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Approximate quasi efficient solutions in multiobjective optimization

by

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Abstract

In this paper we consider unitar concepts for some optimality and efficiency or approximate solutions in scalar optimization and vectorial optimization respectively. In this cases some necessary and/or sufficient conditions for these approximate solutions in the multiobjective case are derived via scalarization and an alternative theorem. Thus are generalized and refined some corresponding results in multiobjective optimization.

Key Words: Multiobjective programming, optimization, approximate solution.
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1 Introduction

The general goal in any optimization is to identify a single or all best solution within a set of feasible points. While it is theoretically possible to identify the complete set of best solutions, finding an exact description of this set often turns out to be practically impossible or at least computationally too expensive, and thus many research efforts focus on approximation concepts and procedures.
Interest in getting approximate solutions of optimization problems has spread greatly during the past 20 years. The first concepts of approximate solution or ε-efficient solution in multiobjective optimization appear in the works of Kutateladze [6], Loridan [10, 11], and White [21]. Loridan [11] define ε-efficient point for a set and, starting from this concept, he extend the ε-optimal solution concept from scalar optimization problems to multiobjective optimization problems defining ε-efficient solution in a Pareto context.
This definition is the same as Kutateladze's notion introduced in [6] and is the concept more used in the literature to study approximate solutions for multiobjective optimization problems. White [21] analyze six different concepts of

ε-efficient solution in the context of five solution methods of multiobjective optimization problems through scalarization. Tanaka [19] proposed a new approximate solution concept and Yokoyama [22] explored relations among three types of approximate solution set. Li [8], Lui [12] and Valyi [20] discussed ε-vector Lagrangians, generalized ε-saddle points and ε-problems for nonsmooth nonconvex multiobjective programming problems. Tammer [18] studied approximate solutions to vector optimization problems via generating the well known Ekeland's variational principle.

Related to general result - the Theorem on Nonconvex Functions of Kaliszewski [5] (a counterpart of the Fundamental Theorem on Convex Functions [15] in the case the convexity assumption does not hold), in this paper we discuss different conditions for $(\varepsilon, \overline{\varepsilon})$-quasi weak efficiency, $(\varepsilon, \overline{\varepsilon})$-quasi proper efficiency and $(\varepsilon, \overline{\varepsilon})$-quasi efficiency for a multiobjective and nonconvex optimization problem. These results generalize and refine some known results in the literature of multiobjective programming. Further we can apply these results in statistics or information theory [1, 2].

We organize this paper as follows: in Section 2 we present the terminology used in this paper, derive necessary and/or sufficient conditions for the $(\varepsilon, \overline{\varepsilon})$-quasi weak efficiency, the $(\varepsilon, \overline{\varepsilon})$-quasi proper efficiency and the $(\varepsilon, \overline{\varepsilon})$-quasi efficiency in Sections 3, 4 and 5 respectively. Finally, in Section 6 we give a few concluding remarks.

2 Preliminaries

Let $\varphi : X \to \mathbb{R}$ and the scalar problem

$$(P)\begin{cases} \inf \varphi(x) \\ x \in X. \end{cases}$$

where X is a nonempty set in a normed linear space.

Definition 1. $[4, 9]$ *Let $\alpha > 0$. A point $x_0 \in X$ is called*

i1) *α-optimal solution of the scalar problem (P) if $\varphi(x_0) \leqq \varphi(x) + \alpha$, for all $x \in X$;*

i2) *α-quasi optimal solution of the scalar problem (P) if*

$$\varphi(x_0) \leqq \varphi(x) + \alpha\|x - x_0\|,$$

for all $x \in X$.

We see that an optimal solution is an α-optimal solution and an optimal solution is an α-quasi solution. When $\alpha = 0$, an α-optimal solution (quasi solution) of (P) is an optimal solution.

Definition 2. *Let* $(\alpha_0, \overline{\alpha}_0) \in \mathbb{R}_+ \times \mathbb{R}_+$ *and* $x_0 \in X$. *We say that* x_0 *is* $(\alpha_0, \overline{\alpha}_0)$-*quasi optimal solution to* (P) *if*

$$\varphi(x_0) \leqq \varphi(x) + \alpha_0 \|x - x_0\| + \overline{\alpha}_0, \ \forall x \in X.$$

We see that for $\alpha = \alpha_0 = 0$ we get the concept of optimal solutions; if $\alpha_0 = 0$ we get $\overline{\alpha}_0$-optimal solution and when $\overline{\alpha}_0 = 0$ we obtain α_0-quasi optimal solution concept.

Consider the following multiobjective optimization problem

$$(VP) \begin{cases} \text{minimize } f(x) = (f_1(x), ..., f_m(x))^T \\ \text{subject to } x \in X, \end{cases}$$

where $f_i : X \to \mathbb{R}$, $i = 1, 2, ..., m$, and $m \geq 2$.

Definition 3. *Let* $x_0 \in X$ *and* $\varepsilon \in \mathbb{R}^n_+$. *Then*

i) *the point* x_0 *is called an* ε *-quasi efficiency solution of* (VP) *if*

$$f(x) + \varepsilon\|x - x_0\| - f(x_0) \nleqq 0, \forall x \in X$$

ii) *the point* x_0*is called an* ε *-quasi weakly efficiency solution of* (VP) *if*

$$f(x) + \varepsilon\|x - x_0\| - f(x_0) \nless 0, \forall x \in X$$

If $\varepsilon = \varepsilon_0 e_m$ where $\varepsilon_0 \in \mathbb{R}^*_+$ and $e_m = (1, ..., 1)^T \in \mathbb{R}^m$, then this definition is reduced to Definition 5.1 [4]. If $\varepsilon = 0$ we get the concepts of efficient solution and weak efficient solution respectively. Similar to ε-properly efficiency we consider ε-quasi properly efficiency.

Definition 4. $x_0 \in X$ *is called* ε*-quasi properly efficient solution of* (VP) *if it is* ε*-quasi efficient and if there exists* $M > 0$ *such that for any* $i \in \mathcal{M} = \{1, 2, ..., m\}$ *and* $x \in X$ *satisfying* $f_i(x) < f_i(x_0) - \varepsilon_i\|x - x_0\|$ *there exists* $j \in \mathcal{M}$ *with* $f_j(x) > f_j(x_0) - \varepsilon_j\|x - x_0\|$ *and*

$$\frac{f_i(x_0) - \varepsilon_i\|x - x_0\| - f_i(x)}{f_j(x) - f_j(x_0) + \varepsilon_j\|x - x_0\|} \leqq M.$$

Remark 1. **(1)** ε*-quasi proper efficiency* $\Rightarrow$ ε*-quasi efficiency* $\Rightarrow$ α*-quasi weakly efficiency*

(2) *when* $\varepsilon = 0$, *an* ε*-(quasi properly, quasi weakly) efficient solution of* (VP) *is a (properly, weakly) efficient solution.*

Definition 5. *Let* $(\varepsilon, \overline{\varepsilon}) \in \mathbb{R}^m_+ \times \mathbb{R}^m_+$ *and* $x_0 \in X$. *We say that* x_0 *is:*

i1) $(\varepsilon, \overline{\varepsilon})$*-quasi efficient to* (VP) *if*

$$f(x) + \varepsilon\|x - x_0\| + \overline{\varepsilon} - f(x_0) \lneqq 0, \forall x \in X$$

i2) $(\varepsilon, \overline{\varepsilon})$*-quasi weakly efficient to* (VP) *if*

$$f(x) + \varepsilon\|x - x_0\| + \overline{\varepsilon} - f(x_0) \not< 0, \forall x \in X$$

i3) $(\varepsilon, \overline{\varepsilon})$*-quasi properly efficient to* (VP)*if exists* $M > 0$ *such that for any* $i \in \mathcal{M} = \{1, 2, ..., m\}$ *and* $x \in X$ *satisfying* $f_i(x) < f_i(x_0) - \varepsilon_i\|x - x_0\| - \overline{\varepsilon}_i$ *there exists* $j \in \mathcal{M}$ *with* $f_j(x) > f_j(x_0) - \varepsilon_j\|x - x_0\| - \overline{\varepsilon}_j$ *and*

$$\frac{f_i(x_0) - \varepsilon_i\|x - x_0\| - \overline{\varepsilon}_i - f_i(x)}{f_j(x) - f_j(x_0) + \varepsilon_j\|x - x_0\| + \overline{\varepsilon}_j} \leqq M.$$

As in [7], to prove our main results, we quote an alternative theorem for nonconvex functions given by Kaliszewski [5, Theorem 3.1].

Theorem Kaliszewski *One and only one of the following alternatives holds:*

(a) *there exists some* $x \in X$ *such that* $f_i(x) < 0$*,* $i \in \mathcal{M}$*;*

(b) *for any negative numbers* $\delta_1, \delta_2, ..., \delta_m$ *there exist positive numbers* $\lambda_i = -\delta_i^{-1}$*,* $i \in \mathcal{M}$ *such that*

$$max_{i\in\mathcal{M}}\lambda_i[f_i(x) - \delta_i] \geqq 1, \textit{for any } x \in X.$$

3 Case of $(\varepsilon, \overline{\varepsilon})$-Quasi Weak Efficiency

In this section, relate to $(\varepsilon, \overline{\varepsilon})$-quasi weak efficiency we give a necessary and sufficient condition.

Theorem 1. *Let* $x_0 \in X$*. Then* x_0 *is an* $(\varepsilon, \overline{\varepsilon})$*-quasi weakly efficient solution of* (VP) *if and only if for any* $y_i^* < f_i(x_0)$*,* $i \in \mathcal{M}$*,* x_0 *is* $(\varepsilon_0, \overline{\varepsilon}_0)$*-quasi optimal solution to the following scalar optimization problem*

$$(SP_1)\begin{cases} \textit{minimize } max_{i\in\mathcal{M}}\lambda_i[f_i(x) - y_i^*], \\ \textit{subject to } x \in X, \end{cases}$$

where $\lambda_i = [f_i(x_0) - y_i^*]^{-1}$*,* $i \in \mathcal{M}$*,* $\varepsilon_0 = max_{i\in\mathcal{M}}\lambda_i\varepsilon_i$ *and* $\overline{\varepsilon}_0 = max_{i\in\mathcal{M}}\lambda_i\overline{\varepsilon}_i$*.*

Proof: Firstly, let $x_0 \in X$ be a $(\varepsilon, \overline{\varepsilon})$-quasi weakly efficient solution (q.w.e.s.) of (VP) and we prove that x_0 is $(\varepsilon_0, \overline{\varepsilon}_0)$-quasi optimal for (SP_1). Hence the system of inequalities

$$f_i(x) < f_i(x_0) - \varepsilon_i\|x - x_0\| - \overline{\varepsilon}_i, \ i \in \mathcal{M}$$

has no solution $x \in X$. Thus by Theorem [Kaliszewski, 5] for any negative numbers $\delta_1, \delta_2, ..., \delta_m$ there exist positive numbers $\lambda_i = -\delta_i^{-1}$, $i \in \mathcal{M}$ such that

$$max_{i\in\mathcal{M}}\lambda_i[f_i(x) - f_i(x_0) + \varepsilon_i\|x - x_0\| + \overline{\varepsilon}_i - \delta_i] \geqq 1, \text{ for any } x \in X.$$

Since

$$max_{i\in\mathcal{M}}\lambda_i[f_i(x) - y_i^* + \varepsilon_i\|x - x_0\| + \overline{\varepsilon}_i] \leqq max_{i\in\mathcal{M}}\lambda_i[f_i(x) - y_i^*] + \varepsilon_0\|x - x_0\| + \overline{\varepsilon}_0$$

for all $x \in X$ and $\lambda_i[f_i(x_0) - y_i^*] = 1$, $i \in \mathcal{M}$, we obtain easily that x_0 is ε_0-quasi optimal for (SP_1).

Conversely, let x_0 be a $(\varepsilon_0, \overline{\varepsilon}_0)$-quasi optimal solution for (SP_1). We have to prove that x_0 is $(\varepsilon, \overline{\varepsilon})$-quasi weakly efficient to (VP). We suppose contrary, i.e. x_0 is not $(\varepsilon, \overline{\varepsilon})$-quasi weakly efficient to (VP). Therefore, there would exist some $\overline{x}_0 \in X$ such that

$$f_i(\overline{x}_0) < f_i(x_0) - \varepsilon_i\|\overline{x}_0 - x_0\| - \overline{\varepsilon}_i, \ i \in \mathcal{M}.$$

Let $y_i^* = f_i(\overline{x}_0)$, $i \in \mathcal{M}$. Then $y_i^* < f_i(x_0)$, $\lambda_i^{-1} = f_i(x_0) - y_i^* = f_i(x_0) - f_i(\overline{x}_0) > \varepsilon_i\|\overline{x}_0 - x_0\| + \overline{\varepsilon}_i$ and hence $\lambda_i\varepsilon_i\|\overline{x}_0 - x_0\| + \lambda_i\overline{\varepsilon}_i < 1$ for $i \in \mathcal{M}$. By assumption that x_0 is $(\varepsilon_0, \overline{\varepsilon}_0)$-quasi optimal to (SP_1), for any $x \in X$ we have

$$max_{i\in\mathcal{M}}\lambda_i[f_i(x) - f_i(\overline{x}_0)] \geqq$$

$$\geqq max_{i\in\mathcal{M}}\lambda_i[f_i(x_0) - f_i(\overline{x}_0)] - max_{i\in\mathcal{M}}\lambda_i(\varepsilon_i\|\overline{x}_0 - x_0\| + \overline{\varepsilon}_i) =$$

$$= max_{i\in\mathcal{M}}\lambda_i[f_i(x_0) - f_i(\overline{x}_0)] - \varepsilon_0\|\overline{x}_0 - x_0\| - \overline{\varepsilon}_0 =$$

$$= 1 - \varepsilon_0\|\overline{x}_0 - x_0\| - \overline{\varepsilon}_0 > 0.$$

Taking $x = \overline{x}_0$ in above inequality, we get $0 > 0$. This contradiction implies that x_0 is $(\varepsilon, \overline{\varepsilon})$-quasi weakly efficient to (VP). This proof is completed. □

4 Case of $(\varepsilon, \overline{\varepsilon})$-Quasi Proper Efficiency

As in [7] we can prove a similar theorem for $(\varepsilon, \overline{\varepsilon})$-quasi proper efficiency.

Lemma 1. *If $x_0 \in X$ is $(\varepsilon, \overline{\varepsilon})$-quasi proper efficient to (VP) then the system*

$$\alpha_i f_i(x) + \rho e^T f(x) <$$

$$< \alpha_i f_i(x_0) + \rho e^T f(x_0) - \alpha_i\varepsilon_i\|x - x_0\| - \alpha_i\overline{\varepsilon}_i - \rho e^T\varepsilon\|x - x_0\| - \rho e^T\overline{\varepsilon}$$

with $i \in \mathcal{M}$ admits no solution $x \in X$ for some $\rho > 0$, where $\alpha_i > 0$, $i \in \mathcal{M}$.

Proof: We suppose that x_0 is $(\varepsilon, \overline{\varepsilon})$-quasi properly efficient to (VP). By the $(\varepsilon, \overline{\varepsilon})$-quasi proper efficiency of x_0, the system

$$f_i(x) < f_i(x_0) - \varepsilon_i \|x - x_0\| - \overline{\varepsilon}_i,\ i \in \mathcal{M} \tag{1}$$

has no solution in X. Therefore, the system

$$\alpha_i f_i(x) < \alpha_i f_i(x_0) - \alpha_i \varepsilon_i \|x - x_0\| - \alpha_i \overline{\varepsilon}_i,\ i \in \mathcal{M}$$

has no solution in X for $\alpha_i > 0$, $i \in \mathcal{M}$. Let $\widehat{x}_0 \in X$ be fixed. We discuss in the following two cases:
Case 1 : We prove that if $e^T f(x_0) - e^T \varepsilon \|\widehat{x}_0 - x_0\| - e^T \overline{\varepsilon} \leq e^T f(\widehat{x}_0)$ then the system of m inequalities

$$\alpha_i f_i(\widehat{x}_0) + \rho e^T f(\widehat{x}_0) <$$

$$< \alpha_i f_i(x_0) + \rho e^T f(x_0) - \alpha_i \varepsilon_i \|\widehat{x}_0 - x_0\| - \alpha_i \overline{\varepsilon}_i - \rho e^T \varepsilon \|\widehat{x}_0 - x_0\| - \rho e^T \overline{\varepsilon}$$

with $i \in \mathcal{M}$ is inconsistent for any $\rho > 0$. This is because if it was not the case, we would have

$$\alpha_i f_i(x_0) - \alpha_i \varepsilon_i \|\widehat{x}_0 - x_0\| - \alpha_i \overline{\varepsilon}_i - \alpha_i f_i(\widehat{x}_0) >$$

$$> \rho[e^T f(\widehat{x}_0) - e^T f(x_0) + e^T \varepsilon \|\widehat{x}_0 - x_0\| + e^T \overline{\varepsilon}]$$

with $i \in \mathcal{M}$ a contradiction to (1).
Case 2 : If $e^T f(x_0) - e^T \varepsilon \|\widehat{x}_0 - x_0\| - e^T \overline{\varepsilon} > e^T f(\widehat{x}_0)$, then

$$I = \{i \in \mathcal{M} | f_i(x_0) - \varepsilon_i \|\widehat{x}_0 - x_0\| - \overline{\varepsilon}_i > f(\widehat{x}_0)\} \neq \emptyset$$

By Remark 1, x_0 is $(\varepsilon, \overline{\varepsilon})$-quasi efficient to (VP). There exists some $j_0 \in \mathcal{M}$ such that $f_{j_0}(\widehat{x}_0) > f_{j_0}(x_0) - \varepsilon_{j_0} \|\widehat{x}_0 - x_0\| - \overline{\varepsilon}_{j_0}$.
Let

$$f_l(\widehat{x}_0) - f_l(x_0) + \varepsilon_l \|\widehat{x}_0 - x_0\| + \overline{\varepsilon}_l = max_{i \in \mathcal{M}}[f_i(\widehat{x}_0) - f_i(x_0) + \varepsilon_i \|\widehat{x}_0 - x_0\| + \overline{\varepsilon_i}].$$

It is clear that $f_l(\widehat{x}_0) - f_l(x_0) + \varepsilon_l \|\widehat{x}_0 - x_0\| + \overline{\varepsilon}_l > 0$. By the $(\varepsilon, \overline{\varepsilon})$-quasi properly efficiency of x_0, there exists an M > 0, such that for any $i \in I$, there exists an $j \in \mathcal{M}$ satisfying $f_j(\widehat{x}_0) > f_j(x_0) - \varepsilon_j \|\widehat{x}_0 - x_0\| - \overline{\varepsilon}_j$ and

$$\frac{f_i(x_0) - \varepsilon_i \|\widehat{x}_0 - x_0\| - \overline{\varepsilon}_i - f_i(\widehat{x}_0)}{f_j(\widehat{x}_0) - f_j(x_0) + \varepsilon_j \|\widehat{x}_0 - x_0\| + \overline{\varepsilon}_j} \leqq \mathrm{M}.$$

Thus,

$$f_i(x_0) - \varepsilon_i \|\widehat{x}_0 - x_0\| - \overline{\varepsilon}_i - f_i(\widehat{x}_0) \leqq \mathrm{M}[f_j(\widehat{x}_0) - f_j(x_0) + \varepsilon_j \|\widehat{x}_0 - x_0\| + \overline{\varepsilon}_j] \leqq$$

$$\leqq \mathrm{M}[f_l(\widehat{x}_0) - f_l(x_0) + \varepsilon_l \|\widehat{x}_0 - x_0\| + \overline{\varepsilon}_l]$$

and hence

$$[f_l(\widehat{x}_0) - f_l(x_0) + \varepsilon_l\|\widehat{x}_0 - x_0\| + \overline{\varepsilon}_l]^{-1}\sum_{i\in I}[f_i(x_0) - \varepsilon_i\|\widehat{x}_0 - x_0\| - \overline{\varepsilon}_i - f_i(\widehat{x}_0)] \leqq$$

$$\leqq \mathrm{M}(m-1)$$

Since

$$0 < e^T[f(x_0) - \varepsilon\|\widehat{x}_0 - x_0\| - \overline{\varepsilon} - f(\widehat{x}_0)] \leqq \sum_{i\in I}[f_i(x_0) - \varepsilon_i\|\widehat{x}_0 - x_0\| - \overline{\varepsilon}_i - f_i(\widehat{x}_0)]$$

we have

$$[f_l(\widehat{x}_0) - f_l(x_0) + \varepsilon_l\|\widehat{x}_0 - x_0\| + \overline{\varepsilon}_l]^{-1}e^T[f(x_0) - \varepsilon\|\widehat{x}_0 - x_0\| - \overline{\varepsilon} - f(\widehat{x}_0)] \leqq \mathrm{M}(m-1).$$

Let $\rho = (min_{i\in\mathcal{M}}\alpha_i)[\mathrm{M}(m-1)]$. Then

$$\rho \leqq \alpha_l[\mathrm{M}(m-1)]^{-1} \leqq$$

$$\leqq \alpha_l[f_l(\widehat{x}_0) - f_l(x_0) + \varepsilon_l\|\widehat{x}_0 - x_0\| + \overline{\varepsilon}_l]\cdot[e^T(f(x_0) - \varepsilon\|\widehat{x}_0 - x_0\| - \overline{\varepsilon} - f(\widehat{x}_0))]^{-1}$$

i.e.

$$\rho e^T[f(x_0) - \varepsilon\|\widehat{x}_0 - x_0\| - \overline{\varepsilon} - f(\widehat{x}_0)] \leq \alpha_l[f_l(\widehat{x}_0) - f_l(x_0) + \varepsilon_l\|\widehat{x}_0 - x_0\| + \overline{\varepsilon}_l].$$

Hence

$$\alpha_l f_l(x_0) + \rho e^T f(x_0) - \alpha_l\varepsilon_l\|\widehat{x}_0 - x_0\| - \alpha_l\overline{\varepsilon}_l - \rho e^T\varepsilon\|\widehat{x}_0 - x_0\| - \rho e^T\overline{\varepsilon} <$$

$$< \alpha_l f_l(\widehat{x}_0) + \rho e^T f(\widehat{x}_0).$$

Therefore, the system of m inequalities

$$\alpha_i f_i(\widehat{x}_0) + \rho e^T f(\widehat{x}_0) <$$

$$< \alpha_i f_i(x_0) + \rho e^T f(x_0) - \alpha_i\varepsilon_i\|\widehat{x}_0 - x_0\| - \alpha_i\overline{\varepsilon}_i - \rho e^T\varepsilon\|\widehat{x}_0 - x_0\| - \rho e^T\overline{\varepsilon}$$

with $i \in \mathcal{M}$ is inconsistent.
Noting that $\widehat{x}_0$ can be any element of X, we conclude that the system

$$\begin{cases} \alpha_i f_i(\widehat{x}_0) + \rho e^T f(\widehat{x}_0) < \alpha_i f_i(x_0) + \rho e^T f(x_0) - \alpha_i\varepsilon_i\|\widehat{x}_0 - x_0\| - \alpha_i\overline{\varepsilon}_i - \\ \qquad - \rho e^T\varepsilon\|\widehat{x}_0 - x_0\| - \rho e^T\overline{\varepsilon},\ i \in \mathcal{M} \\ x \in X. \end{cases} \tag{2}$$

has no solution for some $\rho > 0$. This completes the proof. □

From Theorem Kaliszewski and Lemma 1 we get the following necessary condition for a feasible solution to be an $(\varepsilon, \overline{\varepsilon})$-properly efficient solution to (VP).

Theorem 2. *Let x_0 be a $(\varepsilon, \overline{\varepsilon})$-quasi proper efficient solution of (VP). Then x_0 is an $(\varepsilon_0, \overline{\varepsilon}_0)$-quasi optimal solution to the following scalar optimization problem*

$$(SP_2)\begin{cases} \text{minimize } \ max_{i\in\mathcal{M}}\lambda_i[(f_i(x)-y_i^*)+\rho e^T(f(x)-y^*)] \\ \text{subject to } x\in X, \end{cases}$$

for some $\rho > 0$, when $\varepsilon_0 = max_{i\in\mathcal{M}}\ \lambda_i(\varepsilon_i + \rho e^T\varepsilon)$, $\overline{\varepsilon}_0 = max_{i\in\mathcal{M}}\ \lambda_i(\overline{\varepsilon}_i + \rho e^T\overline{\varepsilon})$ and $\lambda_i = [f_i(x_0) - y_i^* + \rho e^T(f(x_0) - y^*)]^{-1}$.

Proof: Let $\alpha_i = 1$, $i \in \mathcal{M}$. By Lemma 1, there exists a $\rho > 0$ such that the system (2) has no solution. By Theorem Kaliszewski, for any negative numbers $\delta_1, \delta_2, ..., \delta_m$, $\lambda_i = -\delta_i^{-1} > 0$, $i \in \mathcal{M}$ and

$$max_{i\in\mathcal{M}}\lambda_i[(f_i(x)-f_i(x_0))+\rho e^T(f(x)-f(x_0))+\varepsilon_i\|x-x_0\|+\overline{\varepsilon}_i+$$
$$+\rho e^T\varepsilon\|x-x_0\|+\rho e^T\overline{\varepsilon}-\delta_i] \geqq 1$$

for any $x \in X$.
Let y_i^* be a number such that $\delta_i = (y_i^* - f_i(x_0)) + \rho e^T(y^* - f(x_0)) < 0$, $i \in \mathcal{M}$. Denote $\lambda_i = -\delta_i^{-1}$. Then $\lambda_i > 0$, $i \in \mathcal{M}$ and for any $x \in X$,

$$1 \leqq max_{i\in\mathcal{M}}\lambda_i[(f_i(x)-y_i^*)+\rho e^T(f(x)-y^*)+$$
$$+\varepsilon_i\|x-x_0\|+\overline{\varepsilon}_i+\rho e^T\varepsilon\|x-x_0\|+\rho e^T\overline{\varepsilon}] \leqq$$
$$\leqq max_{i\in\mathcal{M}}\lambda_i[(f_i(x)-y_i^*)+\rho e^T(f(x)-y^*)]+$$
$$+max_{i\in\mathcal{M}}\lambda_i[\varepsilon_i\|x-x_0\|+\overline{\varepsilon}_i+\rho e^T\varepsilon\|x-x_0\|+\rho e^T\overline{\varepsilon}] =$$
$$= max_{i\in\mathcal{M}}\lambda_i[(f_i(x)-y_i^*)+\rho e^T(f(x)-y^*)]+$$
$$+max_{i\in\mathcal{M}}\lambda_i(\varepsilon_i+\rho e^T\varepsilon)\|x-x_0\|+max_{i\in\mathcal{M}}\lambda_i(\overline{\varepsilon}_i+\rho e^T\overline{\varepsilon}) =$$
$$= max_{i\in\mathcal{M}}\lambda_i[(f_i(x)-y_i^*)+\rho e^T(f(x)-y^*)]+\varepsilon_0\|x-x_0\|+\overline{\varepsilon}_0.$$

Hence, for any $x \in X$

$$max_{i\in\mathcal{M}}\lambda_i[(f_i(x)-y_i^*)+\rho e^T(f(x)-y^*)] \geqq$$
$$\geqq max_{i\in\mathcal{M}}\lambda_i[(f_i(x_0)-y_i^*)+\rho e^T(f(x_0)-y^*)]-\varepsilon_0\|x-x_0\|-\overline{\varepsilon}_0.$$

The proof is completed. □

Theorem 3. *Let x_0 be a $(\varepsilon, \overline{\varepsilon})$-quasi properly efficient solution of (VP). Then there exists $\rho \in \mathbb{R}_+^*$ such that x_0 is $(\varepsilon_0, \overline{\varepsilon}_0)$-quasi optimal to the following scalar optimization problem*

$$(SP_3)\begin{cases} \text{minimize } \ max_{i\in\mathcal{M}}[\lambda_i(f_i(x)-y_i^*)+\rho e^T(f(x)-y^*)] \\ \text{subject to } x\in X, \end{cases}$$

where $\varepsilon_0 = max_{i\in\mathcal{M}}(\lambda_i\varepsilon_i+\rho e^T\varepsilon)$, $\overline{\varepsilon}_0 = max_{i\in\mathcal{M}}(\lambda_i\overline{\varepsilon}_i+\rho e^T\overline{\varepsilon})$, $\lambda_i = [f_i(x_0)-y_i^*]^{-1}$ *and* y_i^* *is a fixed number such that* $\lambda_i > 0$, $i \in \mathcal{M}$.

Proof: Assume that $x_0 \in X$ is an $(\varepsilon, \overline{\varepsilon})$-quasi properly efficient solution to (VP). Let y_i^* be a fixed number such that $\lambda_i = [f_i(x_0) - y_i^*]^{-1} > 0$, $i \in \mathcal{M}$. By Lemma 1, there exists a $\rho > 0$ such that for any $x \in X$, the system

$$\lambda_i[f_i(x) - y_i^*] + \rho e^T[f(x) - y^*] <$$

$$< \lambda_i[f_i(x_0) - y_i^*] + \rho e^T[f(x_0) - y^*] - \lambda_i\varepsilon_i\|x - x_0\| - \lambda_i\overline{\varepsilon}_i - \rho e^T\varepsilon\|x - x_0\| - \rho e^T\overline{\varepsilon}$$

with $i \in \mathcal{M}$ is inconsistent. Hence, there exists some $i_0 \in \mathcal{M}$ such that

$$\lambda_{i_0}[f_{i_0}(x) - y_{i_0}^*] + \rho e^T[f(x) - y^*] \geqq$$

$$\geqq \lambda_{i_0}[f_{i_0}(x_0) - y_{i_0}^*] + \rho e^T[f(x_0) - y^*]-$$

$$-\lambda_{i_0}\varepsilon_{i_0}\|x - x_0\| - \lambda_{i_0}\overline{\varepsilon}_{i_0} - \rho e^T\varepsilon\|x - x_0\| - \rho e^T\overline{\varepsilon} =$$

$$= 1 + \rho e^T[f(x_0) - y^*] - [\lambda_{i_0}\varepsilon_{i_0}\|x - x_0\| + \lambda_{i_0}\overline{\varepsilon}_{i_0} + \rho e^T\varepsilon\|x - x_0\| + \rho e^T\overline{\varepsilon}] =$$

$$= max_{i\in\mathcal{M}}[\lambda_i(f_i(x_0) - y_i^*) + \rho e^T(f(x_0) - y^*)]-$$

$$-[\lambda_{i_0}\varepsilon_{i_0}\|x - x_0\| + \lambda_{i_0}\overline{\varepsilon}_{i_0} + \rho e^T\varepsilon\|x - x_0\| + \rho e^T\overline{\varepsilon}] \geqq$$

$$\geqq max_{i\in\mathcal{M}}[\lambda_i(f_i(x_0) - y_i^*) + \rho e^T(f(x_0) - y^*)]-$$

$$-max_{i\in\mathcal{M}}[\lambda_i\varepsilon_i\|x - x_0\| + \lambda_i\overline{\varepsilon}_i + \rho e^T\varepsilon\|x - x_0\| + \rho e^T\overline{\varepsilon}] \geqq$$

$$\geqq max_{i\in\mathcal{M}}[\lambda_i(f_i(x_0) - y_i^*) + \rho e^T(f(x_0) - y^*)]-$$

$$-max_{i\in\mathcal{M}}[\lambda_i\varepsilon_i + \rho e^T\varepsilon]\|x - x_0\| - max_{i\in\mathcal{M}}[\lambda_i\overline{\varepsilon}_i + \rho e^T\overline{\varepsilon}] =$$

$$= max_{i\in\mathcal{M}}[\lambda_i(f_i(x_0) - y_i^*) + \rho e^T(f(x_0) - y^*)] - \varepsilon_0\|x - x_0\| - \overline{\varepsilon}_0.$$

Therefore,

$$max_{i\in\mathcal{M}}[\lambda_i(f_i(x) - y_i^*) + \rho e^T(f(x) - y^*)] \geqq$$

$$\geqq max_{i\in\mathcal{M}}[\lambda_i(f_i(x_0) - y_i^*) + \rho e^T(f(x_0) - y^*)] - \varepsilon_0\|x - x_0\| - \overline{\varepsilon}_0.$$

That implies that x_0 is $(\varepsilon_0, \overline{\varepsilon}_0)$-quasi optimal to scalar optimization problem (SP_3). The proof is completed. □

5 Case of $(\varepsilon, \overline{\varepsilon})$-Quasi Efficiency

Theorem 4. *Let x_0 be a $(\varepsilon_0, \overline{\varepsilon}_0)$-quasi optimal solution of (SP_2) where $\varepsilon_0 = max_{i \in \mathcal{M}} \lambda_i(\varepsilon_i + \rho e^T \varepsilon)$, $\overline{\varepsilon}_0 = max_{i \in \mathcal{M}} \lambda_i(\overline{\varepsilon}_i + \rho e^T \overline{\varepsilon})$, $\rho \in \mathbb{R}^*_+$, y^*_i, $i \in \mathcal{M}$ such that*

$$\lambda_i = [f_i(x_0) - y^*_i + \rho e^T(f(x_0) - y^*)]^{-1} > 0$$

with $i \in \mathcal{M}$. Then x_0 is $(\varepsilon, \overline{\varepsilon})$-quasi efficient for (VP).

Proof: Suppose that there exists some $\rho > 0$ such that, for any y^*_i such that $\lambda_i = [f_i(x_0) - y^*_i + \rho e^T(f(x_0) - y^*)]^{-1} > 0$, $i \in \mathcal{M}$, x_0 is $(\varepsilon_0, \overline{\varepsilon}_0)$-quasi optimal to (SP_2). Assume that x_0 was not $(\varepsilon, \overline{\varepsilon})$-quasi efficient to (VP). By Definition 5, there would exist some $\overline{x}_0 \in X$ satisfying

$$f_j(\overline{x}_0) \leqq f_j(x_0) - \varepsilon_j \|\overline{x}_0 - x_0\| - \overline{\varepsilon}_j, \ \ j \in \mathcal{M}$$

with at least one strict inequality. Let $y^*_i = f_i(\overline{x}_0)$, $i \in \mathcal{M}$. Then

$$\lambda_i^{-1} = f_i(x_0) - y^*_i + \rho e^T(f(x_0) - y^*) =$$

$$= f_i(x_0) - f_i(\overline{x}_0) + \rho e^T(f(x_0) - f(\overline{x}_0)) >$$

$$> \varepsilon_i \|\overline{x}_0 - x_0\| + \overline{\varepsilon}_i + \rho e^T \varepsilon \|\overline{x}_0 - x_0\| + \rho e^T \overline{\varepsilon} \geq 0, \ \ i \in \mathcal{M}.$$

We get

$$\lambda_i(\varepsilon_i \|\overline{x}_0 - x_0\| + \overline{\varepsilon}_i + \rho e^T \varepsilon \|\overline{x}_0 - x_0\| + \rho e^T \overline{\varepsilon}) < 1, \ \ i \in \mathcal{M}.$$

Because

$$\varepsilon_0 = max_{i \in \mathcal{M}} \lambda_i(\varepsilon_i + \rho e^T \varepsilon)$$

and

$$\overline{\varepsilon}_0 = max_{i \in \mathcal{M}} \lambda_i(\overline{\varepsilon}_i + \rho e^T \overline{\varepsilon}),$$

then

$$\varepsilon_0 \|\overline{x}_0 - x_0\| + \overline{\varepsilon}_0 < 1.$$

Since x_0 is $(\varepsilon_0, \overline{\varepsilon}_0)$-quasi optimal to (SP_2), we have

$$0 = max_{i \in \mathcal{M}} \lambda_i[(f_i(\overline{x}_0) - y^*_i) + \rho e^T(f(\overline{x}_0) - y^*)] \geq$$

$$\geq max_{i \in \mathcal{M}} \lambda_i[(f_i(x_0) - y^*_i) + \rho e^T(f(x_0) - y^*)] - \varepsilon_0 \|\overline{x}_0 - x_0\| - \overline{\varepsilon}_0 =$$

$$= 1 - \varepsilon_0 \|\overline{x}_0 - x_0\| - \overline{\varepsilon}_0 > 0$$

a contradiction. This implies that x_0 is $(\varepsilon, \overline{\varepsilon})$-quasi efficient to (VP). ∎

Theorem 5. *Let x_0 be a $(\varepsilon_0, \overline{\varepsilon}_0)$-quasi optimal solution of (SP_3) with $\varepsilon_0 = max_{i \in \mathcal{M}}(\lambda_i \varepsilon_i + \rho e^T \varepsilon)$, $\overline{\varepsilon}_0 = max_{i \in \mathcal{M}}(\lambda_i \overline{\varepsilon}_i + \rho e^T \overline{\varepsilon})$, $\rho > 0$, $y^*_i \in \mathbb{R}$, $i \in \mathcal{M}$ such that for any $i \in \mathcal{M}$, $\lambda_i = [f_i(x_0) - y^*_i]^{-1} > 0$ then x_0 is $(\varepsilon, \overline{\varepsilon})$-quasi efficient solution (VP).*

Proof: Suppose that there exists some $\rho > 0$ such that, for any y_i^* satisfying $\lambda_i = [f_i(x_0) - y_i^*]^{-1} > 0$, $i \in \mathcal{M}$, x_0 is $(\varepsilon, \overline{\varepsilon})$-quasi optimal to (SP_3). If x_0 was not $(\varepsilon, \overline{\varepsilon})$-quasi efficient to (VP), there would exist an $\overline{x}_0 \in X$ such that

$$f_i(\overline{x}_0) \leq f_i(x_0) - \varepsilon_i\|\overline{x}_0 - x_0\| - \overline{\varepsilon}_i,\ \ i \in \mathcal{M} \tag{3}$$

with at least one strict inequality. Let $y_i^* = f_i(x_0) - \varepsilon_i\|\overline{x}_0 - x_0\| - \overline{\varepsilon}_i$, $i \in \mathcal{M}$. Then

$$\lambda_i^{-1} = f_i(x_0) - y_i^* = \varepsilon_i\|\overline{x}_0 - x_0\| + \overline{\varepsilon}_i > 0,\ \ i \in \mathcal{M},$$

and $\varepsilon_0 = max_{i\in\mathcal{M}}(\lambda_i\varepsilon_i + \rho e^T\varepsilon)$, $\overline{\varepsilon}_0 = max_{i\in\mathcal{M}}(\lambda_i\overline{\varepsilon}_i + \rho e^T\overline{\varepsilon})$, hence

$$\varepsilon_0\|\overline{x}_0 - x_0\| + \overline{\varepsilon}_0 = max_{i\in\mathcal{M}}(\lambda_i\varepsilon_i + \rho e^T\varepsilon)\|\overline{x}_0 - x_0\| + max_{i\in\mathcal{M}}(\lambda_i\overline{\varepsilon}_i + \rho e^T\overline{\varepsilon}) =$$

$$= max_{i\in\mathcal{M}}(\lambda_i\varepsilon_i\|\overline{x}_0 - x_0\| + \lambda_i\overline{\varepsilon}_i + \rho e^T[\varepsilon\|\overline{x}_0 - x_0\| + \overline{\varepsilon}]) =$$

$$= 1 + \rho e^T[\varepsilon\|\overline{x}_0 - x_0\| + \overline{\varepsilon}].$$

By (3) and $(\varepsilon_0, \overline{\varepsilon}_0)$-quasi optimality of x_0 with respect to (SP_3), we have

$$0 > max_{i\in\mathcal{M}}[\lambda_i(f_i(\overline{x}_0) - y_i^*) + \rho e^T(f(\overline{x}_0) - y^*)] >$$

$$> max_{i\in\mathcal{M}}[\lambda_i(f_i(x_0) - y_i^*) + \rho e^T(f(x_0) - y^*)] - \varepsilon_0\|\overline{x}_0 - x_0\| - \overline{\varepsilon}_0 =$$

$$= max_{i\in\mathcal{M}}(\lambda_i\varepsilon_i\|\overline{x}_0 - x_0\| + \lambda_i\overline{\varepsilon}_i + \rho e^T\varepsilon\|\overline{x}_0 - x_0\| + \rho e^T\overline{\varepsilon}]) - \varepsilon_0\|\overline{x}_0 - x_0\| - \overline{\varepsilon}_0$$

$$= max_{i\in\mathcal{M}}[1 + \rho e^T(\varepsilon\|\overline{x}_0 - x_0\| + \overline{\varepsilon})] - \varepsilon_0\|\overline{x}_0 - x_0\| - \overline{\varepsilon}_0 = 0$$

a contradiction. Therefore, x_0 is $(\varepsilon, \overline{\varepsilon})$-quasi efficient to (VP). The proof is completed. □

Remark 2. *For* $\varepsilon = 0$ *and/or* $\overline{\varepsilon} = 0$ *we get some results stated in* $[3, 4, 5, 7, 9, 13, 14, 16]$.

6 Conclusions

In this paper we give necessary and sufficient conditions for the $(\varepsilon, \overline{\varepsilon})$-quasi weak efficiency, two necessary conditions for the $(\varepsilon, \overline{\varepsilon})$-quasi proper efficiency and two sufficient conditions for $(\varepsilon, \overline{\varepsilon})$-quasi efficiency in multiobjective optimization. No assumption of any convexity is made in this paper and the discussion is carried out in a very general framework. The results generalize the corresponding ones in this field.

We remark that some other results could be obtained, for example in the convex case with ε-subdifferential calculus for scalar functions or locally Lipschitz case by using the generalized gradient of Clarke [17].

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