 1}$.

Let $S_1(x) = \sum\limits_{\substack{k \le x \\ k \text{ powerful}}} c_k$.

Theorem 1. *We have the relation*

$$S_1(x) = \frac{3}{\pi^2}\zeta\left(\frac{4}{3}\right)\sqrt[3]{x^2} + O(\sqrt{x}\ln x).$$

Proof: The sum $S_1(x)$ counts all the squarefree c_k corresponding to the powerful numbers k that do not exceed x. We will count these terms using the second method described before. Taking s a positive integer that does not exceed $\sqrt{x}$, s^2 is a factor for the following powerful numbers not exceeding x:

$$s^2 \cdot f_1^3, s^2 \cdot f_2^3, \ldots, s^2 \cdot f_{h_s}^3, \text{ where } h_s = Q\left(\sqrt[3]{\frac{x}{s^2}}\right).$$

Each of these numbers corresponds to a squarefree term in $S_1(x)$. Therefore

$$S_1(x) + 1 = \sum_{s=1}^{[\sqrt{x}]} \left(f_1 + f_2 + \ldots + f_{Q\left(\sqrt[3]{\frac{x}{s^2}}\right)} \right),$$

(we have added 1 to S_1 because in the sum on the right hand side appears the term f_1 for $s = 1$, which does not correspond to any powerful number in S_1).

Using Gegenbauer's result

$$Q(n) = \frac{6}{\pi^2} n + O(\sqrt{n}),$$

and taking $x = f_n$ we obtain

$$n = Q(f_n) = \frac{6}{\pi^2} f_n + O(\sqrt{f_n}), \text{ so}$$

$$f_n = \frac{\pi^2}{6} n + O(\sqrt{f_n}).$$

It follows that

$$f_n \sim \frac{\pi^2}{6} n, \text{ so}$$

$$O(\sqrt{f_n}) = O(\sqrt{n}).$$

Putting all the above together, we get

$$f_n = \frac{\pi^2}{6} n + O(\sqrt{n}). \tag{3}$$

Using (3), we obtain

$$f_1 + f_2 + \ldots + f_n = \frac{\pi^2}{6} \frac{n(n+1)}{2} + O(\sqrt{1} + \sqrt{2} + \ldots + \sqrt{n}).$$

Using Stolz's Lemma we obtain

$$\sum_{k=1}^{n} \sqrt{k} \sim \frac{2}{3} n\sqrt{n},$$

so

$$f_1 + f_2 + \ldots + f_n = \frac{\pi^2 n^2}{12} + O(n\sqrt{n}).$$

We obtain that

$$S_1(x) = \frac{\pi^2}{12} \sum_{s=1}^{[\sqrt{x}]} \left(\frac{36}{\pi^4} \sqrt[3]{\frac{x^2}{s^4}} + O\left(\sqrt[3]{\frac{x}{s^2}} \sqrt[6]{\frac{x}{s^2}} \right) \right),$$

so

$$S_1(x) = \frac{3}{\pi^2}\sqrt[3]{x^2}\sum_{s=1}^{[\sqrt{x}]}\frac{1}{s^{4/3}} + O\left(\sqrt{x}\sum_{s=1}^{[\sqrt{x}]}\frac{1}{s}\right). \tag{4}$$

Taking into account that

$$\zeta\left(\frac{4}{3}\right) = \sum_{s=1}^{\infty}\frac{1}{s^{4/3}},$$

and since is is easy to prove that there exists a positive number h such that

$$\frac{1}{(x+1)^{4/3}} < h\left(\frac{1}{x^{1/3}} - \frac{1}{(x+1)^{1/3}}\right) \text{ for } x \geq 1,$$

we derive that

$$0 < \zeta\left(\frac{4}{3}\right) - \sum_{l=1}^{[\sqrt{x}]}\frac{1}{l^{4/3}} = \frac{1}{([\sqrt{x}]+1)^{4/3}} + \frac{1}{([\sqrt{x}]+2)^{4/3}} + \ldots <$$

$$< h\left(\frac{1}{[\sqrt{x}]^{1/3}} - \frac{1}{([\sqrt{x}]+1)^{1/3}} + \frac{1}{([\sqrt{x}]+1)^{1/3}} - \frac{1}{([\sqrt{x}]+2)^{1/3}} + \ldots\right) =$$

$$= O\left(\frac{1}{\sqrt[6]{x}}\right).$$

Therefore,

$$\sum_{l=1}^{[\sqrt{x}]}\frac{1}{s^{4/3}} = \zeta\left(\frac{4}{3}\right) + O\left(\frac{1}{\sqrt[6]{x}}\right). \tag{5}$$

On the other hand, the relation

$$\sum_{s=1}^{[\sqrt{x}]}\frac{1}{s} = O(\ln\sqrt{x}) = O(\ln x) \tag{6}$$

is well known.

Using (5) and (6) back in (4), we derive

$$S_1(x) = \frac{3}{\pi^2}\sqrt[3]{x^2}\left(\zeta\left(\frac{4}{3}\right) + O\left(\frac{1}{\sqrt[6]{x}}\right)\right) + O(\sqrt{x}\ln x) =$$

$$= \frac{3}{\pi^2}\zeta\left(\frac{4}{3}\right)\sqrt[3]{x^2} + O(\sqrt{x}) + O(\sqrt{x}\ln x),$$

so we have proved that

$$S_1(x) = \frac{3}{\pi^2}\zeta\left(\frac{4}{3}\right)\sqrt[3]{x^2} + O(\sqrt{x}\ln x).$$

□

Let $S_2(x) = \sum\limits_{\substack{k \le x \\ k \text{ powerful}}} c_k^2$.

Theorem 2. *We have the relation*

$$S_2(x) = \frac{x}{3} + O(\sqrt[6]{x^5}).$$

Proof: Using the same counting method as in Theorem 1, it results that

$$S_2(x) + 1 = \sum_{s=1}^{[\sqrt{x}]} \left(f_1^2 + f_2^2 + \ldots + f_{Q\left(\sqrt[3]{\frac{x}{s^2}}\right)}^2 \right).$$

Using (3) we can write

$$f_1^2 + f_2^2 + \ldots + f_n^2 = \frac{\pi^4}{36} \cdot \frac{n(n+1)(2n+1)}{6} + O(\sum_{k=1}^{n} k\sqrt{k}).$$

Using Stolz's Lemma, we obtain

$$\sum_{k=1}^{n} k\sqrt{k} \sim \frac{2}{5} n^2 \sqrt{n},$$

therefore

$$f_1^2 + f_2^2 + \ldots + f_n^2 \sim \frac{\pi^4}{36} \cdot \frac{n(n+1)(2n+1)}{6} + O(n^2\sqrt{n}).$$

We have this far obtained

$$f_1^2 + f_2^2 + \ldots + f_{Q\left(\sqrt[3]{\frac{x}{s^2}}\right)}^2 \sim \frac{\pi^4}{3 \cdot 6^2} \cdot \frac{6^3}{\pi^6} \cdot \frac{x}{s^2} + O\left(\sqrt[6]{\left(\frac{x}{s^2}\right)^5} \right);$$

Since the series $\sum \frac{1}{s^{5/3}}$ converges,

$$S_2(x) = \frac{2}{\pi^2} \cdot x \sum_{s=1}^{[\sqrt{x}]} \frac{1}{s^2} + O\left(\sqrt[6]{x^5} \sum_{s=1}^{[\sqrt{x}]} \frac{1}{s^{5/3}} \right) = \frac{2}{\pi^2} \cdot x \sum_{s=1}^{[\sqrt{x}]} \frac{1}{s^2} + O(\sqrt[6]{x^5}). \qquad (7)$$

Now

$$0 < \zeta(2) - \sum_{s=1}^{[\sqrt{x}]} \frac{1}{s^2} = \frac{1}{([\sqrt{x}]+1)^2} + \frac{1}{([\sqrt{x}]+2)^2} + \ldots <$$

$$< \frac{1}{(\sqrt{x})^2} + \frac{1}{(\sqrt{x}+1)^2} + \ldots < \frac{1}{\sqrt{x}-1} - \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x}+1} + \ldots = O\left(\frac{1}{\sqrt{x}}\right),$$

(we have used that $x < [x] + 1$ for all x and $\frac{1}{x^2} < \frac{1}{x-1} - \frac{1}{x}$ for $x > 1$).

Therefore,

$$\sum_{s=1}^{[\sqrt{x}]} \frac{1}{s^2} = \zeta(2) + O\left(\frac{1}{\sqrt{x}}\right). \tag{8}$$

We now use (8) in (7), and obtain

$$S_2(x) = \frac{2}{\pi^2} \cdot \zeta(2)x + O(\sqrt{x}) + O(\sqrt[6]{x^5}) = \frac{2}{\pi^2} \cdot \frac{\pi^2}{6} x + O(\sqrt[6]{x^5}).$$

Therefore,

$$S_2(x) = \frac{x}{3} + O(\sqrt[6]{x^5}).$$

∎

Let $S_3(x) = \sum_{\substack{k \le x \\ k \text{ powerful}}} \frac{1}{c_k}$.

Theorem 3. *We have the relation*

$$S_3(x) = M\sqrt{x} + O(\ln x), \text{ where } M = \frac{\zeta\left(\frac{5}{2}\right)}{\zeta(5)}.$$

Proof: We will use the first method of counting the powerful numbers that do not exceed x. For a given squarefree f that does not exceed $\sqrt[3]{x}$, we consider the following powerful numbers:

$$f^3 \cdot 1^2, f^3 \cdot 2^2, \ldots, f^3 \cdot k^2, \text{ where } k = \left[\sqrt{\frac{x}{f^3}}\right].$$

For each one of these numbers, the least positive integer that multiplied to it gives a square is f. Therefore, in the sum $S_3(x)$ every squarefree f appears $\left[\sqrt{\frac{x}{f^3}}\right]$ times. Taking into account that 1 is not a powerful number we obtain:

$$S_3(x) = \sum_{\substack{f \le \sqrt[3]{x} \\ f \text{ squarefree}}} \frac{1}{f}\left[\sqrt{\frac{x}{f^3}}\right] - 1.$$

Since $x - 1 < [x] \le x$, it follows that

$$\sum_{\substack{f \le \sqrt[3]{x} \\ f \text{ squarefree}}} \frac{1}{f}\left(\sqrt{\frac{x}{f^3}} - 1\right) - 1 < S_3(x) \le \sum_{\substack{f \le \sqrt[3]{x} \\ f \text{ squarefree}}} \frac{1}{f}\sqrt{\frac{x}{f^3}},$$

so

$$\sqrt{x} \cdot \sum_{\substack{f \le \sqrt[3]{x} \\ f \text{ squarefree}}} \frac{1}{f^2\sqrt{f}} - \sum_{\substack{f \le \sqrt[3]{x} \\ f \text{ squarefree}}} \frac{1}{f} - 1 < S_3(x) \le \sqrt{x} \cdot \sum_{\substack{f \le \sqrt[3]{x} \\ f \text{ squarefree}}} \frac{1}{f^2\sqrt{f}}. \tag{9}$$

We will evaluate each of the sums in (9).
In the first place we have

$$\sum_{f \text{ squarefree}} \frac{1}{f^{5/2}} = \prod_{p \text{ prime}} \left(1 + \frac{1}{p^{5/2}}\right) = \prod_{p \text{ prime}} \frac{\left(1 - \frac{1}{p^5}\right)}{\left(1 - \frac{1}{p^{5/2}}\right)} = \frac{\zeta\left(\frac{5}{2}\right)}{\zeta(5)} = M.$$

It is easy to prove that there exists a positive number k such that

$$\frac{1}{(x+1)^{5/2}} < k\left(\frac{1}{(x+1)^{3/2}} - \frac{1}{(x+2)^{3/2}}\right) \text{ for all } x \geq 0,$$

so we obtain

$$0 < M - \sum_{\substack{f \leq \sqrt[3]{x} \\ f \text{ squarefree}}} \frac{1}{f^{5/2}} \leq \frac{1}{([\sqrt[3]{x}]+1)^{5/2}} + \frac{1}{([\sqrt[3]{x}]+2)^{5/2}} + \ldots <$$

$$< k\left(\frac{1}{([\sqrt[3]{x}]+1)^{3/2}} - \frac{1}{([\sqrt[3]{x}]+2)^{3/2}} + \frac{1}{([\sqrt[3]{x}]+2)^{3/2}} - \ldots\right) = O\left(\frac{1}{\sqrt{x}}\right).$$

Therefore,

$$\sum_{\substack{f \leq \sqrt[3]{x} \\ f \text{ squarefree}}} \frac{1}{f^{5/2}} = M + O\left(\frac{1}{\sqrt{x}}\right). \tag{10}$$

Secondly, we have

$$0 < \sum_{\substack{f \leq \sqrt[3]{x} \\ f \text{ squarefree}}} \frac{1}{f} < \sum_{t \leq \sqrt[3]{x}} \frac{1}{t} = O(\ln \sqrt[3]{x}) = O(\ln x),$$

so

$$\sum_{\substack{f \leq \sqrt[3]{x} \\ f \text{ squarefree}}} \frac{1}{f} = O(\ln x). \tag{11}$$

Considering (10) and (11),(9) leads to:

$$M\sqrt{x} + O(1) + O(\ln x) - 1 < S_3(x) \leq M\sqrt{x} + O(1).$$

We have thus proved that

$$S_3(x) = M\sqrt{x} + O(\ln x).$$

∎

The last result presented here gives an estimation of the product of the terms of the sequence $(c_k)_{k\geq 1}$.
Let $P(x) = \prod_{\substack{k\leq x\\ k \text{ powerful}}} c_k$.

Theorem 4. *We have the relation*

$$P(x) = e^{a\sqrt{x}+O(\sqrt[3]{x}\ln x)},$$

where $a = \sum_{f \text{ squarefree}} \frac{\ln f}{f^{3/2}}$.

Proof: Using the same counting method as in Theorem 3, we obtain

$$P(x) = \prod_{\substack{f\leq \sqrt[3]{x}\\ f \text{ squarefree}}} f^{\left[\sqrt{\frac{x}{f^3}}\right]},$$

therefore

$$\ln P(x) = \sum_{\substack{f\leq \sqrt[3]{x}\\ f \text{ squarefree}}} \left[\sqrt{\frac{x}{f^3}}\right] \ln f.$$

Taking into account that $x-1 < [x] \leq x$ for all x, this leads to

$$\sqrt{x} \sum_{\substack{f\leq \sqrt[3]{x}\\ f \text{ squarefree}}} \frac{\ln f}{f^{3/2}} - \sum_{\substack{f\leq \sqrt[3]{x}\\ f \text{ squarefree}}} \ln f < \ln P(x) \leq \sqrt{x} \sum_{\substack{f\leq \sqrt[3]{x}\\ f \text{ squarefree}}} \frac{\ln f}{f^{3/2}}. \tag{12}$$

Since $a = \sum_{f \text{ squarefree}} \frac{\ln f}{f^{3/2}}$ and the function $g(x) = \frac{\ln x}{x^{3/2}}$ is decreasing for $x > e^{2/3}$, we have

$$0 < a - \sum_{\substack{f\leq \sqrt[3]{x}\\ f \text{ squarefree}}} \frac{\ln f}{f^{3/2}} < \frac{\ln([\sqrt[3]{x}]+1)}{([\sqrt[3]{x}]+1)^{3/2}} + \frac{\ln([\sqrt[3]{x}]+2)}{([\sqrt[3]{x}]+2)^{3/2}} + \ldots <$$

$$< \frac{\ln\sqrt[3]{x}}{x^{3/2}} + \frac{\ln(\sqrt[3]{x}+1)}{(x+1)^{3/2}} + \ldots$$

Taking into account that there exists a positive number k such that

$$\frac{\ln y}{y^{3/2}} < k\left(\frac{1}{\sqrt[3]{y-1}} - \frac{1}{\sqrt[3]{y}}\right) \text{ for all } y \geq 2,$$

we obtain

$$\sum_{\substack{f\leq \sqrt[3]{x}\\ f \text{ squarefree}}} \frac{\ln f}{f^{3/2}} = a + O\left(\frac{1}{\sqrt{x}}\right) \tag{13}$$

Since

$$\ln n! \sim n \ln n,$$

we obtain

$$0 < \sum_{\substack{f \le \sqrt[3]{x} \\ f \text{ squarefree}}} \ln f \le \sum_{n \le \sqrt[3]{x}} \ln n = \ln[\sqrt[3]{x}]! \sim \sqrt[3]{x} \ln \sqrt[3]{x} = O(\sqrt[3]{x} \ln x).$$

Therefore,

$$\sum_{\substack{f \le \sqrt[3]{x} \\ f \text{ squarefree}}} \ln f = O(\sqrt[3]{x} \ln x). \tag{14}$$

From (12), (13) and (14) we derive that

$$a\sqrt{x} + O(1) + O(\sqrt[3]{x} \ln x) < \ln P(x) \le a\sqrt{x} + O(1),$$

so

$$P(x) = e^{a\sqrt{x} + O(\sqrt[3]{x} \ln x)}.$$

□

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Received: 20.03.2007.

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