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Exact solutions for the rotational flow of generalized Maxwell fluids in a circular cylinder

by
W. Akhtar and M. Nazar

Abstract

In this note the velocity field and the associated tangential stress corresponding to the rotational flow of a generalized Maxwell fluid within an infinite circular cylinder are determined by means of the Laplace and Hankel transforms. At time $t = 0$ the fluid is at rest and the motion is produced by the rotation of the cylinder around its axis. The solutions that have been obtained are presented under integral and series forms in terms of the generalized G-functions. The similar solutions for ordinary Maxwell fluid, performing the same motion, are obtained as particular cases of our solutions for $\beta = 1$.

Key Words: Generalized Maxwell model, velocity fiel, shear stress, Exact solutions.
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1 Introduction

The inadequacy of the classical Navier-Stokes theory to describe the behavior of the rheologically complex fluids such as polymer solutions, heavy oils, blood and many emulsions, has led to the development of models of non-Newtonian fluids. Among them, the models of differential type and rate type have received much attention. The first viscoelastic rate type model is due to Maxwell [1], and this model has had some success in describing the response of some polymeric liquids. In the last ten years, many authors have made use of rheological equations with fractional derivatives to describe the properties of polymers. In general, the constitutive equations with fractional derivative are obtained from known non-Newtonian models by replacing time ordinary derivatives by derivatives of fractional order.

For example [2], in the prediction of the dynamic mechanical properties of a viscous damper containing a viscoelastic fluid in the form of silicon gel, the fluid is modeled by a fractional derivative Maxwell model. The motion of a fluid in the neighborhood of a rotating body is of interest to both academic workers and industry. In this note, we consider the viscoelastic fluid to be modeled as a generalized Maxwell fluid and study the flow starting from rest due to the rotation of the cylinder around its axis with a velocity of constant angular acceleration. The velocity field as well as the adequate shear stress, obtained by means of the Laplace and Hankel transforms, are presented under series forms in terms of the generalized G-functions. The similar solutions for ordinary Maxwell fluids as well as those for Newtonian fluids, performing the same motion, are obtained as limiting cases of our general solutions.

2 Governing equations

The constitutive equations of an incompressible generalized Maxwell fluid, as it results from [3-5] are of the form

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S},\ \mathbf{S} + \lambda(D_t^{\beta}\mathbf{S} + \mathbf{V}\cdot\nabla\mathbf{S} - \mathbf{L}\mathbf{S} - \mathbf{S}\mathbf{L}^T) = \mu\mathbf{A}, \tag{1}$$

where $\mathbf{T}$ is the Cauchy stress tensor, $-p\mathbf{I}$ denotes the indeterminate spherical stress, $\mathbf{S}$ is the extra-stress tensor, $\mathbf{V}$ is the velocity, $\mathbf{L}$ is the velocity gradient, $\mathbf{A} = \mathbf{L} + \mathbf{L}^T$ is the first Rivlin-Ericksen tensor, μ is the dynamic viscosity, λ is the relaxation time, ∇ is the gradient operator, the superscript T denotes the transpose operation and D_t^{β} is Reimann-Liouville fractional derivative of order β [6,7],

$$D_t^{\beta} f(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^{\beta}} d\tau; \quad 0 < \beta \leq 1 \tag{2}$$

where $\Gamma(\cdot)$ is the Gamma function. This model reduces to the ordinary Maxwell model when $\beta = 1$ because $D_t^1 f = df/dt$. In cylindrical coordinates (r, θ, z), the rotational flow velocity is given by [3, 8, 9]

$$\mathbf{V} = \mathbf{V}(r,t) = \omega(r,t)\mathbf{e}_{\theta}, \tag{3}$$

where $\mathbf{e}_{\theta}$ is the transverse unit vector. For such flows the constraint of incompressibility is automatically satisfied. Since the velocity field is independent of θ and z, we also assume that $\mathbf{S}$ depends only of r and t. Furthermore, if the fluid is assumed to be at rest at the moment $t = 0$ then

$$\mathbf{S}(r,0) = \mathbf{0}\,, \tag{4}$$

Equalities $(1)_2$, (3) and (4) imply $S_{rr} = S_{zz} = S_{rz} = S_{\theta z} = 0$ and

$$(1 + \lambda D_t^{\beta})\tau = \mu\left(\frac{\partial \omega}{\partial r} - \frac{1}{r}\omega\right), \tag{5}$$

where $\tau = S_{r\theta}$ is the shear stress which is different of zero.

In the absence of body forces and a pressure gradient in the axial direction, the balance of the linear momentum leads to the relevant equation

$$\rho \frac{\partial \omega}{\partial t} = -\frac{\partial p}{\partial \theta} + \left(\frac{\partial}{\partial r} + \frac{2}{r} \right) \tau \tag{6}$$

where ρ is the constant density of the fluid and $\partial p / \partial \theta$ has to be zero due to the rotational symmetry [7, 10].

Eliminating τ among Eqs. (5) and (6), we attain to the governing equation

$$(1 + \lambda D_t^{\beta}) \frac{\partial \omega(r,t)}{\partial t} = \nu \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \right) \omega(r,t)\,, \tag{7}$$

where $\nu = \mu / \rho$ is the kinematic viscosity of the fluid.

3 Rotating flow in an infinite circular cylinder

Let us consider an incompressible generalized Maxwell fluid at rest in an infinite circular cylinder of radius R. At time $t = 0^+$ the cylinder begins to rotate around its axis with the angular velocity $\Omega\, t$. Owing to the shear the fluid is gradually moved, its velocity being of the form (3) while the governing equation is (7). The appropriate initial and boundary conditions are

$$\omega(r,0) = \frac{\partial \omega(r,0)}{\partial t} = 0\,; \quad r \in [0, R) \tag{8}$$

and

$$\omega(R,t) = R\,\Omega\, t; \quad t \geq 0\,. \tag{9}$$

To solve this problem we shall use as in [9, 11], the Laplace and Hankel transforms.

3.1 Calculation of the velocity field

Applying the Laplace transform to Eq. (7), using the Laplace transform formula for sequential fractional derivatives [7] and having the initial and boundary conditions (8) and (9) in mind, we find that

$$(q + \lambda q^{\beta+1}) \overline{\omega}(r,q) = \nu \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \right) \overline{\omega}(r,q);\ r \in (0, R), \tag{10}$$

where the image function $\overline{\omega}(r,q)$ has to satisfy the condition

$$\overline{\omega}(R,q) = \frac{\Omega\, R}{q^2}. \tag{11}$$

Multiplying both sides of Eqs. (10) by $rJ_1(rr_{1n})$, integrating with respect to r from 0 to R and taking into account the condition (11) and the equality

$$\int_0^R r\left(\frac{\partial^2\omega}{\partial r^2}+\frac{1}{r}\frac{\partial\omega}{\partial r}-\frac{\omega}{r^2}\right)J_1(rr_{1n})dr = Rr_{1n}J_2(Rr_{1n})\omega(R,t)-r_{1n}^2\omega_{_H}(r_{1n},t),$$

we find that

$$\overline{\omega}_{_H}(r_{1n},q)=\nu\Omega R^2 r_{1n}J_2(Rr_{1n})\frac{1}{q^2[q+\lambda q^{\beta+1}+\nu r_{1n}^2]} \tag{12}$$

where

$$\overline{\omega}_{_H}(r_{1n},q)=\int_0^R r\overline{\omega}(r,q)J_1(rr_{1n})dr$$

is the Hankel transform of $\overline{\omega}(r,q)$, while r_{1n} are the positive roots of the transcendental equation $J_1(Rr)=0$. Now for a more suitable presentation of the final results, we rewrite Eq. (12) in the following equivalent form

$$\begin{aligned}\overline{\omega}_H(r_{1n},q) &= \frac{\Omega R^2}{q^2 r_{1n}}J_2(Rr_{1n})-\frac{\Omega R^2}{\nu r_{1n}^3}\left(\frac{1}{q}-\frac{1}{q+\nu r_{1n}^2}\right)J_2(Rr_{1n})- \\ &\quad \nu\,\Omega R^2 r_{1n}J_2(Rr_{1n})\frac{1}{q+\nu r_{1n}^2}\frac{\lambda q^{\beta-1}}{q+\lambda q^{\beta+1}+\nu r_{1n}^2}\end{aligned} \tag{13}$$

Using the following formulae

$$\overline{\omega}(r,q)=\frac{2}{R^2}\sum_{n=1}^{\infty}\overline{\omega}_H(r_{1n},q)\frac{J_1(rr_{1n})}{J_2^2(Rr_{1n})},\qquad \int_0^R r^2 J_1(rr_{1n})dr=\frac{R^2}{r_{1n}}J_2(rr_{1n}),$$

we obtain that

$$\begin{aligned}\overline{\omega}(r,q) &= \frac{r\,\Omega}{q^2}-\frac{2\,\Omega}{\nu}\sum_{n=1}^{\infty}\left(\frac{1}{q}-\frac{1}{q+\nu r_{1n}^2}\right)\frac{J_1(rr_{1n})}{r_{1n}^3 J_2(Rr_{1n})}- \\ &\quad 2\nu\Omega\sum_{n=1}^{\infty}\frac{r_{1n}J_1(rr_{1n})}{J_2(Rr_{1n})}\frac{1}{q+\nu r_{1n}^2}\frac{\lambda q^{\beta-1}}{q+\lambda q^{\beta+1}+\nu r_{1n}^2}\end{aligned} \tag{14}$$

To obtain $\omega(r,t)=L^{-1}\{\overline{\omega}(r,q)\}$ we will apply the discrete inverse Laplace transform method [3, 5, 11]. For this, we use the expansion

$$\frac{\lambda q^{\beta-1}}{q+\lambda q^{\beta+1}+\nu r_{1n}^2}=\sum_{k=0}^{\infty}\left(-\frac{\nu r_{1n}^2}{\lambda}\right)^k\frac{q^{\beta-k-2}}{(q^\beta+\frac{1}{\lambda})^{k+1}} \tag{15}$$

Introducing (15) into (14), applying the discrete inverse Laplace transform and using the following properties

$$L^{-1}\{\overline{F}_1(q)\overline{F}_2(q)\}=(f_1{*}f_2)(t)=\int_0^t f_1(t-s)f_2(s)ds,\quad f_i=L^{-1}\{\overline{F}_i(q)\},\ i=1,2,$$

$$L^{-1}\left\{\frac{q^b}{(q^a-d)^c}\right\}=G_{a,\,b,\,c}(d,t),\ \ Re(ac-b)>0, Re(q)>0;$$

$$L^{-1}\left\{\frac{1}{q+a}\right\}=e^{-at},\ \ a\geq 0,$$

we find for $\omega(r,t)$ the expression

$$\begin{aligned}\omega(r,t) \quad &= \quad \omega_{_N}(r,t)-2\nu\,\Omega\sum_{n=1}^{\infty}\frac{r_{1n}J_1(rr_{1n})}{J_2(Rr_{1n})}\sum_{k=0}^{\infty}\left(-\frac{\nu\,r_{1n}^2}{\lambda}\right)^k \\ &\int_0^t e^{-\nu r_{1n}^2(t-s)}G_{\beta,\beta-k-2,k+1}\left(-\frac{1}{\lambda},s\right)ds\end{aligned} \tag{16}$$

where (cf. [8], Eqs. (5.1) and (5.2) with $\alpha=0$)

$$\omega_{_N}(r,t)=r\Omega t-\frac{2\Omega}{\nu}\sum_{n=1}^{\infty}\left(1-e^{-\nu r_{1n}^2 t}\right)\frac{J_1(rr_{1n})}{r_{1n}^3J_2(Rr_{1n})} \tag{17}$$

is the similar solution for Newtonian fluids performing the same motion, and [12]

$$G_{a,\,b,\,c}(d,t)=\sum_{j=0}^{\infty}\frac{\Gamma(c+j)}{\Gamma(c)}\frac{d^j\ t^{(c+j)a-b-1}}{\Gamma(j+1)\Gamma[(c+j)a-b]},\quad Re(ac-b)>0, \tag{18}$$

is the generalized G-function.

3.2 Calculation of the shear stress

Applying the Laplace transform to Eq. (5) and using the initial condition (4), we find that

$$(1+\lambda q^\beta)\overline{\tau}(r,q)=\mu\left[\frac{\partial\overline{\omega}(r,q)}{\partial r}-\frac{1}{r}\overline{\omega}(r,q)\right], \tag{19}$$

The image function $\overline{\omega}(r,q)$ can be immediately obtained using Eqs. (16), (17) and the formula

$$L\left\{\frac{t^a}{\Gamma(a+1)}\right\}=\frac{1}{q^{a+1}},\qquad a>-1.$$

Consequently, applying the Laplace transform to Eq. (16), differentiating the results with respect to r and using the identity

$$rJ_1'(rr_{1n})-J_1(rr_{1n})=-rr_{1n}J_2(rr_{1n}), \tag{20}$$

we find that

$$\begin{aligned}\frac{\partial\overline{\omega}(r,q)}{\partial r}-\frac{1}{r}\overline{\omega}(r,q) \quad = \quad & \frac{2\Omega}{\nu}\sum_{n=1}^{\infty}\left(\frac{1}{q}-\frac{1}{q+\nu r_{1n}^2}\right)\frac{J_2(rr_{1n})}{r_{1n}^2J_2(Rr_{1n})}+\\ & 2\nu\Omega\sum_{n=1}^{\infty}\frac{r_{1n}^2J_2(rr_{1n})}{J_2(Rr_{1n})}\sum_{j,k=0}^{\infty}\left(-\frac{1}{\lambda}\right)^{k+j}\times\\ & (\nu r_{1n}^2)^k\frac{\Gamma(k+j+1)}{\Gamma(k+1)\Gamma(j+1)}\frac{1}{q+\nu r_{1n}^2}\frac{1}{q^{\beta(k+j)+k+2}} \end{aligned} \tag{21}$$

Introducing (21) into (19), and using the immediate decomposition

$$\frac{1}{1+\lambda q^\beta}=1-\frac{q^\beta}{q^\beta+\frac{1}{\lambda}} \tag{22}$$

we find that

$$\begin{aligned}\overline{\tau}(r,t) \quad = \quad & 2\Omega\rho\sum_{n=1}^{\infty}\left(\frac{1}{q}-\frac{1}{q+\nu r_{1n}^2}\right)\frac{J_2(rr_{1n})}{r_{1n}^2J_2(Rr_{1n})}-\\ & 2\Omega\rho\nu\sum_{n=1}^{\infty}\frac{J_2(rr_{1n})}{J_2(Rr_{1n})}\frac{1}{q+\nu r_{1n}^2}\frac{q^{\beta-1}}{q^\beta+\frac{1}{\lambda}}+2\nu\mu\Omega\sum_{n=1}^{\infty}\frac{r_{1n}^2J_2(rr_{1n})}{\lambda J_2(Rr_{1n})}\times\\ & \sum_{j,k=0}^{\infty}\left(-\frac{1}{\lambda}\right)^{k+j}(\nu r_{1n}^2)^k\frac{\Gamma(k+j+1)}{\Gamma(k+1)\Gamma(j+1)}\times\\ & \left(\frac{1}{q+\nu r_{1n}^2}\right)\frac{1}{q^{\beta(k+j)+k+2}}\cdot\frac{1}{q^\beta+\frac{1}{\lambda}} \end{aligned} \tag{23}$$

Applying the inverse Laplace transform to Eq. (23), we get the shear stress $\tau(r,t)$ under form

$$\begin{aligned}\tau_(r,t) \quad = \quad & \tau_{_N}(r,t)-2\mu\Omega\sum_{n=1}^{\infty}\frac{J_2(rr_{1n})}{J_2(Rr_{1n})}\int_0^t e^{-\nu r_{1n}^2(t-s)}G_{\beta,\beta-1,1}\left(-\frac{1}{\lambda},s\right)ds+\\ & 2\nu\Omega\frac{\mu}{\lambda}\sum_{n=1}^{\infty}\frac{r_{1n}^2J_2(rr_{1n})}{J_2(Rr_{1n})}\sum_{j,k=0}^{\infty}\left(-\frac{1}{\lambda}\right)^{k+j}(\nu\, r_{1n}^2)^k\frac{\Gamma(k+j+1)}{\Gamma(k+1)\Gamma(j+1)}\times\\ & \int_0^t\int_0^\sigma e^{-\nu r_{1n}^2(t-\sigma)}\frac{s^{\beta(k+j)+k+1}}{\Gamma[\beta(k+j)+k+2]}\times\\ & G_{\beta,0,1}\left(-\frac{1}{\lambda},\sigma-s\right)ds\,d\sigma \end{aligned} \tag{24}$$

where (cf. [8], Eqs. (5.1) and (5.2) with $\alpha=0$)

$$\tau_{_N}(r,t)=2\rho\Omega\sum_{n=1}^{\infty}\left(1-e^{-\nu r_{1n}^2t}\right)\frac{J_2(rr_{1n})}{r_{1n}^2J_2(Rr_{1n})} \tag{25}$$

is the shear stress corresponding to a Newtonian fluid performing the same motion.

4 Limiting cases

Making $\beta = 1$ into Eq. (16), we obtain the velocity field

$$\begin{aligned}\omega(r,t) &= \Omega r t - \frac{2\Omega}{\nu}\sum_{n=1}^{\infty}(1-e^{-\nu r_{1n}^2 t})\frac{J_1(rr_{1n})}{r_{1n}^3 J_2(Rr_{1n})} - \\ & 2\Omega\nu\sum_{n=1}^{\infty}\frac{r_{1n}J_1(rr_{1n})}{J_2(Rr_{1n})}\sum_{k=0}^{\infty}\left(\frac{-\nu r_{1n}^2}{\lambda}\right)^k \times \\ & \int_0^t e^{-\nu r_{1n}^2(t-s)}G_{1,-k-1,k+1}\left(-\frac{1}{\lambda},s\right)ds \end{aligned} \tag{26}$$

corresponding to an ordinary Maxwell fluid performing the same motion. Similar, from Eqs. (24) and (25), we obtain the associated shear stress

$$\begin{aligned}\tau(r,t) &= 2\rho\Omega\sum_{n=1}^{\infty}(1-e^{-\nu r_{1n}^2 t})\frac{J_2(rr_{1n})}{r_{1n}^2 J_2(Rr_{1n})} - 2\rho\Omega\nu\sum_{n=1}^{\infty}\frac{J_2(rr_{1n})}{J_2(Rr_{1n})}\times \\ & \int_0^t e^{-\nu r_{1n}^2(t-s)}G_{1,0,1}\left(-\frac{1}{\lambda},s\right)ds + \\ & 2\Omega\nu\frac{\mu}{\lambda}\sum_{n=1}^{\infty}\frac{r_{1n}^2 J_2(rr_{1n})}{J_2(Rr_{1n})}\sum_{j,k=0}^{\infty}\left(-\frac{1}{\lambda}\right)^{k+j}(\nu\, r_{1n}^2)^k \times \\ & \frac{\Gamma(k+j+1).e^{-\nu r_{1n}^2 t}}{\Gamma(k+1)\Gamma(j+1)\Gamma(j+2k+2)} \times \\ & \int_0^t\int_0^\sigma e^{\nu r_{1n}^2\sigma}\, s^{j+2k+1}G_{1,0,1}\left(-\frac{1}{\lambda},0,\sigma-s\right)ds\, d\sigma \end{aligned} \tag{27}$$

Using the following equality

$$G_{1,0,1}(a,t) = \sum_{j=0}^{\infty}\frac{a^j t^j}{\Gamma(j+1)} = \sum_{j=0}^{\infty}\frac{(at)^j}{j!} = e^{at},$$

the above relation becomes

$$\begin{aligned}\tau(r,t) &= 2\rho\Omega\sum_{n=1}^{\infty}(1-e^{-\nu r_{1n}^2 t})\frac{J_2(rr_{1n})}{r_{1n}^2 J_2(Rr_{1n})} - \\ &2\Omega\nu\rho\sum_{n=1}^{\infty}\frac{J_2(rr_{1n})}{J_2(Rr_{1n})}e^{-\nu r_{1n}^2 t}\frac{\lambda}{1-\lambda\nu r_{1n}^2}\times \\ &\left[1-exp\left(-\frac{1-\lambda\nu r_{1n}^2}{\lambda}t\right)\right]+ \\ &2\Omega\nu\frac{\mu}{\lambda}\sum_{n=1}^{\infty}\sum_{j,k=0}^{\infty}\frac{r_{1n}^2 J_2(rr_{1n})}{J_2(Rr_{1n})}\left(-\frac{1}{\lambda}\right)^{k+j}(\nu\, r_{1n}^2)^k\times \\ &\frac{\Gamma(k+j+1).e^{-\nu r_{1n}^2 t}}{\Gamma(k+1)\Gamma(j+1)\Gamma(j+2k+2)}\times \\ &\int_0^t\int_0^\sigma exp\left[(\nu r_{1n}^2-\frac{1}{\lambda})\sigma+\frac{s}{\lambda}\right]s^{2k+j+1}ds\,d\sigma \qquad (28)\end{aligned}$$

5 Conclusion

In this paper, the velocity field and the adequate shear stress corresponding to the rotational flow induced by an infinite circular cylinder in an incompressible generalized Maxwell fluid, have been determined using Hankel and Laplace transforms. The motion is produced by the circular cylinder that at the initial moment begins to rotate around its axis with an angular velocity of constant acceleration. The solutions that have been obtained, written under integral and series forms in terms of the generalized G functions, satisfy all imposed initial and boundary conditions.

Furthermore, they are presented as a sum between the Newtonian solutions and the adequate non-Newtonian contributions. In the special case when $\beta \to 1$, the similar solutions for ordinary Maxwell fluids, performing same motion, are obtained.

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Abdus Salam School of Mathematical Sciences,
GC University Lahore, Pakistan
E-mail: `wasakh75@hotmail.com`