Bull. Math. Soc. Sci. Math. Roumanie
Tome 51(99) No. 3, 2008, 177–182

On the Ramsey numbers for paths and generalized Jahangir graphs $J_{s,m}$

by

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Abstract

For given graphs G and H, the *Ramsey number* $R(G, H)$ is the least natural number n such that for every graph F of order n the following condition holds: either F contains G or the complement of F contains H. In this paper, we determine the Ramsey number of paths versus generalized Jahangir graphs. We also derive the Ramsey number $R(tP_n, H)$, where H is a generalized Jahangir graph $J_{s,m}$ where $s \geq 2$ is even, $m \geq 3$ and $t \geq 1$ is any integer.

Key Words: Ramsey number, path, generalized Jahangir graph.
2000 Mathematics Subject Classification: Primary 05C55, Secondary 05D10.

1 Introduction

The study of Ramsey numbers for (general) graphs have received tremendous efforts in the last two decades, see few related papers [1]-[4], [6, 8] and a nice survey paper [7].

Let $G(V, E)$ be a graph with vertex-set $V(G)$ and edge-set $E(G)$. If $xy \in E(G)$ then x is called *adjacent* to y, and y is a *neighbor* of x and vice versa. For any $A \subseteq V(G)$, we use $N_A(x)$ to denote the set of all neighbors of x in A, namely $N_A(x) = \{y \in A | xy \in E(G)\}$. Let P_n be a path with n vertices, C_n be a cycle with n vertices, W_k be a wheel of $k+1$ vertices, i.e., a graph consisting of a cycle C_k with one additional vertex adjacent to all vertices of C_k. For $s, m \geq 2$, the *generalized Jahangir graph* $J_{s,m}$ is a graph on $sm + 1$ vertices i.e., a graph consisting of a cycle C_{sm} with one additional vertex which is adjacent to m vertices of C_{sm} at distance s to each other on C_{sm}.

Recently, Surahmat and Tomescu [9] studied the Ramsey number of a combination of paths P_n versus $J_{2,m}$, and obtained the following result.

Theorem A. [9].

$$R(P_n, J_{2,m}) = \begin{cases} 6 & \textit{if } (n,m) = (4,2), \\ n+1 & \textit{if } m = 2 \textit{ and } n \geq 5, \\ n+m-1 & \textit{if } m \geq 3 \textit{ and } n \geq (4m-1)(m-1)+1. \end{cases}$$

For the Ramsey number of P_n with respect to wheel W_m, Surahmat and Baskoro [1] showed the following result.

Theorem B. [1].

$$R(P_n, W_m) = \begin{cases} 2n-1 & \textit{if } m \geq 4 \textit{ is even and } n \geq \frac{m}{2}(m-2), \\ \\ 3n-2 & \textit{if } m \geq 5 \textit{ is odd and } n \geq \frac{m-1}{2}(m-3). \end{cases}$$

In this paper, we determine the Ramsey numbers involving paths P_n and generalized Jahangir graphs $J_{s,m}$. We also find the Ramsey number $R(tP_n, H)$, where H is a generalized Jahangir graph $J_{s,m}$ where $s \geq 2$ is even, $m \geq 3$. In the following section we prove our main results.

2 Main Results

Theorem 1. *For even $s \geq 2$ and $m \geq 3$, $R(P_n, J_{s,m}) = n + \frac{sm}{2} - 1$, where $n \geq (2sm-1)(\frac{sm}{2} - 1) + 1$.*

Proof: Let $G = K_{n-1} \bigcup K_{\frac{sm}{2}-1}$. We have $R(P_n, J_{s,m}) \geq n + \frac{sm}{2} - 1$ since $P_n \not\subseteq G$ and $J_{s,m} \not\subseteq \overline{G}$. It remains to prove that $R(P_n, J_{s,m}) \leq n + \frac{sm}{2} - 1$. Let F be a graph of order $n + \frac{sm}{2} - 1$ and containing no path P_n, we will show that $\overline{F} \supseteq J_{s,m}$. Let $L_1 = l_{1,1}, l_{1,2}, \ldots, l_{1,k}$ be the longest path in F and so $k \leq n-1$. If $k = 1$ we have $\overline{F} \cong K_{n+\frac{sm}{2}-1}$, which contains $J_{s,m}$. Suppose that $k \geq 2$ and $J_{s,m} \not\subseteq \overline{F}$. We have $zl_{1,1}, zl_{1,k} \notin E(F)$ for each $z \in V_1 = V(F) \backslash V(L_1)$. We distinguish two cases:

Case 1. $k \leq 2sm - 1$. Let $L_2 = l_{1,2}, l_{2,2}, \ldots, l_{2,t}$ be a longest path in $F[V_1]$. It is clear that $1 \leq t \leq k$. If $t = 1$ then the vertices in V_1 induce a subgraph having only isolated vertices. In this case we shall add an edge uv to F, where $u, v \in V_1$ and denote $L_2 = u, v$. In this way we can define inductively the system of paths $L_1, L_2, \ldots, L_{\frac{sm}{2}-1}$ such that L_i is a longest path in $F[V_{i-1}]$, where $V_{i-1} = V(F) \backslash \bigcup_{j=1}^{i-1} V(L_j)$ or an edge added to F as above. By denoting the set of remaining vertices by B, we have $|B| \geq n + \frac{sm}{2} - 1 - (\frac{sm}{2} - 1)(2sm - 1) \geq \frac{sm}{2} \geq 3$ since $s \geq 2$ and $m \geq 3$. Let $x, y, z \in B$ be three distinct vertices which are not in any L_j for $j = 1, 2, \ldots, \frac{sm}{2} - 1$. Clearly, x, y, z are not adjacent to all endpoints of these L_j. If F_1 denotes the graph F or the graph F plus some edges added in the process of defining the system of paths, it follows that the endpoints of these L_j induce in $\overline{F_1}$ a complete graph K_{sm-2} minus a matching having at most $\frac{sm}{2}-1$ edges if some of the endpoints of same L_j are adjacent in F_1. Since x,y,z are not adjacent to all endpoints of these L_j it is easy to see that vertices x,y,z and endpoints of the paths L_j form a $J_{s,m} \subseteq \overline{F_1} \subseteq \overline{F}$.

Case 2. $k > 2sm-1$. In this case we define $\frac{sm}{2}-1$ quadruple of consecutive vertices of L_1 as follows:

$$\begin{array}{lcl} C_1 & = & \{l_{1,2}, l_{1,3}, l_{1,4}, l_{1,5}\}, \\ C_2 & = & \{l_{1,6}, l_{1,7}, l_{1,8}, l_{1,9}\}, \\ & \vdots & \\ C_{\frac{sm}{2}-1} & = & \{l_{1,2sm-6}, l_{1,2sm-5}, l_{1,2sm-4}, l_{1,2sm-3}\}. \end{array}$$

Let $Y = V(F) \setminus V(L_1)$. We have $|Y| = n + \frac{sm}{2} - 1 - k \geq \frac{sm}{2}$ since $k \leq n-1$. Hence we can consider $\frac{sm}{2}$ distinct elements in $Y: y_1, y_2, \ldots, y_{\frac{sm}{2}}$ and $\frac{sm}{2}-1$ pairs of elements $Y_i = \{y_i, y_{i+1}\}$ for $i = 1, \ldots, \frac{sm}{2}-1$. By the maximality of L_1 it follows that for each $i = 1, \ldots, \frac{sm}{2}-1$ at least one vertex in C_i is not adjacent to any vertex in Y_i. Denote by c_i the vertex in C_i which is not adjacent to any vertex in Y_i for $i = 1, \ldots, \frac{sm}{2}-1$. We have $\overline{F} \supseteq J_{s,m}$, where $J_{s,m}$ consists of the cycle C_{sm} having $V(C_{sm}) = \{y_1, c_1, y_2, c_2, \ldots, y_{\frac{sm}{2}-1}, c_{\frac{sm}{2}-1}, y_{\frac{sm}{2}}, l_{1,k}\}$ and the hub $l_{1,1}$. □

Theorem 2. *For odd $s \geq 3$,*

$$R(P_n, J_{s,m}) = \begin{cases} 2n-1 & \text{if } n \geq \frac{sm}{2}(sm-2), \text{ and } m \geq 2 \text{ is even,} \\ \\ 2n & \text{if } n \geq \frac{sm-1}{2}(sm-1), \text{ and } m \geq 3 \text{ is odd.} \end{cases}$$

Proof: To show the lower bound, consider graphs $2K_{n-1}$ and $K_1 \cup 2K_{n-1}$ for the first and second cases of Theorem respectively.

For the reverse inequality, firstly we will prove the result for the first case of Theorem. Let F be a graph of order $2n-1$ containing no path P_n where $n \geq \frac{sm}{2}(sm-2)$. We will show that $\overline{F} \supseteq J_{s,m}$. Since F does not contain P_n, by Theorem B, $\overline{F}$ will contain a wheel W_{sm}, and so clearly $\overline{F} \supseteq J_{s,m}$.

For the second case, to prove $R(P_n, J_{s,m}) \leq 2n$ let F be a graph on $2n$ vertices containing no P_n. Let $L_1 = (l_{11}, l_{12}, \cdots, l_{1k-1}, l_{1k})$ be a longest path in F and so $k \leq n-1$. If $k=1$ we have $\overline{F} \simeq K_{2n}$, which contains $J_{s,m}$. Suppose that $k \geq 2$ and $\overline{F}$ does not contain $J_{s,m}$. Obviously, zl_{11}, zl_{1k} are not in $E(F)$ for each $z \in V_1$, where $V_1 = V(F) \setminus V(L_1)$. Let $L_2 = (l_{21}, l_{22}, \cdots, l_{2t-1}, l_{2t})$ be a longest path in $F[V_1]$. It is clear that $1 \leq t \leq k$. Let $V_2 = V(F) \setminus (V(L_1) \cup V(L_2))$. We distinguish three cases.

Case 1 : $k < sm-1$. If $t = 1$ then the vertices in V_1 induce a subgraph having only isolated vertices. In this case we shall add an edge uv to F, where $u, v \in V_1$ and denote $L_2 = u, v$. In this way we can define inductively the system of paths $L_1, L_2, \cdots, L_{\frac{sm-1}{2}}$ such that L_i is a longest path in $F[V_{i-1}]$, where $V_{i-1} = V(F) \setminus \bigcup_{j=1}^{i-1} V(L_j)$ or an edge added to F as above. If F_1 denotes the graph F or the graph F plus some edges added in the process of defining the system of paths, it follows that endpoints of these L_j , where $j = 1, 2, \cdots, \frac{sm-1}{2}$ induce in $\overline{F_1}$ a complete graph K_{sm-1} minus a matching having at most $\frac{sm-1}{2}$ edges if some of the endpoints of same L_j are adjacent in F_1. Since $s, m \geq 3$ there exist at least two vertices x, y which are not adjacent to all endpoints of these L_j. Thus, it is easy to see that vertices x, y together with all endpoints of paths L_j form a $J_{s,m} \subseteq \overline{F_1} \subseteq \overline{F}$.

Case 2: $k \geq sm-1$ and $t \geq sm-1$. For $i = 1, 2, \cdots, \frac{sm-3}{2}$ define the couples A_i in path L_1 as follows:

$$A_i = \begin{cases} \{l_{1i+1}, l_{1i+2}\} & \textit{for } i \textit{ odd,} \\ \{l_{1k-i}, l_{1k-i+1}\} & \textit{for } i \textit{ even.} \end{cases}$$

Similarly, define couples B_i in path L_2 as follows:

$$B_i = \begin{cases} \{l_{2i+1}, l_{2i+2}\} & \textit{for } i \textit{ odd,} \\ \{l_{2t-i}, l_{2t-i+1}\} & \textit{for } i \textit{ even.} \end{cases}$$

Since $t \leq k \leq n-1$ and $|F| = 2n$, there exist at least two vertices x, y which are not in $L_1 \cup L_2$. Since L_1 is a longest path in F, there exists one vertex of A_i for each i, say a_i which is not adjacent with x. Similarly, since L_2 is a longest path in $V(F) \setminus V(L_1)$ there must be one vertex, say b_i, in couple B_i which is not adjacent to x for each i. By maximality of path L_1, $b_i a_i$ and $a_i b_{i+1}$ are not in $E(F)$ for each i. Thus $\{l_{11}, b_1, a_1, b_2, a_2, \cdots, b_{\frac{sm-3}{2}}, a_{\frac{sm-3}{2}}, l_{2t}, y\}$ will form a cycle C_{sm} in $\overline{F}$ and since x is adjacent with at least $sm-1$ vertices of cycle C_{sm} in $\overline{F}$, we have a subgraph in $\overline{F}$ which contain $J_{s,m}$, so $J_{s,m} \subseteq \overline{F}$.

Case 3: $k \geq sm-1$ and $t < sm-1$. Since $k \leq n-1$ (F has no P_n), V_1 will have at least $n+1$ vertices. Then, we can define the same process as in Case 1, since $n + 1 - (sm-2)\frac{sm-1}{2} \geq \frac{sm+1}{2} \geq 5$. □

In the following theorem we derive Ramsey number $R(tP_n, J_{s,m})$ for any integer $t \geq 1$, even s and $m \geq 3$, where n is large enough with respect to s and m as follows.

Theorem 3. *$R(tP_n, J_{s,m}) = tn + \frac{sm}{2} - 1$ if $n \geq (\frac{sm}{2} - 1)(2sm-1) + 1$, $s \geq 2$ is even, $m \geq 3$ and t is any positive integer.*

Proof: Since graph $G = K_{\frac{sm}{2}-1} \cup K_{tn-1}$ contains no tP_n and $\overline{G}$ contains no $J_{s,m}$, then $R(tP_n, J_{s,m}) \geq tn + \frac{sm}{2} - 1$. For proving the upper bound, let F be a graph of order $tn + \frac{sm}{2} - 1$ such that $\overline{F}$ contains no $J_{s,m}$. We will show that F contains tP_n. We use induction on t. For $t = 1$ this is true from Theorem 1. Now, let assume that the theorem is true for all $t^{'} \leq t - 1$. Take any graph F of $tn + \frac{sm}{2} - 1$ vertices such that its complement contains no $J_{s,m}$. By the induction hypothesis, F must contain $t - 1$ disjoint copies of P_n. Remove these copies from F, then by Theorem 1 the subgraph $F[H]$ on remaining vertices will induce another P_n in F since $\overline{F} \not\supseteq J_{s,m}$, so $\overline{F[H]} \not\supseteq J_{s,m}$. Therefore $F \supseteq tP_n$. The proof is complete. □

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Received: 1.05.2008.

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