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An existence result for bilocal problems with mixed boundary conditions

by
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Abstract

We prove a Filippov type existence theorem for solutions of a second order differential inclusion with mixed boundary conditions by the application of the contraction principle in the space of the derivatives of solutions instead of the space of solutions.

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1 Introduction

In this note we study the second-order differential inclusion

$$x'' \in F(t,x), \quad a.e.\ (I) \tag{1.1}$$

with boundary conditions of the form

$$\begin{aligned} x(0) - k_1 x'(0) = c_1, \\ x(1) + k_2 x'(1) = c_2, \end{aligned} \tag{1.2}$$

where $I = [0,1]$, $F(.,.) : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ and $k_i \in \mathbf{R}_+$, $c_i \in \mathbf{R}$, $i = 1,2$.

In the theory of ordinary differential equations (i.e., when F is a single valued map) problem (1.1)-(1.2) is well known as a bilocal problem with mixed boundary conditions.

The present note is motivated by a recent paper of Belrabi and Benchohra ([1]) in which several existence results concerning second order nonlinear boundary value problems with integral conditions are obtained via fixed point techniques. The aim of our paper is to provide a Filippov type result concerning the existence of solutions of problem (1.1)-(1.2). Recall that for a differential inclusion defined by a lipschitzian set-valued map with nonconvex values, Filippov's theorem consists in proving the existence of a solution starting from a given almost solution.

Our approach is different from the one in [1] and consists in applying the contraction principle in the space of derivatives of solutions instead of the space of solutions. The idea of applying the set-valued contraction principle due to Covitz and Nadler ([6]) in the space of derivatives of the solutions belongs to Kannai and Tallos ([7]) and it was already used for other results concerning differential inclusions ([3,4,5] etc.).

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel and in Section 3 we prove our main result.

2 Preliminaries

In this short section we sum up some basic facts that we are going to use later.

Let (X,d) be a metric space and consider a set valued map T on X with nonempty closed values in X. T is said to be a λ-contraction if there exists $0<\lambda<1$ such that:

$$d_H(T(x),T(y)) \leq \lambda d(x,y) \quad \forall x,y\in X,$$

where $d_H(.,.)$ denotes the Pompeiu-Hausdorff distance. Recall that Pompeiu-Hausdorff distance of the closed subsets $A,B\subset X$ is defined by

$$d_H(A,B)=\max\{d^*(A,B),d^*(B,A)\}, \quad d^*(A,B)=\sup\{d(a,B);a\in A\},$$

where $d(x,B)=\inf_{y\in B} d(x,y)$.

If X is complete, then every set valued contraction has a fixed point, i.e. a point $z\in X$ such that $z\in T(z)$ ([6]).

We denote by $Fix(T)$ the set of all fixed points of the set-valued map T. Obviously, $Fix(T)$ is closed.

Proposition 2.1.([8])*Let X be a complete metric space and suppose that T_1,T_2 are λ-contractions with closed values in X. Then*

$$d_H(Fix(T_1),Fix(T_2)) \leq \frac{1}{1-\lambda}\sup_{z\in X} d(T_1(z),T_2(z)).$$

By AC^1 we denote the space of differentiable functions $x(.):(0,1)\to \mathbf{R}$ whose first derivative $x'(.)$ is absolutely continuous and by L^1 we denote the Banach space of Lebesgue integrable functions $x(.):[0,1]\to\mathbf{R}$ endowed with the norm $||u(.)||_1=\int_0^1|u(t)|dt$.

A function $x(.)\in AC^1$ is said to be a solution of (1.1)-(1.2) if there exists a function $v(.)\in L^1$ with $v(t)\in F(t,x(t))$, *a.e.* (I) such that $x''(t)=v(t)$, *a.e.* (I) and $x(.)$ satisfies conditions (1.2).

The next statement is well known (e.g. [1]).

Lemma 2.2. *If $v(.) : [0,1] \to \mathbf{R}$ is an integrable function then the problem*

$$\begin{gathered} x''(t) = v(t) \quad a.e.\ (I) \\ x(0) - k_1 x'(0) = c_1, \\ x(1) + k_2 x'(1) = c_2, \end{gathered}$$

has a unique solution $x(.) \in AC^1$ given by

$$x(t) = P_c(t) + \int_0^1 G(t,s) v(s) ds,$$

where if $c = (c_1, c_2) \in \mathbf{R}^2$ we denote

$$P_c(t) = \frac{(1 - t + k_2)c_1 + (k_1 + t)c_2}{1 + k_1 + k_2} \tag{2.1}$$

and

$$G(t,s) = \frac{-1}{1 + k_1 + k_2} \begin{cases} (k_1 + t)(1 - s + k_2) & \text{if} \quad 0 \le t < s \le 1 \\ (k_1 + s)(1 - t + k_2) & \text{if} \quad 0 \le s < t \le 1 \end{cases} \tag{2.2}$$

is the Green function of the problem.

Note that if $a = (a_1, a_2), b = (b_1, b_2) \in \mathbf{R}^2$ we put $||a|| = |a_1| + |a_2|$ and

$$|P_a(t) - P_b(t)| \le ||a - b||.$$

On the other hand, it is well known that $\sup_{t,s \in I} |G(t,s)| = \frac{1+k_1+k_2}{4}$.
In what follows we impose the following conditions on F.

Hypothesis 2.3. *(i) $F(.,.) : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ has nonempty closed values and for every $x \in X$ $F(.,x)$ is measurable.*

(ii) There exists $L(.) \in L^1$ such that for almost all $t \in I, F(t,\cdot)$ is $L(t)$-Lipschitz in the sense that

$$d_H(F(t,x), F(t,y)) \le L(t)|x - y| \quad \forall\ x, y \in \mathbf{R}$$

and $d(0, F(t,0)) \le L(t) \quad a.e.(I)$.

3 The main result

We are now ready to prove the main result of this paper.

Theorem 3.1. *Assume that Hypothesis 2.3 is satisfied, $\lambda := \sup_{t,s \in I} |G(t,s)| \cdot \int_0^1 L(s) ds < 1$ and let $y(.) \in AC^1$ be such that there exists $q(.) \in L^1$*

with $d(y''(t), F(t, y(t))) \leq q(t)$, *a.e.* (I). *Denote* $\tilde{c}_0 = y(0) - k_1 y'(0)$, $\tilde{c}_1 = y(1) + k_2 y'(1)$ *and* $\tilde{c} = (\tilde{c}_1, \tilde{c}_2)$.

Then for every $\varepsilon > 0$ *there exists* $x(.)$ *a solution of (1.1)-(1.2) satisfying for all* $t \in I$

$$|x(t) - y(t)| \leq \frac{1}{1-\lambda}||c - \tilde{c}|| + \frac{\sup_{t,s\in I}|G(t,s)|}{1-\lambda}\int_0^1 q(t)dt + \varepsilon,$$

Proof: For $u(.) \in L^1$ and define the following set valued maps:

$$M_u(t) = F(t, P_c(t) + \int_0^1 G(t,s)u(s)ds), \quad t \in I,$$

$$T(u) = \{\phi(.) \in L^1; \quad \phi(t) \in M_u(t) \quad a.e.\ (I)\}.$$

It follows from the definition and Lemma 2.2 that $x(.)$ is a solution of (1.1)-(1.2) if and only if $x''(.)$ is a fixed point of $T(.)$.

We shall prove first that $T(u)$ is nonempty and closed for every $u \in L^1$. The fact that that the set valued map $M_u(.)$ is measurable is well known. For example the map $t \to P(t) + \int_0^1 G(t,s)u(s)ds$ can be approximated by step functions and we can apply Theorem III. 40 in [2]. Since the values of F are closed with the measurable selection theorem (Theorem III.6 in [2]) we infer that $M_u(.)$ admits a measurable selection ϕ. One has

$$|\phi(t)| \leq d(0, F(t,0)) + d_H(F(t,0), F(t, P_c(t) + \int_0^1 G(t,s)u(s)ds) \leq$$

$$\leq L(t)(1 + |P_c(t)| + \sup_{t,s\in I}|G(t,s)|\int_0^1 |u(s)|ds),$$

which shows that $\phi \in L^1$ and $T(u)$ is nonempty.

On the other hand, the set $T(u)$ is also closed. Indeed, if $\phi_n \in T(u)$ and $||\phi_n - \phi||_1 \to 0$ then we can pass to a subsequence ϕ_{n_k} such that $\phi_{n_k}(t) \to \phi(t)$ for a.e. $t \in I$, and we find that $\phi \in T(u)$.

We show next that $T(.)$ is a contraction on L^1.

Let $u, v \in L^1$ be given, $\phi \in T(u)$ and let $\delta > 0$. Consider the following set-valued map:

$$H(t) = M_v(t) \cap \{x \in \mathbf{R};\ |\phi(t) - x| \leq L(t)|\int_0^1 G(t,s)(u(s) - v(s))ds| + \delta\}.$$

From Proposition III.4 in [2], $H(.)$ is measurable and from Hypothesis 2.2 ii) $H(.)$ has nonempty closed values. Therefore, there exists $\psi(.)$ a measurable selection of $H(.)$. It follows that $\psi \in T(v)$ and according with the definition of the norm we have

$$||\phi - \psi||_1 = \int_0^1 |\phi(t) - \psi(t)|dt \leq \int_0^1 L(t)(\int_0^1 |G(t,s)|.|u(s) - v(s)|ds)dt +$$

$$\int_0^1 \delta dt = \int_0^1 (\int_0^1 L(t)|G(t,s)|dt)|u(s)-v(s)|ds + \delta \leq \lambda ||u-v||_1 + \delta.$$

Since $\delta > 0$ was chosen arbitrarly, we deduce that

$$d(\phi, T(v)) \leq \lambda ||u-v||_1.$$

Replacing u by v we obtain

$$d_H(T(u), T(v)) \leq \lambda ||u-v||_1,$$

thus $T(.)$ is a contraction on L^1.

We consider next the following set-valued maps

$$F_1(t,x) = F(t,x) + q(t)[-1,1], \quad (t,x) \in I \times \mathbf{R},$$

$$P_{\tilde{c}}(t) = \frac{(1-t+k_2)\tilde{c}_1 + (k_1+t)\tilde{c}_2}{1+k_1+k_2},$$

$$M_u^1(t) = F_1(t, P_{\tilde{c}}(t) + \int_0^1 G(t,s)u(s)ds), \quad t \in I, \quad u(.) \in L^1,$$

$$T_1(u) = \{\psi(.) \in L^1; \quad \psi(t) \in M_u^1(t) \quad a.e.\ (I)\}.$$

Obviously, $F_1(.,.)$ satisfies Hypothesis 2.3.

Repeating the previous step of the proof we obtain that T_1 is also a λ-contraction on L^1 with closed nonempty values.

We prove next the following estimate

$$d_H(T(u), T_1(u)) \leq ||c - \tilde{c}|| \int_0^1 L(t)dt + \int_0^1 q(t)dt \tag{3.1}$$

Let $\phi \in T(u), \delta > 0$ and define

$$H_1(t) = M_u^1(t) \cap \{z \in \mathbf{R}; \quad |\phi(t) - z| \leq L(t)|P_c(t) - P_{\tilde{c}}(t)| + q(t) + \delta\}.$$

With the same arguments used for the set valued map $H(.)$, we deduce that $H_1(.)$ is measurable with nonempty closed values. Hence let $\psi(.)$ be a measurable selection of $H_1(.)$. It follows that $\psi \in T_1(u)$ and one has

$$||\phi - \psi||_1 = \int_0^1 |\phi(t) - \psi(t)|dt \leq \int_0^1 [L(t)|P_c(t) - P_{\tilde{c}}(t)| + q(t) + \delta]dt \leq$$

$$\int_0^1 L(t)|P_c(t) - P_{\tilde{c}}(t)|dt + \int_0^1 q(t) + \delta \leq ||c - \tilde{c}|| \int_0^1 L(t)dt + \int_0^1 q(t)dt + \delta.$$

Since δ is arbitrarly, as above we obtain (3.1).

We apply Proposition 2.1 and we infer that

$$d_H(Fix(T), Fix(T_1)) \leq \frac{1}{1-\lambda} ||c - \tilde{c}|| \int_0^1 L(t)dt + \frac{1}{1-\lambda} \int_0^1 q(t)dt.$$

Since $y''(.) \in Fix(T_1)$ it follows that there exists $u(.) \in Fix(T)$ such that for any $\varepsilon > 0$

$$||y'' - u||_1 \leq \frac{1}{1-\lambda}||c - \tilde{c}|| \int_0^1 L(t)dt + \frac{1}{1-\lambda}\int_0^1 q(t)dt + \frac{\varepsilon}{\sup_{t,s\in I}|G(t,s)|}.$$

We define $x(t) = P_c(t) + \int_0^1 G(t,s)u(s)ds$, $t \in I$ and we have

$$|x(t) - y(t)| \leq |P_c(t) - P_{\tilde{c}}(t)| + \int_0^1 |G(t,s)|.|u(s) - y''(s)|ds$$

$$\leq |P_c(t) - P_{\tilde{c}}(t)| + \sup_{t,s\in I}|G(t,s)|.||y'' - u||_1 \leq |P_c(t) - P_{\tilde{c}}(t)| +$$

$$\frac{1}{1-\lambda}\sup_{t,s\in I}|G(t,s)|\int_0^1 L(t)dt||c - \tilde{c}|| + \frac{\sup_{t,s\in I}|G(t,s)|}{1-\lambda}\int_0^1 q(t)dt + \varepsilon$$

$$\leq \frac{1}{1-\lambda}||c - \tilde{c}|| + \frac{\sup_{t,s\in I}|G(t,s)|}{1-\lambda}\int_0^1 q(t)dt + \varepsilon,$$

which completes the proof. □

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