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Disjoint paths spanning simple polytopal graphs

by
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Abstract

In this paper, the definition of traceability is extended to n-path-traceability, i. e. the existence of a spanning set of n disjoint paths. Subsequently, an algorithm is provided to show that for each natural number $n > 1$, there is a simple 3-polytopal graph with at most 44n + 46 vertices which is not n-path-traceable

Key Words: Spanning subgraphs, spanning set of paths, simple 3-polytopes.
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1 Introduction

The question of the existence of hamiltonian cycles in graphs, a historically old question, was in modern times mainly motivated by the well-known 4-colour conjecture. For cubic 3-connected planar graphs, the question was answered through a counterexample by Tutte [6] in 1946. From then on, the question of smallest counterexamples arose.

The corresponding question about smallest *non-traceable* graphs, i. e. without hamiltonian paths, is mentioned by V. Klee [3]. T. Zamfirescu found in 1968 an example with 88 vertices, which remained the smallest known until these days (see [1], [8]). Moreover, he generalized the last question replacing the hamiltonian paths by *n-paths*, which are disjoint unions of at most n paths. He also separately considered the question when the n paths are not necessarily pairwise disjoint. A related problem was treated in [7].

By a *graph* we shall always understand here a cubic 3-connected planar graph, which means by Steinitz' Theorem the 1-skeleton of a simple polytope. A graph is *n-path-traceable* if it has a spanning n-path.

By using the well-known Lederberg-Bosák-Barnette graph T, which is a smallest non-hamiltonian graph [2], one easily gets a non-n-path-traceable graph. It suffices to insert $2n+1$ copies of $T^{'}$, the graph T minus one vertex, into equally

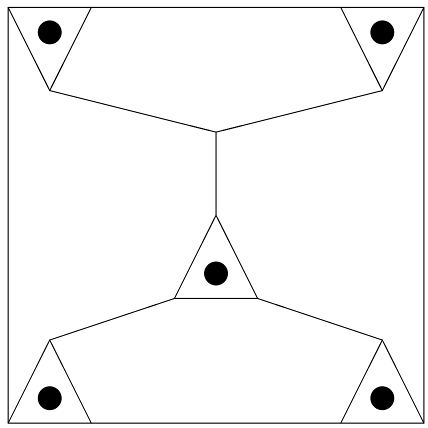

Figure 1: A non-2-path-traceable graph, 186 vertices

many vertices of an arbitrary graph H. To keep the example small, we take H to have $2n+2$ vertices. Since $T^{'}$ contains 37 vertices, the constructed graph has $74n+38$ vertices. (An example for $n=2$ is shown in Figure 1. Each pointed triangle stands for a copy of $T^{'}$)

This paper describes the construction of non-n-path-traceable simple 3-polytopal graphs consisting of at most $44n+46$ vertices. This result has already been announced in [4].

2 The construction

The first graph needed for our construction is shown in Figure 2. It consists of three Tutte triangles (see [6]), contains 43 vertices and will be called T_3.

Lemma 1. *T_3 is not traceable.*

Proof: Assume T_3 contains a spanning path P. Then, P cannot enter and leave the triangle $aa'b'$, since this would lead to a hamilton path from a' to b'. (The same holds true for $a''b''b$.)

Thus, each of the small triangles must contain an endpoint of P. If these triangles are contracted to a respectively b, P would become a hamiltonian path from a to b in the big Tutte triangle, which is impossible. □

Lemma 2. *If a graph G contains a copy I of T_3 and a spanning n-path J, then $I \cap J$ is not connected.*

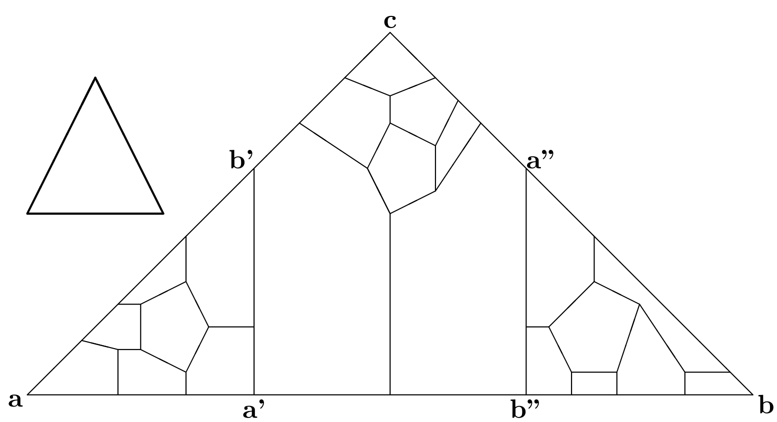

Figure 2: The graph T_3 with its symbol

The proof is easy, since no subpath of J can span I.

Theorem 1. *Let G be the graph of Figure 3, the black-white partition of the vertex set being $\{V_1, V_2\}$, with*

$$card\ V_1 = card\ V_2 = \begin{cases} n+1 \ for\ n\ odd \\ n+2 \ for\ n\ even. \end{cases}$$

If $n+1$ vertices in V_2 are replaced by copies of T_3, then the resulting graph is not n-path-traceable and has $44n+44$ vertices if n is odd and $44n+46$ if n is even.

Proof: Let H be the resulting graph. Assume H contains a spanning n-path J. Then, for each copy I of T_3 in H, $I \cap J$ must consist of more than one component. If each of these components is contracted to a single vertex and all edges of $H \backslash J$ are deleted, then we obtain a union K of n disjoint paths (see Figure 4).

During the process of transformation from G over H to K, the number of black vertices does not change. If all other vertices in K are considered white, then no two white vertices can be adjacent to each other. But since $I \cap J$ consists of more than one component for each copy I of T_3 and H contains $n+1$ such copies, K must contain at least $n+1$ more white than black vertices. Thus, the number of paths must be larger than n, and a contradiction is obtained. $\square$

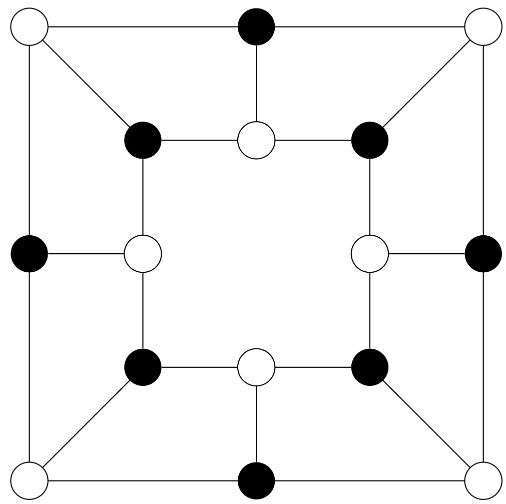

Figure 3: Graph G for $n = 6$ or $n = 7$

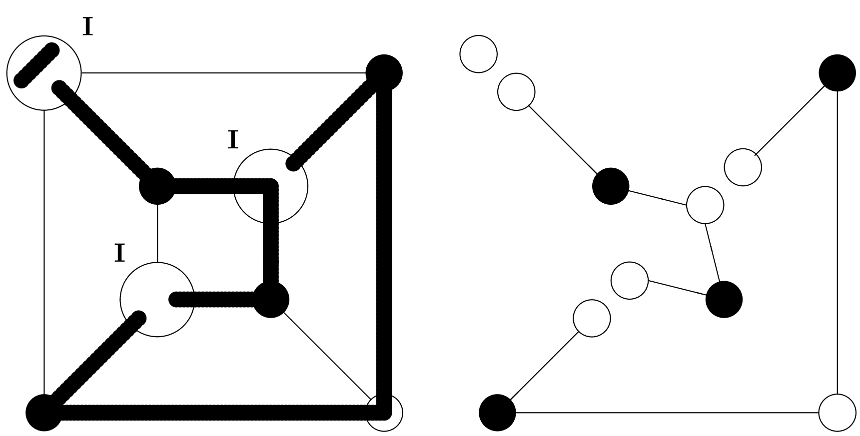

Figure 4: A spanning 3-path is transformed

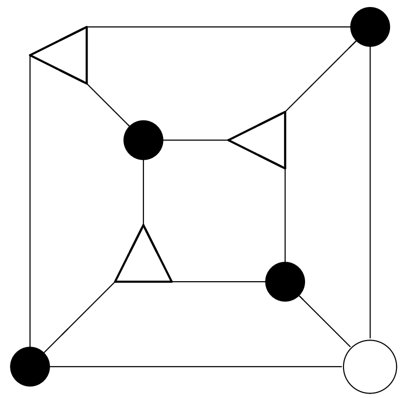

Figure 5: A non-2-path-traceable Graph, 134 vertices

For $n = 2$, the graph described in the Theorem is depicted in Figure 5. It was found by Strauch [5] in 2002.

References

[1] B. Grünbaum Polytopes, graphs and complexes, Bull. Amer. Math. Soc. **76** (1970), pp. 1131-1201

[2] D. A. Holton & B. D. McKay The Smallest Non-Hamiltonian 3-Connected Cubic Planar Graphs Have 38 Vertices, J. Comb. Th. B **45** (1989), pp. 305-319

[3] V. Klee, Paths on polyhedra II, Pac. J. Math. **17** (1966), pp. 249-262

[4] P. Knorr, Aufspannende Kreise und Wege in 3-polytopalen Graphen, Diplomarbeit (2007), Universität Dortmund

[5] C. Strauch, Ein planarer, 3-zusammenhängender, 3-regulärer Graph ohne aufspannenden Y-Baum, Analele Universităţii din Craiova, Seria Matematică-Informatică, Vol. **XXIX** (2002), pp. 23-25

[6] W. T. Tutte, On hamiltonian circuits, Journal London Mathematical Society **21** (1946), pp. 98-101

[7] T. Zamfirescu, On spanning and expanding stars, Atti Accad. Sci. Ist. Bologna, Serie XIII-**1** (1974), pp. 45-47

[8] T. Zamfirescu, Three small cubic graphs with interesting hamiltonian properties, J. Graph Th. **4** (1980), pp. 287-292

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