Bull. Math. Soc. Sci. Math. Roumanie
Tome 51(99) No. 2, 2008, 123–135

Asymptotic behavior of discrete and continuous semigroups on Hilbert spaces*

by
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Abstract

Let $\phi : [0, \infty) \to [0, \infty)$ be a nondecreasing function with $\phi(t) > 0$ for all $t > 0$, H be a complex Hilbert space and let T be a bounded linear operator acting on H. Among our results is the fact that T is power stable (i.e. its spectral radius is less than 1) if

$$\sum_{n=0}^{\infty} \phi(|\langle T^n x, x\rangle|) < \infty$$

for all $x \in H$ with $||x|| \leq 1$.

In the continuous case we prove that a strongly continuous uniformly bounded semigroup of operators acting on a Hilbert space H is spectrally stable (i.e. the spectrum of its infinitesimal generator lies in the open left half plane) if and only if for each $x \in H$ and each $\mu \in \mathbb{R}$ one has:

$$\sup_{s \geq 0} \left| \int_0^s e^{-i\mu t} \langle T(t)x, x\rangle dt \right| < \infty.$$

Key Words: Spectral radius, discrete semigroups, strongly continuous semigroups, uniform exponential stability, Orlicz space.
2000 Mathematics Subject Classification: Primary 47D03, Secondary 11M35.

1 Introduction

Let X be a complex Banach space and X^* be its dual. The resolvent set and the spectrum of a linear operator T (acting on X) will be denoted respectively by $\rho(T)$ and $\sigma(T)$. When T is bounded, the spectrum radius of T is given by the formula

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\} = \lim_{n\to\infty} ||T^n||^{1/n}.$$

*The research of the first author was partially supported by the Grant CNCSIS 546 (2007) of the Romanian Ministry of Education and Research.

Jan van Neerven ([13]) has shown that $r(T) < 1$ if there exists a Banach function space E over $\mathbb{N}$ with

$$\lim_{n\to\infty} ||\chi_{\{0,1,\ldots,n-1\}}||_E = \infty \tag{1.1}$$

such that for each $x \in X$ and $x^* \in X^*$ the map $n \mapsto |\langle T^n x, x^*\rangle|$ belongs to E. By applying this result to an appropriate Orlicz space E, he also able to show that $r(T) < 1$ if there exists a non-decreasing function $\phi : [0,\infty) \to [0,\infty)$ such that $\phi(t) > 0$ for all $t > 0$ and

$$\sum_{n=0}^{\infty} \phi(|\langle T^n x, y\rangle|) < \infty. \tag{1.2}$$

for all $x \in X$ and $y \in X^*$. The particular case where $\phi(t) = t^p$ for some $(1 \le p < \infty)$ has been noticed by G. Weiss ([21]). It is worth to mention the contribution of K. L. Przyulski ([18]) who considered the case where $\phi(t) = t$, in the context of weakly sequentially complete Banach spaces. The similar topics in the continuous case started with a question raised by A. J. Pritchard and J. Zabczyk [20]. Precisely, they asked whether any weakly-L^p-stable semigroup is necessarily uniformly exponentially stable. See the next section for relevant definitions. An account on the research related to this question can be found in [7], [22], [13].

The aim of this paper is to prove the following.

Theorem 1. *Let T be a bounded linear operator acting on a complex Hilbert space H and let $\phi : [0,\infty) \to [0,\infty)$ be a nondecreasing function such that $\phi(t) > 0$ for all $t > 0$ and*

$$\sum_{n=0}^{\infty} \phi(|\langle T^n x, x\rangle|) < \infty. \tag{1.3}$$

for every $x \in H$. Then $r(T) < 1$.

Theorem 2. *Let $\mathbf{T} = \{T(t)\}_{t\ge 0}$ be a strongly continuous semigroup on a complex Hilbert space H and let $\phi : [0,\infty) \to [0,\infty)$ be a non-decreasing function with $\phi(t) > 0$ for all $t > 0$. If $\mathbf{T}$ is uniformly bounded (that is, $\sup_{t\ge 0} ||T(t)|| < \infty$) and for each $x \in H$, one has:*

$$\int_0^{\infty} \phi(|\langle T(t)x, x\rangle|)dt < \infty, \tag{1.4}$$

then the semigroup $\mathbf{T}$ is uniformly exponentially stable, that is, its uniform growth bound $\omega_0(\mathbf{T}) := \inf_{t>0} \frac{\ln ||T(t)||}{t}$ is negative.

Theorem 3. *Let $\mathbf{T} = \{T(t)\}_{t\ge 0}$ be a strongly continuous semigroup on a complex Hilbert space H. If for each $x \in H$ one has*

$$\sup_{\mu\in\mathbb{R}} \sup_{s\ge 0} \left| \int_0^s e^{-i\mu t} \langle T(t)x, x\rangle dt \right| = M(x) < \infty \tag{1.5}$$

then the semigroup $\mathbf{T}$ *is uniformly exponentially stable.*

Theorem 4. *Let* $\mathbf{T} = \{T(t)\}_{t\geq 0}$ *be a strongly continuous and uniformly bounded bounded semigroup on a complex Hilbert space* H*. The following five statements are equivalent:*

1. *The semigroup* $\mathbf{T}$ *is spectrally stable (that is, the spectrum of its infinitesimal generator* $\sigma(A)$ *lies in the open left half plane* $\mathbb{C}_- := \{z \in \mathbb{C} : \Re(z) < 0.\}$*)*

2. *For each* $\mu \in \mathbb{R}$ *and each* $x \in H$*, we have that*

$$\sup_{s\geq 0}\left\|\int_0^s e^{-i\mu t}T(t)x\,dt\right\| = L(\mu, x) < \infty. \tag{1.6}$$

3. *For each* $\mu \in \mathbb{R}$ *and each* $x \in H$*, one has:*

$$\sup_{s\geq 0}\left|\int_0^s \langle e^{-i\mu t}T(t)x, x\rangle dt\right| = M(\mu, x) < \infty.$$

4. $\sigma(A) \cap i\mathbb{R} = \emptyset$.

5. *For each* $\mu \in \mathbb{R}$ *and each* $x \in X$ *the solution of the inhomogeneous Cauchy Problem*

$$\dot{u}(t) = Au(t) + e^{i\mu t}x, \quad u(0) = 0, \tag{A,μ,x}$$

is bounded.

To the best of our knowledge these results are new even if the proofs are not very difficult. The proof of the Theorem 1 is based on a technical lemma stated in the third section of our paper. In fact, if combining this Lemma with the proof of Theorem 3.3 from [13], originally given by Jan van Neerven, we obtain that the our condition (1.3) and condition (1.2) are equivalent.

If in addition to (1.3) assume that the discrete semigroup (T^n) is strongly asymptotically stable (i.e. $T^n x$ tends to 0 when $n \to \infty$ for any $x \in H$) then we can give a completely different proof for Theorem 1 using a very recent result from the operator theory, originally given by V. Müller, see [12].

If ϕ is the identity map and (1.3) is fulfilled then the formal series $(\sum T^n x)$ is convergent in the norm of H, so in particular (T^n) is strongly asymptotically stable. This result remain true even if put T_n instead of T^n, where T_n is an arbitrary bounded linear operator acting on H, such as is stated in Proposition 1 from the third section of this paper.

The paper is organized as follows. Section 2 contains the results in the case of self-adjoint operators. In the Section 3 we prove Theorem 1 and consider some natural consequences, while the last section is devoted to the proof of the Theorems 2, 3 and 4.

2 The case of self-adjoint operators

In this section consider the case when the semigroups are self-adjoint and ϕ is the identity map. We begin with the following well-known lemma. See for example [2] for a proof.

Lemma 1. *Let S be a bounded linear operator acting on a Banach space X and μ be a real number such that*

$$\sup_{N\in\mathbb{N}}\left\|\sum_{n=0}^{N} e^{-i\mu n}S^n\right\| = M_\mu < \infty.$$

Then S is power bounded and $e^{i\mu}\in\rho(T)$.

Theorem 5. *Let T be a self-adjoint operator acting on a complex Hilbert space H. If*

$$\sup_{||x||\le 1}\sum_{n=0}^{\infty}|\langle T^n x, x\rangle| := K < \infty$$

then the spectral radius of T is less then 1.

Proof: For every integer number k and any positive integer number n the operator $e^{-ikn\pi}T^n$ is self-adjoint. Then using a well-known result ([23], Theorem 3, page 201) we get:

$$\begin{aligned}\left\|\sum_{n=0}^{N} e^{-ikn\pi}T^n\right\| &= \sup_{||x||\le 1}\left|\sum_{n=0}^{N} e^{-ikn\pi}\langle T^n x, x\rangle\right| \\ &\le \sup_{||x||\le 1}\sum_{n=0}^{\infty}|\langle T^n x, x\rangle| = K < \infty.\end{aligned}$$

Then the operator T is power bounded and its spectral radius is less than or equal 1. On the other hand the spectrum of T is real. The spectral radius of T is less than 1 because in according with Lemma 1 above, the set $\{-1, 1\}$ belongs to $\rho(T)$. □

Let $1 \le p < \infty$. In order to introduce similar results in the continuous case we recall that a strongly continuous semigroup $\mathbf{T} = \{T(t)\}_{t\ge 0}$ on a Hilbert space H is called *weakly-L^p-stable* if for each $x \in H$ and $y \in H$ one has

$$\int_0^{\infty} |\langle T(t)x, y\rangle|^p dt < \infty.$$

It is known ([7]), [22]) that every weakly-L^p-stable semigroup $\mathbf{T}$ is uniformly exponentially stable, that is, its uniform growth bound

$$\omega_0(\mathbf{T}) := \inf_{t>0}\frac{\ln ||T(t)||}{t}$$

is negative. A possible new proof of this result in the case $p = 1$ can be stated as follows.

Lemma 2. *Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X. If for each $x \in X$ one has*

$$\sup_{t\geq 0} \left\| \int_0^t T(s)x ds \right\| := M(x) < \infty \tag{2.1}$$

then the half-plane $\{\Re(\lambda) > 0\}$ lies in $\rho(A)$. Moreover $0 \in \rho(A)$ and if

$$M := \sup_{t\geq 0} \left\| \int_0^t T(s) ds \right\|_{\mathcal{L}(X)}$$

then $||R(0, A)|| \leq M$.

Proof: See [16]. □

We remark that if (2.1) holds with $e^{-i\mu t}T(t)$ instead of $T(t)$ for all $\mu \in \mathbb{R}$, then there exists a positive constant L such that

$$\sup_{\mathrm{Re}\ (\lambda)\geq 0} ||R(\lambda, A)|| = L < \infty.$$

Combining this with the Gerhart-Prüss theorem (see e.g. [17], [7]) we get the following well known result ([16]):

Corollary 1. *Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup on a Hilbert space H. If for each $x \in H$ one has*

$$\sup_{\mu\in\mathbb{R}} \sup_{t\geq 0} \left\| \int_0^t e^{-i\mu s} T(s)x ds \right\| := N(x) < \infty$$

then the semigroup $\mathbf{T}$ is uniformly exponentially stable.

Suppose now that the semigroup $\mathbf{T}$ acts on the Hilbert space H and it is weakly-L^1-stable. Then for all $x \in H, y \in H$ with $||x|| \leq 1$ and $||y|| \leq 1$ one has:

$$\left\| \int_0^t e^{-i\mu s} T(s) ds \right\| \leq \int_0^\infty |\langle T(s)x, y\rangle| ds \leq \text{ Constant } < \infty.$$

From the Corollary 1 above follows that the semigroup $\mathbf{T}$ is uniformly exponentially stable.

Theorem 6. *Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a strongly continuous and self-adjoint operator semigroup on a Hilbert space H. If*

$$\sup_{||x||\leq 1} \int_0^\infty |\langle T(t)x, x\rangle| dt = K < \infty,$$

then the semigroup $\mathbf{T}$ is uniformly exponentially stable.

Proof: First remark that the above inequality (2.1) holds. Indeed:

$$\begin{aligned}\sup_{t\geq 0}\left\|\int_0^t T(s)ds\right\|_{\mathcal{L}(H)} &= \sup_{t\geq 0}\sup_{||x||\leq 1}|\langle\int_0^t T(s)xds, x\rangle| \\ &\leq \sup_{||x||\leq 1}\int_0^\infty |\langle T(s)x, x\rangle|ds \leq K.\end{aligned}$$

From the above Lemma 2 follows that the spectrum of A lies in the interval $(-\infty, 0)$. On the other hand

$$\sup_{\Re(\lambda)>0}||R(\lambda, A)|| \leq \sup_{\Re(\lambda)>0}\sup_{||x||\leq 1}\int_0^\infty e^{-\Re(\lambda)t}|\langle T(t)x, x\rangle|dt \leq K.$$

Now we can apply the Gerhart-Prüss theorem. $\square$

3 Proof of Theorem 1 and some natural consequences

We recall some well-known facts about Orlicz spaces. For further details we refer to [9], [10], [11], [4] and references therein. Let $\Phi : [0, \infty) \to [0, \infty]$ be a convex, non-decreasing function such that $\Phi(0) = \Phi(0+) = 0$, and Φ is not identically with 0 or with ∞ on $(0, \infty)$. Let $\mathbb{Z}_+$ be the set of all non-negative integers. For each scalar-valued sequence $a = (a_\nu)_{\nu\in\mathbb{Z}_+}$ let us consider $M^\Phi(a) := \sum\limits_{n\in\mathbb{Z}_+}\Phi(|a_n|)$ and the set L^Φ of all sequences (a_ν) for which there exists a positive real number λ such that $I_\Phi(\lambda a) < \infty$. The space L^Φ can be endowed with the Luxemburg norm, given by:

$$||a||_{L^\Phi} := \inf\{\lambda > 0 : M^\Phi(\lambda^{-1}a) \leq 1\}.$$

The Orlicz spaces over $\mathbb{R}_+$ can be defined by a similar manner. Precisely in this case L^Φ is the set of all complex valued measurable functions f defined on $\mathbb{R}_+$ for which there exists a positive λ such that $\int_0^\infty \Phi(\lambda|f(t)|)dt < \infty$.

The Luxemburg norm of a function $f \in L^\Phi$ is defined by

$$\rho^\Phi(f) := \inf\{k > 0 : \int_0^\infty \Phi(k^{-1}|f(t)|)dt \leq 1\}.$$

Some useful identities are collected in the next Lemma.

Lemma 3. *Let H be a complex Hilbert space, x and y in H, μ be a real number, T be a bounded linear operator acting on H and $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup of bounded linear operators on H. For each $t \geq 0$ let $\rho_\mu(t) := e^{-i\mu t}T(t)$. The following identities are fulfilled:*

$$\begin{aligned}\langle Tx, y\rangle &= \tfrac{1}{2i}[(1-i)(\langle Tx, x\rangle + \langle Ty, y\rangle)+ \\ &+ i\langle T(x+y), x+y\rangle - \langle T(x+iy), x+iy\rangle]\end{aligned} \tag{3.1}$$

and

$$\int_0^s \langle \rho_\mu(t)x, y\rangle dt = \frac{1}{2i}(1-i)\left(\int_0^s \langle \rho_\mu(t)x, x\rangle dt + \int_0^s \langle \rho_\mu(t)y, y\rangle dt\right) +$$

$$+\frac{1}{2}\int_0^s \langle \rho_\mu(t)(x+y), x+y\rangle dt + \frac{1}{2i}\int_0^s \langle \rho_\mu(t)(x+iy), x+iy\rangle dt. \tag{3.2}$$

Proof of the Theorem 1

Using the identity (3.1) it is easily to establish the following inequality:

$$|\langle T^n x, y\rangle| \leq \frac{\sqrt{2}}{2}(|\langle T^n x, x\rangle| + |\langle T^n y, y\rangle|) +$$

$$+\frac{1}{2}(|\langle T^n(x+y), x+y\rangle| + |\langle T^n(x+iy), x+iy\rangle|). \tag{3.3}$$

In view of (1.3) follows that for each $x \in H$, $\phi(|\langle T^n x, x\rangle|)$ tends to 0 when n tends to ∞ and then $|\langle T^n x, x\rangle|$ tends to 0 as well. As a consequence, the maps $n \mapsto |\langle T^n z, z\rangle|$ with $z \in \{x, y, x+y, x+iy\}$ are bounded and moreover them belong to a same Orlicz space E satisfying the condition (1.1), see the proof of Theorem 3.3 by [13]. In view of (3.3), and using the fact the every Orlicz space has the ideal property, follows that for each x and y in H the map $n \mapsto |\langle T^n x, y\rangle|$ belongs to E. Now we can apply van Neerven's theorem, which was reminded in the beginning of our paper, in order to obtain that $r(T) < 1$.

In particular, from (1.3) follows that $||T^n x||$ decays to 0 for any $x \in H$. In the case when ϕ is the identity map, we generalize the latter result, as follows:

Proposition 1. *Let (T_n) be a sequence of bounded linear operators acting on a complex Hilbert space H. The following two statements are equivalent:*

(i) *For each $x \in H$ the series $(\sum_{n\geq 0} |\langle T_n x, x\rangle|)$ is convergent.*

(ii) *For each $x \in H$ and each $y \in H$ the series $(\sum_{n\geq 0} |\langle T_n x, y\rangle|)$ is convergent.*

Moreover, these statements imply the fact that the (formal) series $(\sum_{n\geq 0} T_n x)$ is convergent in the norm of H.

Proof:

$$|\langle T_n x, y\rangle| \leq \frac{\sqrt{2}}{2}(|\langle T_n x, x\rangle| + |\langle T_n y, y\rangle|) +$$

$$+\frac{1}{2}(|\langle T_n(x+y), x+y\rangle| + |\langle T_n(x+iy), x+iy\rangle|).$$

Now, it is clear that the statement **(ii)** is a consequence of **(i)**. On the other hand by the inequality

$$|\sum_{n\geq 0} \langle T_n x, y\rangle| \leq \sum_{n\geq 0} |\langle T_n x, y\rangle|$$

and the statement (**ii**) follows that any subseries of the series

$$(\sum_{n\geq 0}\langle T_n x, y\rangle)$$

is convergent. Then the latter assertion follows by the Orlicz-Pettis theorem, see [5], page 22. $\square$

We are very grateful to Nigel Kalton for pointing out of the above result [8].

With the supplementary hypothesis that $||T^n x||$ decays to 0 for any $x \in H$ we can give another proof of Theorem 1. We use a recent result of V. Müler, [12], which reads as follows: *Let T be a linear and bounded operator acting on a complex Hilbert space H such that $1 \in \sigma(T)$ and $||T^n x||$ decays to 0 for all $x \in H$. Let $(a_n)_{n=1}^{\infty}$ be a non-increasing sequence with $\lim_{n\to\infty} a_n = 0$ and $\sup a_n < 1$. Then there exists $x \in H$ of norm one such that Re $\langle T^n x, x\rangle > a_n$ for all $n \geq 1$.* In such circumstances the operator T is power bounded hence its spectral radius $r(T)$ is less than or equal one. Suppose for the contrary that $r(T) = 1$. Then there exists $\mu \in \mathbb{R}$ such that $e^{i\mu} \in \sigma(T)$. In order to apply the above result we may suppose that $1 \in \sigma(T)$. Indeed, if it is not true, put $S := e^{-i\mu}T$ instead of T. It is clear that $\phi(0) = 0$. We may suppose that $\phi(1) = 1$ and that ϕ is a strictly increasing and continuous function on $\mathbb{R}_+$. If not, put a multiple of $\bar{\phi}$ instead of ϕ, where

$$\bar{\phi}(t) := \int_0^t \phi(s)ds \text{ if } 0 \leq t \leq 1 \text{ and } \bar{\phi}(t) := \frac{at}{at+1-a} \text{ if } t > 1.$$

Here $a := \int_0^1 \phi(s)ds$. Let us consider $a_n := \phi^{-1}\left(\frac{1}{n+1}\right)$. Then as stated in the above Müler result, there exists a $x_0 \in H$, of norm one, such that

$$\sum_{n=1}^{\infty} \phi(|\langle T^n x_0, x_0\rangle|) \geq \sum_{n=1}^{\infty} \phi(a_n) = \infty$$

which is a contradiction.

We can complete the result from Theorem 1 in the following way:

Proposition 2. *Let T be a bounded linear operator acting on a complex Hilbert space H. The following two statements are equivalent:*

(i) *There exists $\varepsilon > 0$ such that for all positive integer n there is a norm one vector $x \in X$ such that*

$$card\ \{k = 0, 1, \ldots : |\langle T^k x, x\rangle| \geq \varepsilon\} \geq n.$$

(ii) *For every non-decreasing function ϕ on $\mathbb{R}_+$, with $\phi(t) > 0$ for all $t > 0$, there exists a norm one vector $x \in X$ such that*

$$\sum_{k=1}^{\infty} \phi(|\langle T^k x, x\rangle|) = \infty.$$

Proof: (**i**) $\Rightarrow$ (**ii**) Follows using the inclusion:

$$\{k \in \mathbb{N} : |\langle T^k x, x\rangle| \geq \varepsilon\} \subset \{j \in \mathbb{N} : \phi(|\langle T^j x, x\rangle|) \geq \phi(\varepsilon)\}.$$

(**ii**) $\Rightarrow$ (**i**) Assume the contrary. For each $n = 1, 2, \cdots$ let us consider

$$w_n := \sup_{||x||=1} \text{ card } \{k \in \mathbb{N} : |\langle T^k x, x\rangle| \geq \frac{1}{2^n}\}.$$

The function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ given by $\phi(0) = 0$ and

$$\phi(t) = \frac{1}{w_1} 1_{[\frac{1}{2},\infty)}(t) + \sum_{n=1}^{\infty} \frac{1}{2^{n+1} w_n} 1_{[\frac{1}{2^{n+1}},\frac{1}{2^n})}(t)$$

for $t > 0$ is non-decreasing, $\phi(t) > 0$ for each $t > 0$, and for each norm one vector $x \in H$, one has:

$$\sum_{n=1}^{\infty} \phi(|\langle T^n x, x\rangle|) \leq 1.$$

This is a contradiction. $\square$

It is clear that if $r(T) \geq 1$ then the above statements **(i)** and **(ii)** are fulfilled.

In particular, follows that *if T is an isometry on a complex Hilbert space H and $\phi : [0,\infty) \to [0,\infty)$ is a non-decreasing function with $\phi(t) > 0$ for all $t > 0$ then there exists a norm one $x \in H$ such that:*

$$\sum_{n=0}^{\infty} \phi(|\langle T^n x, x\rangle|) = \infty.$$

We can compare this result with a similar one given by Jan van Neerven in [14] which states that *if T is a non-unitary isometry on a real or complex Hilbert space H then for all $\varepsilon > 0$ and all $\alpha \in c_0$ of norm one, there exists a norm one vector x such that*

$$|\langle T^n x, x\rangle| \geq (1 - \varepsilon)|\alpha_n| \quad \forall n \in \mathbb{N}.$$

This result do not holds for unitary isometry, (cf. [14], Example 2.4).

4 Proofs of the Theorems 2, 3 and 4

Proof of the Theorem 2. Applying (3.1) with $T(t)$ instead of T we get the following inequality.

$$
\begin{aligned}
|\langle T(t)x, y\rangle| \leq \frac{\sqrt{2}}{2}(|\langle T(t)x, x\rangle| + |\langle T(t)y, y\rangle|)+\\
+\frac{1}{2}(|\langle T(t)(x+y), x+y\rangle| + |\langle T(t)(x+iy), x+iy\rangle|).
\end{aligned}
$$

In view of (1.3), and using the Lemma 3.2 from [13], it follows that the maps $t \mapsto |\langle T(t)z, z\rangle|$ with $z \in \{x, y, x+y, x+iy\}$ belong to the same Orlicz space E over $\mathbb{R}_+$ which satisfies the condition $\lim\limits_{t\to\infty} ||1_{[0,t]}||_E = \infty$. Then the map $t \mapsto |\langle T(t)x, y\rangle|$ belongs to E and the desired assertion follows immediately. See also [15], Theorem 4.6.3 (ii).

Remarks:

1. We leave open the question whether the uniform boundedness condition on the semigroup can be dropped.

2. Under the different assumptions on the function ϕ the uniform boundedness condition on the semigroup can be dropped, see [3].

3. If the semigroup $\mathbf{T}$ acts on the finite dimensional space $\mathbf{C}^n$ then the uniform boundedness condition can be dropped.

Proof of the Theorem 3.

In view of (3.2) and (1.4) we get:

$$
\begin{aligned}
|\int_0^s \langle \rho_\mu(t)x, y\rangle dt| \leq \frac{\sqrt{2}}{2}[|\int_0^s \langle \rho_\mu(t)x, x\rangle dt| + |\int_0^s \langle \rho_\mu(t)y, y\rangle dt|]+\\
+\frac{1}{2}[|\int_0^s \langle \rho_\mu(t)(x+y), x+y\rangle dt| + |\int_0^s \langle \rho_\mu(t)(x+iy), x+iy\rangle dt|]\\
\leq M||x||||y|| < \infty.
\end{aligned}
$$

Now we obtain

$$
\begin{aligned}
\sup_{\mu\in\mathbb{R}} \sup_{s\geq 0} \left\|\int_0^s e^{-i\mu s}T(s)xds\right\| &\leq \sup_{||y||\leq 1} \sup_{s\geq 0} |\int_0^s \langle e^{-i\mu t}T(t)x, y\rangle dt|\\
&\leq M||x|| < \infty
\end{aligned}
$$

and we can apply Corollary 1 to end the proof.

Proof of Theorem 4. The implication **1**. $\Rightarrow$ **2**. was stated in [19] without the assumption of uniform boundedness on $\mathbf{T}$. A proof of **2**. $\Rightarrow$ **1**. can be found in [1]. In fact by the identity

$$
\int_0^s e^{-i\mu t}T(t)(A - i\mu I)(A - i\mu I)^{-1}x = T(s)(A - i\mu I)^{-1}x - (A - i\mu I)^{-1}x
$$

follows that

$$\sup_{s\geq 0}\left\|\int_0^s e^{-\mu t}T(t)xdt\right\| \leq ||(A-i\mu I)^{-1}||(1+\sup_{s\geq 0}||T(s)||)||x||.$$

It is clear that the second statement implies the third one. On the other hand from (3.2) follows that for each $\mu \in \mathbb{R}$ and each $x, y \in H$ have that

$$\sup_{s\geq 0}\left|\langle\int_0^s e^{-i\mu t}T(t)xdt, y\rangle\right| = N(\mu, x, y) < \infty,$$

and then (1.6) is fulfilled. The equivalences between **1.** and **4.** and between **2.** and **5.** are obvious.

The semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$, or its infinitesimal generator A, is called *strongly stable* if $\lim\limits_{t\to\infty} T(t)x = 0$ for every $x \in X$. The *point spectrum* of A, denoted by $\sigma_p(A)$ is the set of all complex scalars λ for which there exists a non-zero vector x such that $Ax = \lambda x$, while that the *residual spectrum* of A, denoted by $\sigma_r(A)$, is the set of all scalar $\lambda \in \sigma(A)$ such that the range of $(\lambda I - A)$ is not dense in X. As consequence of the Hahn-Banach theorem $\sigma_p(A^*) = \sigma_r(A)$. In particular, the punctual spectrum of A^* is a subset of $\sigma(A)$. We recall here the very famous and well-known stability theorem of Arendt-Batty-Lyubich-Vũ.

Theorem 7. *Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a strongly continuous and uniformly bounded semigroup on a Banach space X and let A its infinitesimal generator. If*

(i) *$\sigma(A) \cap (i\mathbb{R})$ is a countable set and*

(ii) *$\sigma_p(A^*) \cap (i\mathbb{R}) = \emptyset$, then the semigroup $\mathbf{T}$ is strongly stable.*

Combining this theorem with the Theorem 4 above, shall obtain:

Corollary 2. *Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a strongly continuous uniformly bounded semigroup on a complex space H and let A its infinitesimal generator. If $\sigma(A) \cap (i\mathbb{R}) = \emptyset$ or if for each $x \in H$ and each $\mu \in \mathbb{R}$ one has:*

$$\sup_{s\geq 0}\left|\int_0^s \langle e^{-i\mu t}T(t)x, x\rangle dt\right| = M(\mu, x) < \infty,$$

then the semigroup $\mathbf{T}$ is strongly stable.

Acknowledgement

The authors are very grateful to Nigel Kalton (University of Missouri) and Constantin P. Niculescu (University of Craiova) for stimulating discussions on the topic of this article.

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Received: 28.02.2007.

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