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Separately locally holomorphic functions and their singular sets

by
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Abstract

The main aim of the present note is to generatize the Siciak's result to separately locally holomorphic functions on the crosses.

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The classical Hartogs Theorem states that every separately holomorphic functions on products of domains in complex Euclidian spaces is holomorphic. This famous theorem was generalized by several authors (see for example [AZ], [JP], [Si1], [Si2]). Recently Siciak [Si2] and after Blocki [BL] have considered the above theorem for separately real analytic sets. For example in [Si2] Siciak has proved that if f is a separately real analytic set in the product $U \times V$ in $\mathbb{R}^p \times \mathbb{R}^q$, the set of points at which f is not analytic is pluripolar in $\mathbb{C}^p \times \mathbb{C}^q$.
Namely in notions of §1 we prove the following

Theorem A: *Let K and L be connected non-pluripolar sets of type F_σ in $\mathbb{C}^p$ and $\mathbb{C}^q$ respectively, $E \subset K$ and $F \subset L$ be non-pluripolar. Let $f : (E \times L) \cup (K \times F) \to \mathbb{C}$ be a separately locally holomorphic function. Then there exist pluripolar sets $E' \subset E$ and $F' \subset F$ such that f is locally holomorphic on $((E \setminus E') \times L) \cup (K \times (F \setminus F'))$.*

Theorem B: *Let K and L be non-pluripolar convex sets in $\mathbb{C}^p$ and $\mathbb{C}^q$ respectively, $E \subset K$ and $F \subset L$ be non-pluripolar. Let $f : (E \times L) \cup (K \times F) \to \mathbb{C}$ be separately locally holomorphic. Then there exist pluripolar sets $E' \subset E$ and $F' \subset F$ such that f is extended holomorphically to a neighbourhood of $((E \setminus E') \times L) \cup (K \times (F \setminus F'))$.*

1 Preliminaries

In this section we recall some notions used in complex analysis and complex pluripotential theory.

1.1. Let $f : X \to \mathbb{C}$, where $X \subset \mathbb{C}^p$. We say that f is locally holomorphic on X if for every $x \in X$ there exist a neighbourbood U of x on $\mathbb{C}^p$ and a holomorphic function $\widetilde{f}$ on U such that $\widetilde{f} = f$ on $U \cup X$. By $\Omega(f)$ we denote the set of points of X at which f is holomorphic. The set $S(f) := X \setminus \Omega(f)$ is called the singular set of f.

1.2. Let $E \subset K \subset \mathbb{C}^p$ and $F \subset L \subset \mathbb{C}^q$ be arbitrary subsets. The set

$$W(E, F, K, L) = (E \times L) \cup (K \times F)$$

is called the cross of E, F, K and L. The function $f : W(E, F, K, L) \to \mathbb{C}$ is said to be locally holomorphic if

i) For every $x \in E$, the function $f^x(y) := f(x, y)$ is locally holomorphic on L

ii) For every $y \in F$, the function $f_y(x) = f(x, y)$ is locally holomorphic on K.

1.3. Let Ω be an open set in $\mathbb{C}^p$. By $\mathrm{PSH}(\Omega)$ we denote the set of all plurisubharmonic (psh) functions on Ω. A subset X of $\mathbb{C}^p$ is called pluripolar if for every $z \in X$ there exist a neighbourbood U of z and $\varphi \in \mathrm{PSH}(U), \varphi \not\equiv -\infty$ such that

$$X \cap U \subset \{z' \in U : \varphi(z') = -\infty\}.$$

It is well known that X is pluripolar if and only if there exists
$\varphi \in \mathrm{PSH}(\mathbb{C}^p), \varphi \not\equiv -\infty$ such that

$$X \subset \{z \in \mathbb{C}^p : \varphi(z) = -\infty\}.$$

1.4. Let $K \subset \Omega$ with Ω is an open set in $\mathbb{C}^p$. We denote $u_{K,\Omega}$ the relatively extremal function of the couple (K, Ω) defined by

$$u_{K,\Omega}(z) = \sup\{u(z) : u \in \mathrm{PSH}(\Omega), u \leqslant -1 \text{ on } K \text{ and } u \leqslant 0 \text{ on } \Omega\}.$$

Let $\omega(., K, \Omega) := u_{K,\Omega}$ be the upper-semicontinous regularization of $u_{K,\Omega}$:

$$\omega(z, K, \Omega) = \overline{\lim_{\Omega \ni z' \to z}} u_{K,\Omega}(z'), z \in \Omega.$$

The point $a \in \Omega$ is called the pluri-regular point of K if $a \in \overline{K} \cap \Omega$ and $\omega(a, K \cap U, U) = 0$ for all neighbourbood U of a in Ω. Denote K^* the set of all pluri-regular points of K (in Ω). If K is not pluripolar, then Theorem 7.1 in [BT] implies that $K \setminus K^*$ is pluripolar.

2 Proof of the Theorem A

Let $\{K_j\}_{j\geqslant 1}$ and $\{L_j\}$ be increasing sequences of compact sets, K_j and L_j in K and L respectively such that $K = \bigcup\limits_{j\geqslant 1}^{\infty} K_j$ and $L = \bigcup\limits_{j\geqslant 1}^{\infty} L_j$. We may assume that K_j and L_j are non-pluripolar. For each $x \in E$ and $y \in F$, put

$$\varepsilon_x^j = \inf_{y\in L_j}\{r_y : f_x \in \mathrm{Hol}(\mathbb{B}(y, r_y)\}$$
$$\varepsilon_y^j = \inf_{x\in K_j}\{r_x : f^y \in \mathrm{Hol}(\mathbb{B}(x, r_x)\},$$

where

$$r_y = \sup\{r > 0 : f_x \in \mathrm{Hol}(\mathbb{B}(y, r)\}$$

and

$$r_x = \sup\{r > 0 : f^y \in \mathrm{Hol}(\mathbb{B}(x, r)\}.$$

$\mathbb{B}(y, r_y) \subset \mathbb{C}^q$, $\mathbb{B}(x, r_x) \subset \mathbb{C}^p$ are balls centered at y and x with radius r_y and r_x in $\mathbb{C}^p$ and $\mathbb{C}^q$ respectively.

Moreover $\mathrm{Hol}(\mathbb{B}(y, r_y))$ and $\mathrm{Hol}(\mathbb{B}(x, r_x))$ denote the spaces of holomorphic functions on $\mathbb{B}(y, r_y)$ and $\mathbb{B}(x, r_x)$ respectively. It is easy to see that $\varepsilon_x^j > 0$ and $\varepsilon_y^j > 0$ for $x \in E$ and $y \in F$ and $j \geqslant 1$ because if $f_x \in \mathrm{Hol}(\mathbb{B}(y, r))$ and $f^y \in \mathrm{Hol}(\mathbb{B}(x, r))$ then $f_x \in \mathrm{Hol}(\mathbb{B}(y', \frac{r}{2}))$ for $||y'-y|| < \frac{r}{2}$ and $f^y \in \mathrm{Hol}(\mathbb{B}(x', \frac{r}{2}))$ for $||x' - x|| < \frac{r}{2}$. For $j \geqslant 1$, put

$$A_n^j = \big\{x \in E : \varepsilon_x^j \geqslant \frac{1}{n}\big\}$$

and

$$B_n^j = \big\{y \in F : \varepsilon_y^j \geqslant \frac{1}{n}\big\}.$$

Then $E = \bigcup\limits_{n=1}^{\infty} A_n^j$ and $F = \bigcup\limits_{n=1}^{\infty} B_n^j$. Since E and F are non-pluripolar sets, we can find $n \geqslant 1$ such that $A_j := A_n^j$ and $B_j := B_n^j$ are non-pluripolar. For $x \in A_j^*$ and $y \in B_j^*$ consider

$$W = \big((A_j^* \cap \mathbb{B}\big(x, \frac{1}{n}\big)\big) \times \mathbb{B}\big(y, \frac{1}{n}\big)\big) \cup \big(\mathbb{B}\big(x, \frac{1}{n}\big) \times \big(B_j^* \cap \mathbb{B}\big(y, \frac{1}{n}\big)\big)\big)$$

and $\widehat{f} : W \to \mathbb{C}$ defined by

$$\widehat{f}(x', y') = \begin{cases} \widehat{f}_{x'}(y') \text{ for } (x', y') \in (A_j^* \cap \mathbb{B}\big(x, \frac{1}{n}\big)\big) \times \mathbb{B}\big(y, \frac{1}{n}\big) \\ \widehat{f}^{y'}(x') \text{ for } (x', y') \in \mathbb{B}\big(x, \frac{1}{n}\big) \times \big(B_j^* \cap \mathbb{B}\big(y, \frac{1}{n}\big)\big). \end{cases}$$

It follows that $\widehat{f}$ is separately holomorphic on W. Theorem 2.2.4 in [AZ] implies that $\big(x \times \mathbb{B}(y, \frac{1}{n})\big) \cup \big(\mathbb{B}(x, \frac{1}{n}) \times y\big) \subset \Omega(f)$. Thus

$$\bigcup_{x \in A_j^*,\ y \in B_j^*} \big[\big(x \times \mathbb{B}(y, \frac{1}{n})\big) \cup \big(\mathbb{B}(x, \frac{1}{n}) \times y\big)\big] \subset \Omega(f).$$

For $x \in \widetilde{A} := \bigcup\limits_{j=1}^{\infty} A_j^*$,

$$G_x = \{y \in L : (x, y) \in \Omega(f)\}.$$

Since K is not pluripolar by a result of Bedford - Taylor (see theorem 7.1 [BT]) $\widetilde{A} \neq \varnothing$ and hence $G_x \neq \varnothing$ are open. Given $y \in \partial G_x \cap L$, take $j \geqslant 1$ such that $x \in A_j^*$, $y \in L_j$ and $y' \in G_x \cap \mathbb{B}(y, \varepsilon_x^j)$. Then there exists $\delta > 0$ such that f is holomorphic on $\mathbb{B}(x, \delta) \times \mathbb{B}(y', \delta)$. Consider the cross

$$M = (A_j^* \cap \mathbb{B}(x, \delta) \times \mathbb{B}(y, \delta)) \cup (\mathbb{B}(x, \delta) \times \mathbb{B}(y', \delta)).$$

Theorem 2.2.4 in [AZ] implies that $y \in G_x$. By the connectedness of L we deduce that $G_x = L$ and hence $\widetilde{A} \times L \subset \Omega(f)$. Similary $K \times \widetilde{B} \subset \Omega(f)$. Thus

$$\Big(\bigcup_{(j,n) \in I} (A_n^j)^* \times L\Big) \cup \Big(K \times \bigcup_{(j,n) \in I} (B_n^j)^*\Big) \subset \Omega(f)$$

where

$$I = \{(j, n) : A_n^j \text{ and } B_n^j \text{ are non-pluripolar}\}.$$

Put

$$E' = \bigcup_{(j,n) \in I} (A_n^j \setminus A_n^{j*}) \text{ and } F' = \bigcup_{(j,n) \in I} (B_n^j \setminus B_n^{j*}).$$

Then E', F' are pluripolar and

$$(E \setminus E') \cup (K \times (F \setminus F')) \subset \Omega(f).$$

The theorem is proved.

3 Proof of the Theorem B

By Theorem A we can find pluripolar sets $E' \subset E$ and $F' \subset F$ such that

$$W(E \setminus E', F \setminus F', K, L) \subset \Omega(f).$$

By the convexity and non-pluripolarity of L every $f_x,\ x \in E \setminus E'$ has a holomorphic extension $\widehat{f}_x$ to a convex neighbourhood $\mathcal{O}_x \times V_x$ of $x \times L$. Similarly every f^y, $y \in F \setminus F'$ has a holomorphic extension $\widehat{f}^y$ to a convex neighbourhood $U_y \times G_x$ of $K \times y$. Let $x_1, x_2 \in E \setminus E'$ with $\mathcal{O}_{x_1} \cap \mathcal{O}_{x_2} \neq \varnothing$. Then $E \cap \mathcal{O}_{x_1} \cap \mathcal{O}_{x_2}$ is not pluripolar. Since E is convex and non-pluripolar, it follows that E is locally non-pluripolar. This yields that

$$(E \setminus E') \cap \mathcal{O}_{x_1} \cap \mathcal{O}_{x_2} \neq \varnothing.$$

For $y \in L$ and $x \in (E \setminus E') \cap \mathcal{O}_{x_1} \cap \mathcal{O}_{x_2}$ we have

$$\widehat{f}_{x_1}(x, y) = \widehat{f}_{x_2}(x, y).$$

This implies by the non-pluripolarity of $(E \setminus E') \cap \mathcal{O}_{x_1} \cap \mathcal{O}_{x_2}$ that

$$\widehat{f}_{x_1}(x, y) = \widehat{f}_{x_2}(x, y)$$

for $x \in \mathcal{O}_{x_1} \cap \mathcal{O}_{x_2}$.

Again by the non-pluripolarity of L we have

$$\widehat{f}_{x_1}(x, y) = \widehat{f}_{x_2}(x, y)$$

for $x \in \mathcal{O}_{x_1} \cap \mathcal{O}_{x_2}$ and $y \in V_{x_1} \cap V_{x_2}$. Thus the family of holomorphic functions $\{\widehat{f}_x\}_{x \in E \setminus E'}$ defines a holomorphic function h on $\mathcal{O} \times L$, where

$$\mathcal{O} = \bigcup \{\mathcal{O}_x : x \in E \setminus E'\} \text{ such that } h = f \text{ on } (E \setminus E') \times L.$$

Since L is compact and convex, it follows that h can be extended to a holomorphic function $\widehat{h}$ on a neighbourhood $\mathcal{O} \times V$ of $\mathcal{O} \times L$, where V is a convex neighbourhood of L.

Similarly the family $\{\widehat{f}^y\}_{y \in F \setminus F'}$ defines a holomorphic function $\widehat{g}$ on a neighbourhood $U \times G$ of $K \times G$, where $U \times G$ a convex neighbourhood of K and $G = \bigcup \{G_y : y \in F \setminus F'\}$. By the connectedness of $\mathcal{O}_x \cap U \times G_y \cap V$ and the non-pluripolarity of $\mathcal{O}_x \cap K$, $G_y \cap L, x \in E \setminus E', y \in F \setminus F'$ and by $h = f = g$ on $\mathcal{O}_x \cap K \times G_y \cap L$, it follows that $\widehat{h} = \widehat{g}$ on $\mathcal{O}_x \cap U \times G_y \cap V$. Hence $\widehat{h} = \widehat{g}$ on

$$\bigcup \{\mathcal{O}_x \cap U \times G_y \cap V : x \in E \setminus E', y \in F \setminus F'\}.$$

Consequently $\widehat{h}$ and $\widehat{g}$ define a holomorphic extension of f to $(\mathcal{O} \times V) \cup (U \times G)$, a neighbourhood of $W(E \setminus E', F \setminus F', K, L)$. The theorem is proved.

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