Sublaplacians on CR manifolds

by

ELISABETTA BARLETTA AND SORIN DRAGOMIR

Abstract

We study the sublaplacian $\Delta_b$ on a strictly pseudoconvex CR manifold endowed with a contact form. $\Delta_b$ is approximated by a continuous family of second order elliptic operators $\{\Delta_\epsilon\}_{\epsilon>0}$. If $\{\Delta_\epsilon\}_{\epsilon>0}$ is uniformly $K$-positive definite (in the sense of W.V. Petryshyn, [14]) then we produce generalized solutions to $\Delta_b u = f$.

Key Words: CR manifold, Webster metric, sub-Riemannian gradient, sublaplacian.

2000 Mathematics Subject Classification: Primary 32V20, Secondary 35H20, 53C17.

1 Introduction

Any strictly pseudoconvex CR manifold $M$ of CR dimension $n$ carries a natural formally selfadjoint second order differential operator $\Delta_b$ associated to a choice of contact form $\theta$ on $M$ (the sublaplacian of $(M, \theta)$). $\Delta_b$ is similar to the Laplace-Beltrami operator in Riemannian geometry yet its symbol is given by

$$\sigma_2(\Delta_b)\omega v = \left[\|\omega\|^2 - \omega(T_x)^2\right] v$$

hence ellipticity of $\Delta_b$ fails in the characteristic direction $T$ of $d\theta$. As it turns out $\Delta_b$ is a degenerate elliptic operator (in the sense of J.M. Bony, [4]) i.e. locally

$$\Delta_b u = -\sum_{i,j=1}^{2n+1} \frac{\partial}{\partial x^i} \left(a^{ij} \frac{\partial u}{\partial x^j}\right) + \sum_{j=1}^{2n+1} c^j \frac{\partial u}{\partial x^j}$$

where $a_{ij}$ is but positive semi-definite and $\Delta_b = \sum_{a=1}^{2n} X_a^2 + Y$ for some smooth real vector fields $\{X_a, Y : 1 \leq a \leq 2n\}$ generating a Lie algebra $\mathcal{L}(X_1, \cdots, X_{2n}, Y)$ of rank $2n$. Also $\Delta_b$ is subelliptic of order $\epsilon = 1/2$ i.e. for each point $x \in M$ there is an open neighborhood $U \subseteq M$ and a constant $C > 0$ such that

$$\|u\|^2_{L^2(M)} \leq C \left[\langle \Delta_b u, u \rangle_{L^2(M)} + \|u\|^2_{L^2(M)}\right]$$
for any $u \in C_0^\infty(U)$. Here $\| \cdot \|$ is the Sobolev norm of order $\epsilon$. Consequently (by a classical result of L. Hörmander, [10]) $\Delta_b$ is hypoelliptic i.e. if $f$ is smooth then any distribution solution $u$ to $\Delta_b u = f$ is smooth as well. More generally $\Delta_b$ satisfies the \textit{a priori} estimates

$$\|u\|_{s+1}^2 \leq C_s \left( \|\Delta_b u\|_s^2 + \|u\|_{L^2(\Omega)}^2 \right), \quad u \in C_0^\infty(U), \quad s \geq 0.$$ 

Recently $\Delta_b$ was seen to appear as the principal part in nonlinear subelliptic systems of variational origin arising in pseudohermitian geometry (cf. E. Barletta & S. Dragomir & H. Urakawa, [1], E. Barletta, [2]-[3]) and extending the harmonic maps theory (cf. e.g. [8]) from the elliptic to the at least hypoelliptic setting. Similar PDE systems were considered within the theory of Hörmander systems of vector fields (cf. J. Jost & C-J. Xu, [11], Z.R. Zhou, [19]-[20]) and in particular on Carnot groups (cf. C. Wang, [16], L. Capogna & N. Garofalo, [5]) where the methods of calculus of variations predominate (while [1]-[3] are devoted mainly to the investigation of differential geometric properties). The theory of Hörmander systems of vector fields may be looked at as the local point of view in CR and pseudohermitian geometry (and then nondegeneracy of a CR manifold is very close to Hörmander’s rank condition on the Lie algebra generated by the given system of vector fields) yet strictly speaking the two theories interlap only partially. Indeed the maximally complex, or Levi, distribution of a nondegenerate CR manifold admits local frames which are Hörmander systems of vector fields defined on open subsets of $\mathbb{R}^{2n+1}$ while the general theory of Hörmander systems is built on domains in $\mathbb{R}^N$ with arbitrary $N$ and allows for vector fields which aren’t necessarily independent at each point or even smooth. The purpose of this paper is to start a systematic application of the methods of functional analysis and calculus of variations to sublaplacians on strictly pseudoconvex CR manifolds. The material is organized as follows. In Sections 2 to 4 we introduce appropriate (Sobolev type) function spaces and develop a calculus with $L^2$ functions on CR manifolds admitting weak horizontal, or sub-Riemannian, gradients. In Section 5 we build an example of sublaplacian (on the lowest dimensional Heisenberg group) which is strictly positive yet not positive definite. In Section 6 we consider regularity fields for sublaplacians and use general results in functional analysis to demonstrate that $\Delta_b : D(\Delta_b) \subset L^2(\Omega) \to L^2(\Omega)$ possesses a selfadjoint extension. Sections 7 and 8 provide a sub-Riemannian approach to the study of sublaplacians on CR manifolds i.e. $\Delta_b$ is approximated by the family $\{\Delta_{b,\epsilon}\}_{\epsilon>0}$ of Laplace-Beltrami operators associated to a family of Riemannian metrics $\{g_{\epsilon}\}_{\epsilon>0}$ contracting the Levi form of $(M, \theta)$ (very much the way Hörmander’s operator $H$ is approximated by the second order elliptic operators $\{H_{\epsilon}\}_{\epsilon>0}$ in [11], p. 4635). In particular we may exploit certain ideas of W.V. Petryshyn, [13]-[14], to produce generalized solutions to $\Delta_b u = f$ as limits of minimizing sequences for the functional

$$\mathcal{F}(u) = \langle \Delta_b u, Ku \rangle_{L^2(\Omega)} - \langle f, Ku \rangle_{L^2(\Omega)} - \langle Ku, f \rangle_{L^2(\Omega)}$$
where \( K : \mathcal{D}(K) \subset L^2(\Omega) \to L^2(\Omega) \) is a preclosed operator such that \( \mathcal{D}(K) \supseteq \mathcal{D}(\Delta_b) \cap \mathcal{D}(T^*T) \). As to the needed material on CR and pseudohermitian geometry we adopt the notations and conventions in [7]. The beautiful book by G. Dincă, [6], was a source of inspiration for the themes treated in this paper.

Contents

1 Introduction 3

2 Weak horizontal gradients 5

3 Local calculations 8

4 Sublaplacians and conformal transformations 11

5 A strictly positive non positive definite sublaplacian 14

6 Regular points and the defect index 17

7 Sublaplacians and sub-Riemannian geometry 18

8 Generalized solutions to \( \Delta_b u = f \). 24

2 Weak horizontal gradients

Let \( \pi : E \to M \) be a Hermitian vector bundle, with the Hermitian bundle metric \( h \), over a strictly pseudoconvex CR manifold \( M \) of CR dimension \( n \). Let \( \theta \) be a contact form on \( M \) and \( dv = \theta \wedge (d\theta)^n \) the corresponding volume form. Let \( \Omega \subseteq M \) be a domain with \( C^2 \) boundary. Let \( L^2(E_\Omega) \) denote the space of all \( L^2 \) sections in \( E_\Omega = \pi^{-1}(\Omega) \) (the portion of \( E \) over \( \Omega \)) that is \( s \in L^2(E_\Omega) \) if \( h(s, s) \in L^1(\Omega) \) i.e. \( \int_\Omega h(s, s) \, dv < \infty \). If \( \Omega \times \mathbb{C} \) is the trivial vector bundle over \( \Omega \) we write simply \( L^2(\Omega) = L^2(\Omega \times \mathbb{C}) \). Let \( H = H(M) \) be the maximally complex distribution on \( M \) and \( G_\theta \) the Levi form (cf. e.g. (1.16) in [7], p. 6). If \( u \in C^1(\Omega) \) and \( X \in \Gamma^\infty(\Omega, H \otimes \mathbb{C}) \) then (by Green’s lemma)

\[
\int_\Omega G_\theta(\nabla^H u, X) \, dv = \int_\Omega X(u) \, dv = \int_{\partial \Omega} u g_\theta(\nu, X) \, da - \int_\Omega u \text{div}(X) \, dv = - \int_\Omega u \text{div}(X) \, dv.
\]

Here \( \nabla^H u = \Pi_H \nabla u \) is the horizontal gradient of \( u \) that is \( \Pi_H : T(M) \to H \) is the projection associated to the direct sum decomposition \( T(M) = H \oplus RT \) and \( \nabla u \) is the ordinary gradient with respect to the Webster metric \( g_\theta \) (cf. Definition 1.10 in [7], p. 9) i.e. \( g_\theta(\nabla u, Y) = Y(u) \) for any \( Y \in \mathfrak{X}^\infty(M) \). Also \( \nu \) is the outward unit normal on \( \partial \Omega \) and \( T \) is the characteristic direction of \( d\theta \). The divergence of \( X \)
is computed with respect to the volume form \( d v \) i.e. \( L_X(d v) = \text{div}(X) d v \). The calculation above suggests the following natural definition. A function \( u \in L^2(\Omega) \) is \textit{weakly differentiable along} \( H \) if there is \( Y_u \in \Gamma(H \Omega \otimes \mathbb{C}) \) such that \( \|Y_u\| = G_\theta(Y_u, \nabla u)^{1/2} \in L^1_{\text{loc}}(\Omega) \) and

\[
\int_{\Omega} G_\theta(Y_u, X) d v = - \int_{\Omega} u \text{div}(X) d v
\]

for any \( X \in \Gamma_0^\infty(\Omega \otimes \mathbb{C}) \). Such \( Y_u \) is unique up to a set of measure zero and is denoted again by \( Y_u = \nabla^H u \) (the \textit{weak horizontal gradient} of \( u \)). Let \( \mathcal{D}(\nabla^H) = W^{1,2}_H(\Omega) \) consist of all \( u \in L^2(\Omega) \) such that \( u \) is weakly differentiable along \( H \) and \( \nabla^H u \in L^2(\Omega \otimes \mathbb{C}) \). Then we may regard the weak horizontal gradient as a linear operator \( \nabla^H : \mathcal{D}(\nabla^H) \subset L^2(\Omega) \to L^2(\Omega \otimes \mathbb{C}) \). As \( C_0^\infty(\Omega) \subset \mathcal{D}(\nabla^H) \) it follows that \( \mathcal{D}(\nabla^H) \) is a dense subspace of \( L^2(\Omega) \). Let

\[
(\nabla^H)^* : \mathcal{D} \left[ (\nabla^H)^* \right] \subset L^2(\Omega \otimes \mathbb{C}) \to L^2(\Omega)
\]

be the adjoint of \( \nabla^H \) i.e. i) \( \mathcal{D} \left[ (\nabla^H)^* \right] \) consists of all \( X \in L^2(\Omega \otimes \mathbb{C}) \) such that

\[
\int_{\Omega} G_\theta(\nabla^H u, X) d v = \int_{\Omega} u X^* d v
\]

for some \( X^* \in L^2(\Omega) \) and any \( u \in \mathcal{D}(\nabla^H) \), and ii) \( (\nabla^H)^* X = X^* \). Then \( \Gamma_0^\infty(\Omega \otimes \mathbb{C}) \subset \mathcal{D} \left[ (\nabla^H)^* \right] \) and the restriction of \( (\nabla^H)^* \) to \( \Gamma_0^\infty(\Omega \otimes \mathbb{C}) \) is \( - \text{div} \).

In particular \( (\nabla^H)^* \) is densely defined. The \textit{sublaplacian} of \((\Omega, \theta)\) is the linear operator

\[
\Delta_b : \mathcal{D}(\Delta_b) \subset L^2(\Omega) \to L^2(\Omega)
\]

given by

\[
\mathcal{D}(\Delta_b) = \{ u \in \mathcal{D}(\nabla^H) : \nabla^H u \in \mathcal{D} \left[ (\nabla^H)^* \right] \},
\]

\[
\Delta_b = (\nabla^H)^* \circ \nabla^H.
\]

Note that

\[
(\Delta_b u, u)_{L^2(\Omega)} = \|\nabla^H u\|_{L^2(\Omega \otimes \mathbb{C})}^2 \geq 0
\]

for any \( u \in \mathcal{D}(\Delta_b) \) i.e. the sublaplacian \((1)-(2)\) is positive. In particular (by Proposition 1.12 in [6], p. 54) the sublaplacian \((1)-(2)\) is symmetric i.e. i) its domain is a dense subspace of \( L^2(\Omega) \) and ii) \( \Delta_b \subset \Delta_b^* \) [equivalently i] \( \mathcal{D}(\Delta_b) \) is dense and ii) \( (\Delta_b u, v)_{L^2(\Omega)} = (u, \Delta_b^* v)_{L^2(\Omega)} \) for any \( u, v \in \mathcal{D}(\Delta_b) \). As customary the notation \( A \subset B \) means that \( \mathcal{D}(A) \subset \mathcal{D}(B) \) and the restriction of \( B \) to \( \mathcal{D}(A) \) is \( A \).

The restriction \((\Delta_b)_0\) of \( \Delta_b \) to \( C_0^\infty(\Omega) \) is referred to as the \textit{Lagrange sublaplacian} of \((M, \theta)\).
Proposition 1. The Lagrange sublaplacian is a strictly positive operator i.e. 
\((\Delta_b)^0 u, u\)_{L^2(\Omega)} \geq 0 \text{ and } (\Delta_b)^0 u, u\)_{L^2(\Omega)} = 0 \text{ if and only if } u = 0 \text{ in } \Omega.

**Proof:** Let \(u \in C_0^\infty(\Omega)\) such that \(\nabla^H u = 0\). Let \(\nabla\) be the Tanaka-Webster connection of \((M, \theta)\) (cf. e.g. Definition 1.25 in [7], p. 26). Then (by the purity property of the torsion of \(\nabla\), cf. (1.37) in [7], p. 25)

\[
2iG_\theta(Z, \overline{Z})T(u) = (\nabla_Z \overline{Z} - \nabla_{\overline{Z}} Z - [Z, \overline{Z}]) u = 0
\]

for any \(Z \in T_{1,0}(M) \setminus \{0\}\) hence \(T(u) = 0\) in \(\Omega\) so that \(u\) is constant. Yet \(u(x) \to 0\) as \(x \to \partial \Omega\) hence \(u = 0\).

Proposition 2. \(\nabla^H : \mathcal{D}(\nabla^H) \subset L^2(\Omega) \to L^2(H_\Omega)\) is preclosed. In particular \(\nabla^H\) admits a closed minimal extension.

**Proof:** Let \((u_k)_{k \geq 1}\) be a sequence in \(\mathcal{D}(\nabla^H)\) converging strongly to \(0 \in L^2(\Omega)\). Let us assume that \(\nabla^H u_k \to Y\) as \(k \to \infty\) for some \(Y \in L^2(H_\Omega \otimes \mathbb{C})\). Then for any \(X \in \Gamma^\infty_0(H_\Omega \otimes \mathbb{C})\)

\[
(\nabla^H u_k, X)_{L^2(H_\Omega \otimes \mathbb{C})} = -\int_\Omega u_k \text{div}(X) \, dv
\]

hence as \(k \to \infty\) one has \((Y, X)_{L^2(H_\Omega \otimes \mathbb{C})} = 0\) so that \(Y = 0\).

As \(\nabla^H\) is preclosed one may consider (cf. Proposition 1.16 in [6], p. 58) its closed minimal extension

\[
\overline{\nabla^H} : \mathcal{D}(\overline{\nabla^H}) \subset L^2(\Omega) \to L^2(H_\Omega \otimes \mathbb{C})
\]

i.e. \(\mathcal{D}(\overline{\nabla^H})\) consists of all \(u \in L^2(\Omega)\) such that there is a sequence \((u_k)_{k \geq 1} \subset \mathcal{D}(\nabla^H)\) with \(u_k \to u\) as \(k \to \infty\) and \((\nabla^H u_k)_{k \geq 1}\) is strongly convergent in \(L^2(H_\Omega \otimes \mathbb{C})\) and \(\overline{\nabla^H} u = \lim_{k \to \infty} \nabla^H u_k\). Then i) \(\overline{\nabla^H}\) is closed and ii) if \(A\) is a closed extension of \(\nabla^H\) then \(\overline{\nabla^H} \subset A\). In particular (by Proposition 1.17 in [6], p. 59)

\[
(\overline{\nabla^H})^* = (\nabla^H)^*.
\]

Proposition 3. \(W^{1,2}_H(\Omega)\) is a Hilbert space with the inner product

\[
(f, g)_{1,2} = \int_\Omega f \overline{g} \, dv + \int_\Omega G_\theta(\nabla^H f, \overline{\nabla^H g}) \, dv \tag{3}
\]

for any \(f, g \in W^{1,2}_H(\Omega)\). In particular \(W^{1,2}_H(\Omega)\) is reflexive.
Proof: Let \( \|u\|_{1,2} = (u, u)_{1,2}^{1/2} \) be the norm associated to the inner product (3). Let \( (u_k)_{k \geq 1} \subset W_{H}^{1,2}(\Omega) \) be a Cauchy sequence. Then \( (u_k)_{k \geq 1} \) is Cauchy in \( L^2(\Omega) \) hence \( u_k \to u \) as \( k \to \infty \) for some \( u \in L^2(\Omega) \). Also \( (\nabla^H u_k)_{k \geq 1} \) is Cauchy in \( L^2(H_{\Omega} \otimes \mathbb{C}) \) hence \( \nabla^H u_k \to Y \) as \( k \to \infty \) for some \( Y \in L^2(H_{\Omega} \otimes \mathbb{C}) \). Moreover
\[
- \int_{\Omega} u \text{div}(X) \, dv = - \lim_{k \to \infty} \int_{\Omega} u_k \text{div}(X) \, dv = \lim_{k \to \infty} \int_{\Omega} G_\theta(\nabla^H u_k, X) \, dv = \int_{\Omega} G_\theta(Y, X) \, dv
\]
for any \( X \in \Gamma_{\infty}^0(H_{\Omega} \otimes \mathbb{C}) \) hence \( u \in W_{H}^{1,2}(\Omega) \).

3 Local calculations

Let \( \mathbb{H}_n = \mathbb{C}^n \times \mathbb{R} \approx \mathbb{R}^{2n+1} \) be the Heisenberg group with coordinates \((z, t) = (z^1, \ldots, z^n, t), z^n = x^n + iy^n, 1 \leq \alpha \leq n, i = \sqrt{-1}, \) and let
\[
Z_{\alpha} = \frac{\partial}{\partial z_{\alpha}} - iz_{\alpha} \frac{\partial}{\partial t}, \quad 1 \leq \alpha \leq n,
\]
be the \textit{Lewy operators} (cf. \[7], p. 11). Then \( \mathbb{H}_n \) is a strictly pseudoconvex CR manifold with the CR structure \( T_{0,1}(\mathbb{H}_n) \) spanned by \( \{Z_{\alpha} : 1 \leq \alpha \leq n\} \). The real differential 1-form
\[
\theta_0 = dt + i \sum_{\alpha=1}^{n} (z_{\alpha} \, dz_{\alpha} - \overline{z}_{\alpha} \, dz^\alpha)
\]
is a contact form on \( \mathbb{H}_n \) such that the corresponding Levi form \( G_{\theta_0} \) is positive definite. Here \( z_{\alpha} = z^\alpha \). Let us consider the left invariant frame of \( H(\mathbb{H}_n) \) given by
\[
X_{\alpha} = \frac{\sqrt{2}}{2} (Z_{\alpha} + Z_{\alpha}^*), \quad X_{\alpha+n} = \frac{i\sqrt{2}}{2} (Z_{\alpha} - Z_{\alpha}^*),
\]
so that \( g_{\theta_0}(X_{\alpha}, X_b) = \delta_{ab} \). Here \( Z_{\alpha} = Z_{\alpha}^* \). Let \( X^*_{\alpha} \) be the formal adjoint of \( X_{\alpha} = b^i_{\alpha} \partial/\partial x^i \) (with \( \{x^i : 1 \leq i \leq 2n + 1\} = \{x^\alpha, y^\alpha, t : 1 \leq \alpha \leq n\} \)) i.e.
\[
X^*_{\alpha} f = - \frac{\partial}{\partial x^i} (b^i_{\alpha} f), \quad f \in C^1_0(\mathbb{H}_n),
\]
and let \( H \) be the \textit{Hörmander operator}
\[
Hu = \sum_{a=1}^{2n} X^*_a X_a u = - \frac{\partial}{\partial x^i} \left( a^{ij} \frac{\partial u}{\partial x^j} \right).
\]
Here \( a^{ij} = \sum_{a=1}^{2n} b^i_{\alpha} b^j_{\alpha} \) and
\[
[b^i_{\alpha}] = \begin{pmatrix}
\frac{\sqrt{2}}{2} \delta^i_{\alpha} & 0 & \sqrt{2} y_{\alpha} \\
0 & \frac{\sqrt{2}}{2} \delta^i_{\alpha} & -\sqrt{2} x_{\alpha}
\end{pmatrix}
\]
so that
\[
H = -\sum_{\alpha=1}^{n} \left( \frac{\partial^2 u}{\partial x_\alpha^2} + \frac{\partial^2 u}{\partial y_\alpha^2} \right) + 2 \frac{\partial}{\partial t} \left( x^\alpha \frac{\partial u}{\partial y^\alpha} - y^\alpha \frac{\partial u}{\partial x^\alpha} \right) - 2|z|^2 \frac{\partial^2 u}{\partial t^2}
\]
where $|z|^2 = z^\alpha z_\alpha$. Note that $X^*_\alpha = -X_\alpha$ hence $H = -\sum_{\alpha=1}^{2n} X^2_\alpha$ so that $H$ is a degenerate elliptic operator (in the sense of [4]).

**Remark 1.** If $M = \mathbb{H}_n$ and $\theta = \theta_0$ then $\Delta_b = H$.

Let $\Omega \subseteq \mathbb{H}_n$ be an open subset. We shall need the (Sobolev type) function spaces
\[
W^{k,p}_X(\Omega) = \{ f \in L^p(\Omega) : X^J f \in L^p(\Omega), \ |J| \leq k, \ J = (a_1, \cdots, a_s), \ 1 \leq a_\ell \leq 2n, \ |J| = s, \ X^J f = X_{a_1} \cdots X_{a_s} f \}.
\]
By a result of C-J. Xu (cf. Theorem 1 in [18], p. 149) if $1 \leq p < \infty$ then $W^{k,p}_X(\Omega)$ is a separable Banach space with the norm
\[
\|f\|_{W^{k,p}_X(\Omega)} = \left( \sum_{|J| \leq k} \|X^J f\|_{L^p(\Omega)}^p \right)^{1/p}.
\]
If $1 < p < \infty$ then $W^{k,p}_X(\Omega)$ is also reflexive. If $p = 2$ then $W^1_X(\Omega) = W^{2,2}_X(\Omega)$ is a separable Hilbert space. Let $W^k_{X,0}(\Omega)$ be the completion of $C^\infty_0(\Omega)$ in $W^k_X(\Omega)$. Note that \{ $X_a : 1 \leq a \leq 2n$ \} is a system of vector fields on $\mathbb{H}_n$ satisfying Hörmander’s condition (i.e. the vector fields $X_a$ and their commutators up to length $r = 2$ span the tangent space $T_x(\mathbb{H}_n)$ at any point $x \in \mathbb{H}_n$). Consequently $W^k_{X,0}(\Omega) \subset H^{k/2}(\Omega)$ where $H^j(\Omega)$ are the ordinary Sobolev spaces (cf. also (3.2) in [11], p. 4635). Similar considerations hold locally on an arbitrary strictly pseudoconvex CR manifold. Let \{ $X_a : 1 \leq a \leq 2n$ \} be a local $G_\theta$-orthonormal frame in $H$ defined on the open subset $\mathbb{U} \subseteq M$. If $u \in C^1(\mathbb{U})$ then (by Green’s lemma)
\[
\int_{\mathbb{U}} X_a(u) \overline{\varphi} \, dv = \int_{\mathbb{U}} \{ X_a(u \overline{\varphi}) - u X_a(\overline{\varphi}) \} \, dv = -\int_{\mathbb{U}} u \{ \overline{\varphi} \text{div}(X_a) + X_a(\overline{\varphi}) \} \, dv = -\int_{\mathbb{U}} \text{div}(\overline{\varphi} X_a) \, dv
\]
for any $\varphi \in C^\infty_0(\mathbb{U})$. We may then adopt the following definition. Let $\Omega \subseteq M$ be a domain such that $\Omega \cap \mathbb{U} \neq \emptyset$. A function $u \in L^2(\Omega \cap \mathbb{U})$ is weakly differentiable in the direction $X_a$ if there is $v_a \in L^1_{\text{loc}}(\Omega \cap \mathbb{U})$ such that
\[
\int_{\Omega \cap \mathbb{U}} v_a \overline{\varphi} \, dv = -\int_{\Omega \cap \mathbb{U}} u \text{div}(\overline{\varphi} X_a) \, dv
\]
for any $\varphi \in C_0^\infty(\Omega \cap U)$. Such $v_a$ is unique up to a set of measure zero and denoted by $v_a = X_a(u)$. Let $\mathcal{D}(X_a)$ consist of all $u \in L^2(\Omega \cap U)$ such that $u$ is weakly differentiable in the direction $X_a$ and $X_a(u) \in L^2(\Omega \cap U)$. We set

$$W_X^{1,2}(\Omega \cap U) = \bigcap_{a=1}^{2n} \mathcal{D}(X_a).$$

We wish to investigate the relationship among the spaces $W_H^{1,2}$ and $W_X^{1,2}$. Let $u \in W_H^{1,2}(\Omega \cap U)$ and let us set $v_a = G_\theta(\nabla^H u, X_a)$. Then

$$\int_{\Omega \cap U} |v_a|^2 \, dv \leq \int_{\Omega \cap U} \|\nabla^H u\|_2^2 \|X_a\|_2^2 \, dv = \|\nabla^H u\|_{L^2(\Omega \cap U)}^2$$

so that $v_a \in L^2(\Omega \cap U)$. Next

$$\int_{\Omega \cap U} v_a \varphi \, dv = \int_{\Omega \cap U} G_\theta(\nabla^H u, \varphi X_a) \, dv = -\int_{\Omega \cap U} u \div(\varphi X_a) \, dv$$

for any $\varphi \in C_0^\infty(\Omega \cap U)$ hence $u \in \mathcal{D}(X_a)$ for any $1 \leq a \leq 2n$. Viceversa, let $u \in W_X^{1,2}(\Omega \cap U)$ and let us set $Y = \sum_{a=1}^{2n} X_a(u)X_a$. For arbitrary $X \in \Gamma_0^\infty(H_{\Omega \cap U} \otimes \mathbb{C})$ let $\varphi_a = G_\theta(X_a, X) \in C_0^\infty(\Omega \cap U)$. Then

$$\int_{\Omega \cap U} G_\theta(Y, X) \, dv = \sum_a \int_{\Omega \cap U} X_a(u) \varphi_a \, dv =$$

$$= -\sum_a \int_{\Omega \cap U} u \div(\varphi_a X_a) \, dv = -\int_{\Omega \cap U} u \div(X) \, dv$$

hence $u \in W_H^{1,2}(\Omega \cap U)$ and $\nabla^H u = \sum_a X_a(u)X_a$. We may state

**Proposition 4.** The restriction to $\Omega \cap U$ induces a natural continuous map $W_H^{1,2}(\Omega) \to W_H^{1,2}(\Omega \cap U)$ and $W_X^{1,2}(\Omega \cap U) = W_X^{1,2}(\Omega \cap U)$.

**Proof:** Only the first statement needs to be checked. The restriction to $\Omega \cap U$ induces natural maps $L^2(\Omega) \to L^2(\Omega \cap U)$ and $L^2(H_{\Omega} \otimes \mathbb{C}) \to L^2(H_{\Omega \cap U} \otimes \mathbb{C})$. Let $u \in W_H^{1,2}(\Omega)$ and $X \in \Gamma_0^\infty(H_{\Omega \cap U} \otimes \mathbb{C})$. Let us set $\bar{X}(x) = X(x)$ for any $x \in \Omega \cap U$ and $\bar{X}(x) = 0$ for any $x \in \Omega \setminus U$ so that $\bar{X} \in \Gamma_0^\infty(H_{\Omega} \otimes \mathbb{C})$. Let $Y \in L^2(H_{\Omega \cap U} \otimes \mathbb{C})$ be the restriction of $\nabla^H u$ to $\Omega \cap U$. Then

$$\int_{\Omega \cap U} G_\theta(Y, \bar{X}) \, dv = \int_{\Omega} G_\theta(\nabla^H u, \bar{X}) \, dv =$$

$$= -\int_{\Omega} u \div(\bar{X}) \, dv = -\int_{\Omega \cap U} u \div(\bar{X}) \, dv$$

hence $u|_{\Omega \cap U} \in W_H^{1,2}(\Omega \cap U)$ and $\nabla^H (u|_{\Omega \cap U}) = Y$. \hfill $\Box$
4 Sublaplacians and conformal transformations

Let $M$ be a strictly pseudoconvex CR manifold of CR dimension $n$. Let $\hat{\theta} = e^{f}\theta$, with $f \in C^{\infty}(\Omega)$ real valued, be another contact form on $M$ and $\hat{\Delta}_b$ the corresponding sublaplacian.

**Proposition 5.** Let $\Omega \subseteq M$ be a domain and $f \in C^{\infty}(\Omega)$ a real valued function. Let us assume that either $\Omega$ is bounded or $f \in L^{\infty}(\Omega)$ and $\nabla^{H}f \in L^{\infty}(H\Omega \otimes \mathbb{C})$. Then $D(\hat{\Delta}_b) = D(\Delta_b)$ and

$$e^{f}\hat{\Delta}_b u = \Delta_b u - n(\nabla^{H}u)f$$

for any $u \in D(\Delta_b)$.

We need the following

**Lemma 1.** Let $\{X_a : 1 \leq a \leq 2n\}$ be a local $G_{\theta}$-orthonormal frame of $H$, defined on the open subset $U \subseteq M$, such that $X_{\alpha + n} = JX_{\alpha}$ for any $1 \leq \alpha \leq n$. Then

$$e^{f}\hat{T} = T + \frac{1}{2} \sum_{a=1}^{n} \{f_{\alpha + n}X_{\alpha} - f_{\alpha}X_{\alpha + n}\}$$

(5)
on $U \cap \Omega$, where $f_a = X_a(f)$. In particular $e^{f}J\hat{T} = \frac{1}{2} \nabla^{H}f$ and

$$\hat{\Pi}_{H} = \Pi_{H} + \frac{1}{2} \theta \otimes J\nabla^{H}f$$

(6)

where $\hat{\Pi}_{H} : T(M) \rightarrow H$ is the projection associated to the decomposition $T(M) = H \oplus \mathbb{R}\hat{T}$. Also $\hat{T}$ is the characteristic direction of $d\hat{\theta}$ and $J : H \rightarrow H$ is the complex structure along $H$ given by $J(Z + \overline{Z}) = i(Z - \overline{Z})$ for any $Z \in T_{1,0}(M)$. Moreover

$$\hat{\text{div}}(X) = \text{div}(X) + (n + 1)X(f)$$

(7)

for any $X \in \mathfrak{X}^{\infty}(\Omega)$.

**Proof:** The identity (5) follows from $\hat{\theta}(\hat{T}) = 1$ and $\hat{T} \cdot d\hat{\theta} = 0$ while (7) is a consequence of $\hat{\omega}v = \hat{\theta} \wedge (d\hat{\theta})^{n} = e^{(n+1)f} \, dv$.

Let $u \in D(\nabla^{H}) = W^{1,2}_{H}(\Omega)$. Then (by (7) in Lemma 1)

$$\int_{\Omega} G_{\alpha}(\nabla^{H}u, \overline{X}) \, dv = - \int_{\Omega} u \text{div}(\overline{X}) \, dv =$$

$$= - \int_{\Omega} u \left[ \text{div}(X) - (n + 1)X(f) \right] e^{-(n+1)f} \, dv =$$

$$= - \int_{\Omega} u \text{div} \left( e^{-(n+1)f}X \right) \, dv$$
for any $X \in \Gamma_0^\infty(H_\Omega \otimes \mathbb{C})$. Therefore $u \in D(\hat{\nabla}^H)$ and

$$\hat{\nabla}^H u = e^{-f} \nabla^H u.$$  \hfill (8)

Moreover it is elementary that $\varphi u \in D(\nabla^H)$ for any $\varphi \in C^\infty(\Omega) \cap L^\infty(\Omega)$ with $\nabla^H \varphi \in L^\infty(H_\Omega \otimes \mathbb{C})$ and any $u \in D(\nabla^H)$. Indeed

$$-\int_{\Omega} \varphi u \text{div}(X) \, dv = -\int_{\Omega} u \left[ \text{div}(\varphi X) - X(\varphi) \right] \, dv =$$

$$\int_{\Omega} G_\theta(\nabla^H u, \varphi X) \, dv + \int_{\Omega} u G_\theta(\nabla^H \varphi, X) \, dv$$

for any $X \in \Gamma_0^\infty(H_\Omega \otimes \mathbb{C})$. In particular $\nabla^H(\varphi u) = \varphi \nabla^H u + u \nabla^H \varphi \in L^2(H_\Omega \otimes \mathbb{C})$.

Let $u \in D(\Delta_b)$. Then $u \in D(\nabla^H)$ and $\nabla^H u \in D[(\nabla^H)^*]$ i.e. there is $w \in L^2(\Omega)$ (here $w = \Delta_b u$) such that

$$\int_{\Omega} \psi w \, dv = \int_{\Omega} G_\theta(\nabla^H \psi, \nabla^H u) \, dv = \int_{\Omega} e^{-nf} G_\theta(\hat{\nabla}^H \psi, \hat{\nabla}^H u) \, \hat{dv}$$

for any $\psi \in D(\nabla^H)$. Let us set $\varphi = e^{-nf} \psi \in D(\nabla^H)$. Then

$$\int_{\Omega} \varphi \left[ e^{-f} w - n G_\theta(\nabla^H f, \hat{\nabla}^H u) \right] \, \hat{dv} = \int_{\Omega} G_\theta(\hat{\nabla}^H \varphi, \hat{\nabla}^H u) \, \hat{dv}$$

so that $\hat{\nabla}^H u \in D[(\nabla^H)^*]$ and

$$(\hat{\nabla}^H)^* \hat{\nabla}^H u = e^{-f} w - n(\nabla^H u)f$$

which (by (8)) yields (4). Proposition 5 is proved.

Remark 2. It may be easily shown that (4) holds for any $u \in C^2(\Omega)$ as well. Indeed let $x_0 \in \Omega$ and let $\{X_a : 1 \leq a \leq 2n\}$ be a local $G_\theta$-orthonormal frame of $H$ defined on an open neighborhood $U \subseteq M$ of $x_0$. Then\(^1\) (cf. (2.6) in [7], p. 112)

$$\Delta_b u = -\sum_{a=1}^{2n} \{X_a(X_a u) - (\nabla X_a X_a)u\}$$

on $\Omega \cap U$. We set $\hat{X}_a = e^{-f/2} X_a$ for any $1 \leq a \leq 2n$. Then

$$\hat{X}_a(\hat{X}_a u) = e^{-f} \{X_a(X_a u) - \frac{1}{2} X_a(f) X_a(u)\},$$

$$(\hat{\nabla} X_a \hat{X}_a)u = e^{-f} \{(\nabla X_a X_a)u - \frac{1}{2} X_a(f) X_a(u)\},$$

\(^1\)The sublaplacian in this work and in reference [7] differ by a sign.
where $\hat{\nabla}$ is the Tanaka-Webster connection of $(M, \hat{\theta})$. Then
\[ e^f \hat{\Delta}_b u = \Delta_b u + L_f u \] (9)
where $L_f$ is the first order differential operator given by
\[ L_f = \sum_{a=1}^{2n} \{ \hat{\nabla}_a X_a - \nabla_a X_a \}. \]

Since $\nabla g_\theta = 0$ one has
\[ X(g_\theta(Y, Z)) + Y(g_\theta(Z, X)) - Z(g_\theta(X, Y)) = 2g_\theta(\nabla_X Y, Z) + \]
\[ + g_\theta(Z, [Y, X]) - g_\theta(Y, [Z, X]) - g_\theta(X, [Z, Y]) + \]
\[ + g_\theta(Z, T_\nabla(Y, X)) - g_\theta(Y, T_\nabla(Z, X)) - g_\theta(X, T_\nabla(Z, Y)) \]
for any $X, Y, Z \in X^\infty(M)$, where $T_\nabla$ is the torsion tensor field of $\nabla$. Recall that $T_\nabla(X, Y) = (d\theta)(X, Y)\hat{T}$ for any $X, Y \in \Gamma^\infty(H)$ (cf. Lemma 1.3 in [7], p. 37).

Let $X, Z \in \Gamma^\infty(H)$ and $Y = X$. Then
\[ 2X(g_\theta(X, Z)) = \sum_{a=1}^{2n} \hat{\theta}([X_a, X_b])g_\theta(X_a, \hat{T}) \]
(10)

Next (as $H$ is parallel with respect to $\nabla$)
\[ 2e^f g_\theta(\nabla_X X_a, X_b) = 2\delta_{ab} X_a(e^f) - X_b(e^f) + 2g_\theta(X_a, [X_b, X_a]) \]
or
\[ g_\theta(\nabla_X X_a, X_b) = \delta_{ab} X_a(f) - \frac{1}{2} X_b(f) + g_\theta(X_a, \hat{T})_H[X_b, X_a]. \] (11)

Subtracting in (10) from (11)
\[ g_\theta(L_f, X_b) = -(n-1)X_b(f) + \sum_{a=1}^{2n} \hat{\theta}([X_a, X_b])g_\theta(X_a, \hat{T}). \] (12)

By (5) in Lemma 1
\[ \sum_{a} \hat{\theta}([X_a, X_b])g_\theta(X_a, \hat{T}) = 2e^f g_\theta(JX_b, \hat{T}) = -f_b \]
hence (12) becomes $g_\theta(L_f, X_b) = -nf_b$ so that $L_f = -n \nabla^H f$ i.e. (9) implies (4) for any smooth $u$. 

Sublaplacians on CR manifolds

\[ e^f \hat{\Delta}_b u = \Delta_b u + L_f u \] (9)
5 A strictly positive non positive definite sublaplacian

Let $\mathbb{H}_1 = \mathbb{C} \times \mathbb{R}$ be the lowest dimensional Heisenberg group endowed with the canonical contact form $\theta = dt + i(zdz - zd\overline{z})$. Let $\Delta_b$ be the corresponding sublaplacian. Let $\overline{B}(0, R) = \{x \in \mathbb{H}_1 : |x| \leq R\}$ where $|x| = (|z|^2 + t^2)^{1/4}$ is the Heisenberg norm of $x = (z, t) \in \mathbb{H}_1$. Let $\Omega = \mathbb{H}_1 \setminus \overline{B}(0, 1)$. Let $\mathcal{M}$ consist of all $u \in L^2(\Omega)$ such that i) $u \in C^2(\Omega)$, ii) $u|_{\Sigma(0, 1)} = 0$, iii) $\int_{\Sigma(0, R)} \mathbb{g}_\theta(\nabla^H u, \nu) \, da \to 0$ as $R \to \infty$, and iv) $\Delta_b u \in L^2(\Omega)$, where $\Sigma(0, R) = \{x \in \mathbb{H}_1 : |x| = R\}$. Also $\nu$ is the outward unit normal field (with respect to the Webster metric of $(\mathbb{H}_1, \theta)$) on the boundary of $\overline{B}(0, R)$. Let $\Omega_R = \{x \in \mathbb{H}_1 : 1 < |x| < R\}$ with $R > 1$. For any $u \in \mathcal{M}$ (by Green’s lemma)

\[
\int_{\Omega_R} (\Delta_b u) \mathbb{g} \, dv = -\int_{\Omega_R} \operatorname{div}(\nabla^H u) \mathbb{g} \, dv = -\int_{\Omega_R} \{\operatorname{div}(\mathbb{g} \nabla^H u) - (\nabla^H u)(\mathbb{g})\} \, dv = -\int_{\Sigma(0, R)} \mathbb{g} \theta(\nabla^H u, \nu) \, da + \int_{\Omega_R} \mathbb{g} \theta(\nabla^H u, \nabla^H \mathbb{g}) \, dv.
\]

Let $R \to \infty$. By (iii) above

\[
(\Delta_b u, u)_{L^2(\Omega)} = ||\nabla^H u||_{L^2(\Omega)}^2 \geq 0, \quad u \in \mathcal{M},
\]

with equality if and only if $u = \text{constant}$ on $\Omega$ (as $\mathbb{H}_1$ is a nondegenerate CR manifold) and then $u = 0$ (by (ii) above). Hence $\Delta_b : \mathcal{M} \subset L^2(\Omega) \to L^2(\Omega)$ is a strictly positive operator. Next let

\[
u \alpha(x) = (|x| - 1)|x|^4 e^{-\alpha |x|}, \quad \alpha > 0.
\]

Clearly $u_\alpha \in C^2(\Omega)$ and $u_\alpha|_{\partial \Omega} = 0$. To compute the limit in (iii) we need to study the geometry of a Heisenberg sphere $\Sigma(0, R)$ in the ambient space $(\mathbb{H}_1, \mathbb{g}_\theta)$. Let us consider the half spaces $\mathbb{H}_1^+ = \{(z, t) \in \mathbb{H}_1 : t > 0\}$ and $\mathbb{H}_1^- = \{(z, t) \in \mathbb{H}_1 : t < 0\}$ and set $U_\pm = \Sigma(0, R) \cap \mathbb{H}_1^\pm$. Let us consider the parametrizations $\psi_\pm : D(0, R) \to U_\pm$ given by $\psi_\pm(z) = (z, \pm \sqrt{R^2 - |z|^2})$ where $D(0, R) = \{z \in \mathbb{C} : |z| < R\}$. Let

\[
X = \frac{\partial}{\partial x} + 2yT, \quad Y = \frac{\partial}{\partial y} - 2xT,
\]

be the left invariant vector fields spanning the Levi distribution (a Hörmander system on $\mathbb{R}^3$) where $z = x + iy$ and $T = \partial/\partial t$. Then

\[
\hat{X} = X - 2 \left( y + \frac{xz^2}{t} \right) T, \quad \hat{Y} = Y + 2 \left( x - \frac{y|z|^2}{t} \right) T,
\]

span the portion of $T(\Sigma(0, R))$ over $U = U_+ \cup U_-$. As

\[
g_\theta(X, X) = g_\theta(Y, Y) = 2, \quad g_\theta(X, Y) = 0,
\]
Sublaplacians on CR manifolds

\[ g_\theta(X, T) = g_\theta(Y, T) = 0, \quad g_\theta(T, T) = 1, \]
a unit normal field \( \nu \) on \( U \) (i.e. \( g_\theta(\tilde{X}, \nu) = g_\theta(\tilde{Y}, \nu) = 0 \) and \( \|\nu\| = 1 \)) is given by
\[
\nu = c \left( T + \left( \frac{x|z|^2}{t} + y \right) X + \left( \frac{y|z|^2}{t} - x \right) Y \right),
\]
\[ c \in \left\{ \pm \frac{t}{\sqrt{t^2 + 2|x|^2z^2}} \right\}. \]

The components \( g_{ij} \) of the first fundamental form of \( \iota : U_+ \hookrightarrow \mathbb{H}_1 \) are given by
\[
g_\theta \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \right) = 2(1 + 2y^2), \quad g_\theta \left( \frac{\partial}{\partial y}, \frac{\partial}{\partial y} \right) = 2(1 + 2x^2),
\]
\[
g_\theta \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) = -4xy,
\]
hence \( \det[g_{ij}] = 4(1 + 2|z|^2) \) and then \( da = 2\sqrt{1 + 2|z|^2} \) \( dx \wedge dy \). We set \( \varphi(z, t) = (|z|^4 + t^2)^{1/4} \) for simplicity. Then
\[
X(\varphi) = \varphi^{-3}(x|z|^2 + yt), \quad Y(\varphi) = \varphi^{-3}(y|z|^2 - xt),
\]
\[
X(u_\alpha) = -\varphi^3 e^{-\alpha\varphi} Q_\alpha(\varphi)X(\varphi), \quad Y(u_\alpha) = -\varphi^3 e^{-\alpha\varphi} Q_\alpha(\varphi)Y(\varphi),
\]
where \( Q_\alpha(R) = \alpha R^2 - (\alpha + 5)R + 4 \). Consequently
\[
\| \nabla^H u_\alpha \|^2 = \frac{1}{2} [X(u_\alpha)^2 + Y(u_\alpha)^2] = \frac{1}{2} |z|^2 \varphi^4 e^{-2\alpha\varphi} Q_\alpha(\varphi)^2.
\]

It follows that \( \| \nabla^H u_\alpha \|^2 \circ \psi_+ = \frac{1}{2} |z|^2 R^4 e^{-2\alpha R} Q_\alpha(R)^2 \) so that for any \( R > 1 \)
\[
\left| \int_{U_+} u_\alpha g_\theta(\nabla^H u_\alpha, \nu) \, da \right| \leq \int_{U_+} |u_\alpha| \| \nabla^H u_\alpha \| \, da =
\]
\[
= 2 \int_{D(0,R)} (u_\alpha \circ \psi_+) \left( \| \nabla^H u_\alpha \| \circ \psi_+ \right) \sqrt{1 + 2|z|^2} \, dx \wedge dy =
\]
\[
= \sqrt{2}(R - 1) R^6 e^{-2\alpha R} |Q_\alpha(R)| \int_{D(0,R)} |z| \sqrt{1 + 2|z|^2} \, dx \wedge dy.
\]
Hence there is a function \( f_\alpha(R) \) such that \( 0 \leq f_\alpha(R) \leq CR^{13} \) for some constant \( C > 0 \) and then
\[
\left| \int_{U_+} u_\alpha g_\theta(\nabla^H u_\alpha, \nu) \, da \right| \leq f_\alpha(R) e^{-2\alpha R} \to 0
\]
as \( R \to \infty \). By a similar calculation \( \lim_{R \to \infty} \int_{U} u_\alpha \varphi_\theta(\nabla^H u_\alpha, \nu) \, da = 0 \) as well. Finally \( \| \nabla^H \varphi \|^2 = \frac{1}{2} \varphi^{-2} |z|^2 \) and \( \Delta_b \varphi = -\frac{3}{2} \varphi^{-3} |z|^2 \) hence

\[
\Delta_b u_\alpha = -\frac{1}{2} |z|^2 e^{-\alpha \varphi} A_\alpha(\varphi) \in L^2(\Omega)
\]

for some polynomial \( A_\alpha \) so that \( \Delta_b u_\alpha \in L^2(\Omega) \). Summing up the information got so far \( u_\alpha \in M \) for any \( \alpha > 0 \). At this point we may show that \( \Delta_b : M \subset L^2(\Omega) \to L^2(\Omega) \) is not positive definite. Note first that the area of the Heisenberg sphere \( \Sigma(0, R) \) is

\[
\int_{\Sigma(0, R)} da = \frac{4\pi}{3} \left[ (1 + 2R^2)^{3/2} - 1 \right].
\]

Moreover

\[
\| u_\alpha \|^2_{L^2(\Omega)} = \int_{\Omega} (|x| - 1)^2 |x|^8 e^{-2\alpha |x|} \, dv(x) =
\]

\[
= \int_1^\infty dR \int_{|x|=R} (R - 1)^2 R^8 e^{-2\alpha R} \, da(x) \geq
\]

\[
\geq \frac{4\pi}{3} \int_1^\infty R^8 (R - 1)^2 (R^3 - 1) e^{-2\alpha R} \, dR =
\]

\[
= \frac{4\pi}{3} [F(2\alpha) - \int_0^1 f(t)e^{-2\alpha t} \, dt]
\]

where \( F(s) \) is the Laplace transform of \( f(t) = (t - 1)^2(t^3 - 1)t^8 \). Similarly

\[
(\Delta_b u_\alpha, u_\alpha)_{L^2(\Omega)} = \frac{1}{2} \int_{\Omega} |z|^2 |x|^4 e^{-2\alpha |x|} Q_\alpha(|x|)^2 \, dv(x) \leq
\]

\[
\leq \int_{\Omega} |x|^6 Q_\alpha(|x|)^2 e^{-2\alpha |x|} \, dv(x) =
\]

\[
= \int_1^\infty dR \int_{|x|=R} R^6 Q_\alpha(R)^2 e^{-2\alpha R} \, da(x) \leq
\]

\[
\leq \frac{8\pi}{3} \int_1^\infty R^7 (2R^2 + R + 1) Q_\alpha(R)^2 e^{-2\alpha R} \, dR =
\]

\[
= \frac{8\pi}{3} [G(2\alpha) - \int_0^1 g(t)e^{-2\alpha t} \, dt]
\]

where \( G(s) \) is the Laplace transform of \( g(t) = t^7(2t^2 + t + 1)Q_\alpha(t)^2 \). There is a polynomial \( p(\alpha) \) such that \( p(0) = 13! \) and

\[
\frac{(\Delta_b u_\alpha, u_\alpha)_{L^2(\Omega)}}{\| u_\alpha \|^2_{L^2(\Omega)}} \leq \frac{O(\alpha) - (2\alpha)^{14} \int_0^1 g(t)e^{-2\alpha t} \, dt}{p(\alpha) - (2\alpha)^{14} \int_0^1 f(t)e^{-2\alpha t} \, dt},
\]

\(^2\)An uninspiring calculation shows that \( A_\alpha(R) = \alpha^2 R^3 - \alpha(\alpha + 13) R^2 + (11\alpha + 35) R - 24 \).
Sublaplacians on CR manifolds

\[
\left( \Delta_b u_\alpha, u_\alpha \right)_{L^2(\Omega)} \rightarrow 0, \quad \alpha \rightarrow 0.
\]

Then for any \( \gamma > 0 \) there is \( \alpha = \alpha(\gamma) \) such that \( \left( \Delta_b u_\alpha, u_\alpha \right)_{L^2(\Omega)} < \gamma^2 \| u_\alpha \|_{L^2(\Omega)}^2 \).

6 Regular points and the defect index

Let \( \text{Reg}(\Delta_b) \subseteq \mathbb{C} \) be the regularity field of the sublaplacian (1)-(2) i.e. if \( \lambda \in \text{Reg}(\Delta_b) \) then there is \( \gamma = \gamma(\lambda) > 0 \) such that

\[
\| (\Delta_b - \lambda I) u \|_{L^2(\Omega)} \geq \gamma \| u \|_{L^2(\Omega)}
\]

for any \( u \in D(\Delta_b) \). Let \( \lambda = a + ib \) be the real and imaginary parts of \( \lambda \in \mathbb{C} \). As \( A = \Delta_b - aI \) is a symmetric operator

\[
\| (\Delta_b - \lambda I) u \|_{L^2(\Omega)}^2 = \| (A - ibI) u \|_{L^2(\Omega)}^2 = \| Au \|_{L^2(\Omega)}^2 + b^2 \| u \|_{L^2(\Omega)}^2
\]

hence \( \| (\Delta_b - \lambda I) u \|_{L^2(\Omega)} \geq |\text{Im}(\lambda)| \| u \|_{L^2(\Omega)} \) for any \( u \in D(\Delta_b) \). Consequently \( \mathbb{C}_+ \subseteq \text{Reg}(\Delta_b) \) where \( \mathbb{C}_+ = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \} \) and \( \mathbb{C}_- = \{ z \in \mathbb{C} : \text{Im}(z) < 0 \} \).

Let \( \Gamma \subseteq \text{Reg}(\Delta_b) \) be a connected component. By Theorem 1.22 in [6], p. 76, the spaces

\[
L^2(\Omega) \ominus \mathcal{R}(\Delta_b - \lambda I), \quad \lambda \in \Gamma,
\]

have the same dimension (as Hilbert spaces). The defect index of \( \Delta_b \) corresponding to the connected component \( \Gamma \) is

\[
\text{def}_\Gamma(\Delta_b) = \dim \left[ L^2(\Omega) \ominus \mathcal{R}(\Delta_b - \lambda I) \right], \quad \lambda \in \Gamma.
\]

Note that \( \Delta_b : D(\Delta_b) \subset L^2(\Omega) \rightarrow L^2(\Omega) \) is a real operator i.e. \( u \in D(\Delta_b) \implies \overline{u} \in D(\Delta_b) \) and \( \Delta_b \overline{u} = \overline{\Delta_b u} \) for any \( u \in D(\Delta_b) \). Then (by Theorem 1.31 in [6], p. 89)

\[
\text{def}_{\mathbb{C}_-}(\Delta_b) = \text{def}_{\mathbb{C}_+}(\Delta_b).
\]

(14)

At this point one may use the von Neumann theorem (cf. e.g. Theorem 1.28 in [6], p. 84) to conclude that

**Theorem 1.** The sublaplacian \( \Delta_b : D(\Delta_b) \subset L^2(\Omega) \rightarrow L^2(\Omega) \) admits a selfadjoint extension.

As \( \Delta_b \) is symmetric its Cayley transform

\[
C_{\Delta_b} : \mathcal{R}(\Delta_b + iI) \subset L^2(\Omega) \rightarrow L^2(\Omega), \quad C_{\Delta_b} = (\Delta_b - iI) \circ (\Delta_b + iI)^{-1},
\]

(an isometry) is well defined. The defect indices of \( \Delta_b \) coincide with those of its Cayley transform (cf. Theorem 1.27 in [6], p. 83). Then (by (14) and Theorem 1.26 in [6], p. 82) \( C_{\Delta_b} \) admits a unitary (i.e. maximal isometric) extension \( \tilde{C}_{\Delta_b} \) and

\[
\tilde{\Delta}_b = i \left( I + \tilde{C}_{\Delta_b} \right) \circ \left( I - \tilde{C}_{\Delta_b} \right)^{-1}
\]
Elisabetta Barletta and Sorin Dragomir

is a maximal symmetric extension of $\Delta_b$.

Let $S^1 \to C(M) \xrightarrow{\pi} M$ be the canonical circle bundle over $M$ and $F_\theta$ the Fefferman metric associated to a contact form $\theta$ (a Lorentzian metric on $C(M)$, cf. Definition 2.15 in [7], p. 128). Let $\mu$ be the canonical measure on $C(M)$ associated to $F_\theta$. Let $\Box$ be the Laplace-Beltrami operator of $(C(M), F_\theta)$ (the wave operator). If $D \subseteq C(M)$ is an open set we regard the wave operator as a linear operator $\Box : D(\Box) \subset L^2(D, \mu) \to L^2(D, \mu)$ with domain $D(\Box) = C_0^\infty(D)$. As $S^1 \subset \text{Isom}(C(M), F_\theta)$ the wave operator is $S^1$-invariant (see also Proposition 2.7 in [7], p. 140). Consequently the pushforward of $\Box(\pi^* \Box)(u) = \Box(u \circ \pi)$, $u \in C^2(M)$,
is well defined. By a result of J.M. Lee, [12], $\pi^* \Box = \Delta_b$ on $C^2(M)$ (here we adopt the conventions $^3$ in [7]).

**Proposition 6.** Let $\Omega \subseteq M$ be a domain and $D = \pi^{-1}(\Omega)$. If $(\Delta_b)_0$ is the Lagrange sublaplacian of $(M, \theta)$ then $\text{Reg}(\Box) \subseteq \text{Reg}[(\Delta_b)_0]$.

**Proof:** If $U \subseteq M$ is an open set then $^4$

$$\int_{\pi^{-1}(U)} (f \circ \pi) \, d\mu = 2\pi \int_U f \, dv \tag{15}$$

for any function $f \in C^\infty(U)$ supported in $U$ (by integration along the fibre). Let $u \in C^\infty_0(\Omega)$. Then $\text{Supp}(u \circ \pi)$ is contained in $\pi^{-1}(\text{Supp}(u))$ (a compact set, as each fibre $\pi^{-1}(x) \approx S^1$ is compact). Therefore $u \circ \pi \in C^\infty(D)$. Let $\lambda \in \text{Reg}(\Box)$. Then there is $\gamma = \gamma(\lambda)$ such that (by (15))

$$\gamma \sqrt{\pi} \|u\|_{L^2(\Omega)} = \gamma \|u \circ \pi\|_{L^2(D)} \leq \|(\Box - \lambda I)(u \circ \pi)\|_{L^2(D)} =$$

$$= \sqrt{\pi} \|[(\Delta_b)_0 - \lambda I]u\|_{L^2(\Omega)}$$

hence $\lambda \in \text{Reg}[(\Delta_b)_0]$. \(\Box\)

**Remark 3.** The behavior of $\text{Reg}(\Delta_b)$ under a transformation $\hat{\theta} = e^f \theta$ is unknown.

### 7 Sublaplacians and sub-Riemannian geometry

Let $M$ be a strictly pseudoconvex CR manifold and $\theta$ a contact form on $M$ with the Levi form $G_\theta$ positive definite. As $H$ possesses the bracket generating property $(M, H, G_\theta)$ is a sub-Riemannian manifold (in the sense of R.S. Strichartz, [15]). Let $\{X_a : 1 \leq a \leq 2n\}$ be a local $\theta$-orthonormal frame of $H$ defined

---

$^3$Under the conventions in [12] $\pi_* \Box = \frac{1}{2} \Delta_b$.

$^4$The symbol $\pi$ in the right hand side of (15) is the irrational number $\pi \in \mathbb{R} \setminus \mathbb{Q}$. 
on the open set $U$. Let $T$ be the characteristic direction of $d\theta$. Let $(U, x^i)$ be a local coordinate system on $M$. We set $X_a = b_a^i \partial_i$ and $T = T^i \partial_i$ for some $b_a^i, T^i \in C^\infty(U)$, where $\partial_i = \partial/\partial x^i$. Let us consider the positive semi-definite matrix
\[ a_{ij} = \sum_{a=1}^{2n} b_a^i b_a^j \in C^\infty(U), \quad 1 \leq i, j \leq 2n + 1. \]

If $g_{ij} = g_\theta(\partial_i, \partial_j)$ and $[g^{ij}] = [g_{ij}]^{-1}$ then (cf. [7], p. 113)
\[ a_{ij} = g_{ij} - T^i T^j. \]

For each $\epsilon > 0$ let $g_\epsilon$ be the Riemannian metric on $M$ locally given by $g_\epsilon(X, Y) = g_\epsilon^{ij} (\partial_i, \partial_j)$ where
\[ [g_\epsilon^{ij}] = [g_\epsilon^{ij}]^{-1}, \quad g_\epsilon^{ij} = a^{ij} + \epsilon g^{ij}. \]

**Lemma 2.** The Riemannian metric $g_\epsilon$ and the Webster metric are related by
\[ g_\epsilon^{ij} = g^{ij} - T^i T^j. \]

**Proposition 7.** Let $\nabla^\epsilon$ be the Levi-Civita connection of $(M, g_\epsilon)$. Then
\[ \nabla^\epsilon_X Y = \nabla_X Y + \{ \Omega(X, Y) - \frac{\epsilon}{1 + \epsilon} A(X, Y) \} T, \]
\[ \nabla^\epsilon_X T = \tau(X) + \left( 1 + \frac{1}{\epsilon} \right) JX, \]
\[ \nabla^\epsilon_T X = \nabla_T X + \left( 1 + \frac{1}{\epsilon} \right) JX, \]
\[ \nabla^\epsilon_T T = 0, \]
for any $X, Y \in H$. Consequently the Laplace-Beltrami operator of $(M, g_\epsilon)$ is given by
\[ \Delta_\epsilon u = (1 + \epsilon) \Delta u - \epsilon T(T(u)), \]
for any $u \in C^2(M)$. 

*Sublaplacians on CR manifolds*
Here $\Omega = -d\theta$, $\tau$ is the pseudohermitian torsion of $\nabla$ (cf. [7], p. 36-37) and $A(X, Y) = g_\theta(\tau X, Y)$ for any $X, Y \in \mathfrak{X}^\infty(M)$. As a consequence of (23) the sublaplacian of $(M, \theta)$ is approximated by the second order elliptic operator $\Delta_\epsilon$.

**Remark 4.** If $M = \mathbb{H}_n$ then $\Delta_\epsilon = (1 + \epsilon)H - \epsilon \frac{\partial^2}{\partial t^2}$.

**Proof of Proposition 7.**

**Proof:** Using (16) and $\nabla^\epsilon g = \nabla g_\theta = 0$ one derives
\[
2g_\epsilon(\nabla_X Y, Z) - \frac{2}{\epsilon(1 + \epsilon)} X(\theta(Y))\theta(Z) = \frac{1}{1 + \epsilon} \{2g_\theta(\nabla_X Y, Z) - 2g_\epsilon(\nabla_X Y, Z) + g_\theta(\tau X, Y) + g_\theta(X, T_\theta Y) + g_\theta(T_\theta X, Y) - \theta(Z)(d\theta)(X, Y))\}
\]
for any $X, Y, Z \in \mathfrak{X}^\infty(M)$. Next the identity (24) and $T_\nabla Y = 2(\theta \wedge \tau - \Omega \otimes T)$ (cf. [7], p. 37) yield (19)-(22). Finally we set
\[
X_\epsilon^a = \sqrt{1 + \epsilon} X_a, \quad 1 \leq a \leq 2n, \quad X_{2n+1}^\epsilon = \sqrt{\epsilon} T,
\]
so that $\{X_\epsilon^i : 1 \leq i \leq 2n + 1\}$ is a local $g_\epsilon$-orthonormal frame of $T(M)$. Then
\[
\Delta_\epsilon u = - \sum_{i=1}^{2n+1} \{X_\epsilon^i(X_\epsilon^i u) - (\nabla_\epsilon^{X_\epsilon^i} X_\epsilon^i u)\}
\]
and (19)-(22) lead to (23).

Let $\text{div}_\epsilon$ be the divergence operator with respect to $g_\epsilon$ i.e. for any $C^1$ vector field $Y \in \mathfrak{X}(M)$
\[
\text{div}_\epsilon(Y) = \sum_{i=1}^{2n+1} g_\epsilon(\nabla^{X_\epsilon^i} Y, X_\epsilon^i) = (1 + \epsilon) \sum_{a=1}^{2n} g_\epsilon(\nabla^{X_a} Y, X_a) + \epsilon g_\epsilon(\nabla_\epsilon T, T)
\]
on $U$. Using the decomposition $Y = \Pi_H Y + \theta(Y)T$ and the formulae (19)-(22) one may easily show that
\[
\nabla^{X_\epsilon} Y = \nabla_X Y + \theta(Y)\{\tau X + \frac{1 + \epsilon}{\epsilon} JX\} + \\
+ \{\Omega(X, Y) - \frac{\epsilon}{1 + \epsilon} A(X, Y)\}T,
\]
\[
\nabla^{X_\epsilon} Y = \nabla_T Y + \frac{1 + \epsilon}{\epsilon} \phi Y,
\]
for any $X \in \Gamma^\infty(H)$. Here $J : H \to H$ was extended to a bundle endomorphism
\( \phi : T(M) \to T(M) \) by requesting that $\phi = J$ on $H$ and $\phi T = 0$. Consequently
(by (16))
\[
g_\epsilon(\nabla_X Y, X) = \frac{1}{1 + \epsilon} \{ g_\theta(\nabla_X Y, X) + \theta(Y)A(X, X) \},
\]
\[
g_\epsilon(\nabla_T Y, T) = \frac{1}{\epsilon} g_\theta(\nabla_T Y, T),
\]
so that (by trace\( C\theta(A) = 0 \))
\[
\text{div}_\epsilon(Y) = \text{div}(Y).
\] (25)

Our considerations require that we dissociate the underlying integration measure
from the domain metric (cf. e.g. [11], p. 4638). Precisely (by taking into account
(25)) we say that $u \in L^2(\Omega)$ admits a weak $\nabla^\epsilon$-gradient if there is
$X_\epsilon \in \mathfrak{X}(\Omega) \otimes \mathbb{C}$ such that $g_\theta(X_\epsilon, X_\epsilon)^{1/2} \in L^1_{\text{loc}}(\Omega)$ and
\[
\int g_\epsilon(X_\epsilon, Y) \, dv = - \int \Omega u \text{div}(Y) \, dv
\] (26)
for any $Y \in \mathfrak{X}^\infty_0(\Omega)$. The vector field $X_\epsilon$ is uniquely determined up to a set of
measure zero and we adopt the notation $\nabla^\epsilon u = X_\epsilon$. Let now $\mathcal{D}(\nabla^\epsilon)$ consist of all
$u \in L^2(\Omega)$ admitting a weak $\nabla^\epsilon$-gradient such that $\nabla^\epsilon u \in L^2(T(\Omega) \otimes \mathbb{C})$. Also
let $\mathcal{D}[(\nabla^\epsilon)^*]$ consist of all $X \in L^2(T(\Omega) \otimes \mathbb{C})$ such that
\[
\int \Omega g_\epsilon(\nabla^\epsilon u, X) \, dv = \int \Omega u \overline{u}X \, dv
\]
for some $u_X \in L^2(\Omega)$ and any $u \in \mathcal{D}(\nabla^\epsilon)$. The adjoint $(\nabla^\epsilon)^* : \mathcal{D}[(\nabla^\epsilon)^*] \subset
L^2(T(\Omega) \otimes \mathbb{C}) \to L^2(\Omega)$ is given by $(\nabla^\epsilon)^* X = u_X$. Finally we set $\mathcal{D}(\Delta_\epsilon) = \{ u \in
\mathcal{D}(\nabla^\epsilon) : \nabla^\epsilon u \in \mathcal{D}[(\nabla^\epsilon)^*] \}$. We shall use the operator $\Delta_\epsilon : \mathcal{D}(\Delta_\epsilon) \subset L^2(\Omega) \to
L^2(\Omega)$ given by $\Delta_\epsilon u = (\nabla^\epsilon)^* \phi \nabla^\epsilon$.

Let $u \in C^1(\Omega)$. For any $\varphi \in C^\infty_0(\Omega)$
\[
\int \Omega T(u)\varphi \, dv = \int \Omega \{ T(u\varphi) - uT(\varphi) \} \, dv = \int \Omega \{ \text{div}(u\varphi T) - u\varphi \text{div}(T) - uT(\varphi) \} \, dv = - \int \Omega uT(\varphi) \, dv
\]
by Green’s lemma and $\text{div}(T) = 0$ (as $T$ is $\nabla$-parallel). Therefore we may adopt
the following definition. A function $u \in L^2(\Omega)$ is weakly differentiable in the
$T$-direction if
\[
\int \Omega u_1 \varphi \, dv = - \int \Omega uT(\varphi) \, dv
\]
for some $u_1 \in L^1_{\text{loc}}(\Omega)$ and any $\varphi \in C^\infty_0(\Omega)$. Such $u_1$ is unique up to a set
of measure zero and we adopt the customary notation $u_1 = T(u)$. Next let
\(\mathcal{D}(T)\) consist of all \(u \in L^2(\Omega)\) such that \(u\) is weakly differentiable in the \(T\)-direction and \(T(u) \in L^2(\Omega)\). We may then regard \(T\) as a (densely defined) linear operator \(T : \mathcal{D}(T) \subset L^2(\Omega) \to L^2(\Omega)\). Let \(T^* : \mathcal{D}(T^*) \subset L^2(\Omega) \to L^2(\Omega)\) be the adjoint of \(T\). Then \(C_0^\infty(\Omega) \subseteq \mathcal{D}(T^*)\) and the restriction of \(T^*\) to \(C_0^\infty(\Omega)\) is \(-T\). We shall need the operator \(T^*T : \mathcal{D}(T^*T) \subset L^2(\Omega) \to L^2(\Omega)\) defined on \(\mathcal{D}(T^*T) = \{u \in \mathcal{D}(T) : T(u) \in \mathcal{D}(T^*)\}\).

**Remark 5.** We adopt the notations and conventions in [17], p. 113-116. Let \(\sigma_2(\Delta_c) \in \text{Smbl}_2(E, E)\) be the symbol of \(\Delta_c\), where \(E = \Omega \times \mathbb{C}\) is the trivial bundle over \(\Omega\). Let \(T'(\Omega) = T^*(\Omega) \setminus \{0\}\) and \(\pi : T'(\Omega) \to \Omega\) the projection. We wish to compute

\[
\sigma_2(\Delta_c)_\omega : (\pi^{-1}E)_\omega \to (\pi^{-1}E)_\omega, \quad \omega \in T'(\Omega).
\]

Let \(v \in E_x = (\pi^{-1}E)\), where \(x = \pi(\omega)\) and let us consider \(f \in C^\infty(\Omega)\) and \(s \in \Gamma^\infty(E)\) such that \((df)_x = \omega\) and \(s(x) = v\). Then

\[
\sigma_2(\Delta_c)_\omega(v) = \Delta_c \left[ \frac{1}{2} (f - f(x))^2 s \right](x) = \left[(1 + \epsilon)\|\omega\|^2 - \omega(T_x)^2\right] v
\]

so that \(\sigma_2(\Delta_c)_\omega : E_x \to E_x\) is a \(\mathbb{C}\)-linear isomorphism (accounting for the ellipticity of \(\Delta_c\)). As \(\epsilon \to 0\) this gives the formula for \(\sigma_2(\Delta_u)\) reported in the Introduction. In particular \(\sigma_2(\Delta_c)_{\theta_x}(v) = \epsilon v\).

**Remark 6.** The coefficients of \(\Delta_c\) (with respect to a local coordinate system \((U, x^i)\) on \(M\)) are jointly continuous in \(x \in U\) and \(\epsilon > 0\). Consequently (by Theorem 4.13 in [17], p. 142) the function \(\epsilon \mapsto \dim_{\mathbb{C}} \ker(\Delta_c)\) is upper semicontinuous. Also for each \(\epsilon_0 > 0\) there is \(\delta > 0\) such that \(\dim_{\mathbb{C}} \ker(\Delta_c) \leq \dim_{\mathbb{C}} \ker(\Delta_{c_0})\) for any \(|\epsilon - \epsilon_0| < \delta\).

The family of second order differential operators \(\{\Delta_c\}_{c>0}\) is said to be **uniformly \(K\)-positive definite** (in a neighborhood of the origin) if there exist \(\epsilon_0 > 0\), a preclosed operator \(K : \mathcal{D}(K) \subset L^2(\Omega) \to L^2(\Omega)\), and two constants \(\alpha > 0\) and \(\beta > 0\) such that for any \(0 < \epsilon \leq \epsilon_0\) the following properties are satisfied

1) \(\mathcal{D}(\Delta_c) \subseteq \mathcal{D}(K), \quad \overline{K(\mathcal{D}(\Delta_c))} = L^2(\Omega),\)

2) \((\Delta_c u, Ku)_{L^2(\Omega)} \geq \alpha^2 \|u\|^2_{L^2(\Omega)}\),

3) \(\|Ku\|^2_{L^2(\Omega)} \leq \beta^2 (\Delta_c u, Ku)_{L^2(\Omega)}\),

for any \(u \in \mathcal{D}(\Delta_c)\). As we shall shortly see if \(\{\Delta_c\}_{c>0}\) is uniformly \(K\)-positive definite then one may produce generalized solutions to the equation \(\Delta_c u = f\). Note that (together with W.V. Petryshyn, [13],[14]) for each \(\epsilon > 0\) and \(f \in L^2(\Omega)\) we may consider the functional \(\mathcal{F}_\epsilon : \mathcal{D}(\Delta_c) \to \mathbb{R}\) given by

\[
\mathcal{F}_\epsilon(u) = (\Delta_c u, Ku)_{L^2(\Omega)} - (f, Ku)_{L^2(\Omega)} - (Ku, f)_{L^2(\Omega)}.
\]
Then (by a result in [14]) the minimum points of $F_\epsilon$ are (under the assumptions (i)-(ii) above) solutions to $\Delta_\epsilon u = f$. In general however these may depend on $\epsilon$ (and one may not employ the identity (30) below to produce solutions to $\Delta_b u = f$). In turn we base our discussion on the functional $F$ (see (28) below) got as $F_\epsilon(u) \to F(u)$ for $\epsilon \to 0$.

We need to establish the following weak version of the identity (17) in Lemma 2.

**Lemma 3.** $D(\nabla^\epsilon) = D(\nabla^H) \cap D(T)$ and

$$\nabla^\epsilon u = (1 + \epsilon)\nabla^H u + \epsilon T(u)T$$

(27)

for any $u \in D(\nabla^\epsilon)$.

**Proof:** Let $u \in D(\nabla^\epsilon)$. Then (26) for $Y \in \Gamma^\infty_0(H)$ may be written

$$\frac{1}{1 + \epsilon} \int_{\Omega} g_\theta(\Pi_H \nabla^\epsilon u, Y) \, dv = - \int_{\Omega} u \text{div}(Y) \, dv$$

hence $u$ is weakly differentiable along $H$ and its weak horizontal gradient is given by $\nabla^H u = (1/(1 + \epsilon))\Pi_H \nabla^\epsilon u$. Moreover

$$\int_{\Omega} \|\nabla^H u\|^2 \, dv = \frac{1}{(1 + \epsilon)^2} \int_{\Omega} \|\Pi_H \nabla^\epsilon u\|^2 \, dv \leq$$

$$\leq \frac{1}{(1 + \epsilon)^2} \int_{\Omega} \|\nabla^\epsilon u\|^2 \, dv \leq \|\nabla^\epsilon u\|^2_{L^2(T(\Omega) \otimes C)} < \infty$$

hence $u \in D(\nabla^H)$. Similarly let $\varphi \in C^\infty_0(\Omega)$ and let us use (26) for $Y = \varphi T$ i.e.

$$\int_{\Omega} \varphi \theta(\nabla^\epsilon u, T) \, dv = - \int_{\Omega} u \text{div}(\varphi T) \, dv$$

or (by $\text{div}(\varphi T) = T(\varphi)$)

$$\frac{1}{\epsilon} \int_{\Omega} \varphi \theta(\nabla^\epsilon u) \, dv = - \int_{\Omega} u T(\varphi) \, dv$$

hence $u$ is weakly differentiable in the $T$-direction and its weak $T$-derivative is given by $T(u) = (1/\epsilon) \theta(\nabla^\epsilon u)$. Also

$$\int_{\Omega} |T(u)|^2 \, dv = \frac{1}{\epsilon^2} \int_{\Omega} |g_\theta(\nabla^\epsilon u, T)|^2 \, dv \leq$$

$$\leq \frac{1}{\epsilon^2} \int_{\Omega} \|\nabla^\epsilon u\|^2 \|T\|^2 \, dv = \frac{1}{\epsilon^2} \|\nabla^\epsilon u\|^2_{L^2(T(\Omega) \otimes C)}$$

hence $u \in D(T)$. Summing up the information got so far $D(\nabla^\epsilon) \subseteq D(\nabla^H) \cap D(T)$ and (17) weak holds for any $u \in D(\nabla^\epsilon)$. Viceversa let $u \in D(\nabla^H) \cap D(T)$. Let
Elisabetta Barletta and Sorin Dragomir

\( Y \in \mathfrak{X}_{\infty}^\infty(\Omega) \) be an arbitrary test vector field decomposed as \( Y = Y_H + \varphi T \) where \( Y_H = \Pi_H Y \) and \( \varphi = \theta(Y) \). Clearly \( Y_H \in \Gamma_{0}^\infty(H_\Omega) \) and \( \varphi \in C_{0}^\infty(\Omega) \). Then

\[
\int_\Omega u \text{div}(Y) \, dv = \int_\Omega u \{\text{div}(Y_H) + T(\varphi)\} \, dv =
\]

\[
= - \int_\Omega g_\theta(\nabla^H u, Y_H) \, dv - \int_\Omega T(u)\varphi \, dv =
\]

\[
= -(1 + \epsilon) \int_\Omega g_\epsilon(\nabla^H u, Y) \, dv - \epsilon \int_\Omega T(u)g_\epsilon(T,Y)
\]

hence \( u \) admits a weak \( \nabla^\epsilon \)-gradient and \( \nabla^\epsilon u = (1 + \epsilon)\nabla^H u + \epsilon T(u)T \). Finally

\[
\frac{1}{2} \int_\Omega \|\nabla^\epsilon u\|^2 \, dv \leq \int_\Omega \{(1 + \epsilon)^2\|\nabla^H u\|^2 + \epsilon^2|T(u)|^2\|T\|^2\} \, dv =
\]

\[
= (1 + \epsilon)^2\|\nabla^H u\|^2_{L^2(H_\Omega \otimes \mathbb{C})} + \epsilon^2\|T(u)\|^2_{L^2(\Omega)} < \infty
\]

hence \( u \in D(\nabla^\epsilon) \).

8 Generalized solutions to \( \Delta_b u = f \).

Let \((M, \theta)\) be a strictly pseudoconvex CR manifold and \( \Omega \subset M \) a domain. Let \( \theta \) be a contact for on \( M \) and \( \Delta_b \) the sublaplacian of \((M, \theta)\). Let \( K : D(K) \subset L^2(\Omega) \to L^2(\Omega) \) be a preclosed operator such that \( D(\Delta_b) \subseteq D(K) \). Let \( f \in L^2(\Omega) \) and let \( \mathcal{F} \) be the functional given by

\[
\mathcal{F}(u) = \int_\Omega (\Delta_b u)\overline{K}u \, dv - 2\text{Re} \int_\Omega f \overline{K}u \, dv, \quad u \in D(\Delta_b).
\]

(28)

We shall establish the following

**Theorem 2.** If the family \( \{\Delta_\epsilon\}_{\epsilon > 0} \) of Laplace-Beltrami operators of the Riemannian metrics \( \{g_\epsilon\}_{\epsilon > 0} \) is uniformly \( K \)-positive definite then any solution \( u_0 \in S = D(\Delta_b) \cap D(T^*T) \) to the equation

\[
\Delta_b u = f
\]

(29)
is a minimum point of the functional \( \mathcal{F} : S \to \mathbb{R} \). Viceversa if i) \( K(S) \) is dense in \( L^2(\Omega) \) and ii) \( u_0 \in S \) is a minimum point of \( \mathcal{F} : S \to \mathbb{R} \) then \( u_0 \) satisfies (29).

The proof of Theorem 2 relies on the weak version of the identity (23) in Proposition 7 above i.e.
Lemma 4. For each $\epsilon > 0$ one has $\mathcal{D}(\Delta_b) \cap \mathcal{D}(T^*T) \subseteq \mathcal{D}(\Delta_\epsilon)$ and the following identity holds

$$\Delta_\epsilon f = (1 + \epsilon)\Delta_b f + \epsilon T^*[T(f)]$$

(30)

for any $f \in \mathcal{D}(\Delta_b) \cap \mathcal{D}(T^*T)$.

Proof: We set $S = \mathcal{D}(\Delta_b) \cap \mathcal{D}(T^*T)$ for simplicity. If $f \in S$ then $f \in \mathcal{D}(\nabla^H) \cap \mathcal{D}(T)$ i.e. $f \in \mathcal{D}(\nabla^\epsilon)$ and

$$\nabla^H f \in \mathcal{D}[(\nabla^H)^*], \quad T(f) \in \mathcal{D}(T^*).$$

Then for each $u \in \mathcal{D}(\nabla^\epsilon)$ (by (27))

$$\int_\Omega g_\epsilon(\nabla^\epsilon u, \nabla^\epsilon f) \, dv =$$

$$= (1 + \epsilon) \int_\Omega g_\theta(\nabla^H u, \nabla^H f) \, dv + \epsilon \int_\Omega T(u)T(f) \, dv =$$

$$= (1 + \epsilon) \int_\Omega u \, \Delta_b f \, dv + \epsilon \int_\Omega u \, T^*(T(f)) \, dv$$

and $(1 + \epsilon)\Delta_b f + \epsilon T^*(T(f)) \in L^2(\Omega)$ hence $\nabla^\epsilon f \in \mathcal{D}[(\nabla^\epsilon)^*]$ and (30) holds good.

Now let us assume that the family $\{\Delta_\epsilon\}_{\epsilon > 0}$ is uniformly $K$-positive definite for some preclosed operator $K : \mathcal{D}(K) \subseteq L^2(\Omega) \rightarrow L^2(\Omega)$. Then $S \subseteq \mathcal{D}(K)$ and there is a constant $\alpha > 0$ such that for each $u \in S$ (by (30))

$$\alpha^2 \|u\|^2_{L^2(\Omega)} \leq (\Delta_\epsilon u, Ku)_{L^2(\Omega)} =$$

$$(1 + \epsilon) (\Delta_b u, Ku)_{L^2(\Omega)} + \epsilon (T^*Tu, Ku)_{L^2(\Omega)}$$

and then for $\epsilon \rightarrow 0$

$$\alpha^2 \|u\|^2_{L^2(\Omega)} \leq (\Delta_b u, Ku)_{L^2(\Omega)}.$$

(31)

Similarly

$$\|Ku\|^2_{L^2(\Omega)} \leq \beta^2 (\Delta_b u, Ku)_{L^2(\Omega)},$$

(32)

for any $u \in S$.

To prove Theorem 2 let us observe first that the sublaplacian $\Delta_b$ is $K$-symmetric on $S$ i.e.

$$(\Delta_b f, Kg)_{L^2(\Omega)} = (Kf, \Delta_b g)_{L^2(\Omega)}, \quad f, g \in S.$$

Indeed

$$(\Delta_b (f + g), K(f + g))_{L^2(\Omega)} = (\Delta_b f - g, K(f - g))_{L^2(\Omega)} +$$
\[ + i \{ (\Delta_b (f + ig), K(f + ig))_{L^2(\Omega)} - (\Delta_b (f - ig), K(f - ig))_{L^2(\Omega)} \} = 4 (\Delta_b f, Kg)_{L^2(\Omega)}. \]

At this point one may interchange \( f \) and \( g \) to get another identity of the sort and exploit \( (\Delta_b u, Ku)_{L^2(\Omega)} \in \mathbb{R} \) for any \( u \in S \) (a consequence of (31)) to conclude.

Next let \( u_0 \in S \) such that \( \Delta_b u_0 = f \). Then

\[ (\Delta_b (u - u_0), K(u - u_0))_{L^2(\Omega)} - (\Delta_b u_0, Ku_0)_{L^2(\Omega)} = \mathcal{F}(u) \]

so that (again by (31))

\[ \mathcal{F}(u_0) = - (\Delta_b u_0, Ku_0)_{L^2(\Omega)} = \min_{u \in S} \mathcal{F}(u). \]

Vice versa let \( u_0 \) be a minimum point of the functional \( \mathcal{F} : S \to \mathbb{R} \). As \( \Delta_b \) is \( K \)-symmetric on \( S \)

\[ \mathcal{F}(u_0 + tu) = \mathcal{F}(u_0) + 2t \text{Re} (\Delta_b u_0 - f, Ku) + O(t^2) \]

hence

\[ \text{Re} \int_{\Omega} (\Delta_b u_0 - f) \overline{Ku} \, dv = 0 \]

for any \( u \in S \). Let us replace \( u \) by \( i u \) so that to obtain

\[ \text{Im} \int_{\Omega} (\Delta_b u_0 - f) \overline{Ku} \, dv = 0. \]

Therefore \( \Delta_b u_0 - f \) is orthogonal on \( K(S) \) and \( \overline{K(S)} = L^2(\Omega) \). Theorem 2 is proved.

**Theorem 3.** Under the assumptions of Theorem 2 one has

i) The functional \( \mathcal{F} \) is bounded from below on \( S \).

ii) For any \( g, h \in S \) and any \( 0 \leq t \leq 1 \)

\[ t \mathcal{F}(g) + (1 - t) \mathcal{F}(h) - \mathcal{F}[tg + (1 - t)h] \geq \alpha^2 t (1 - t) \| g - h \|^2_{L^2(\Omega)}. \] (33)

iii) All minimizing sequences of \( \mathcal{F} : S \to \mathbb{R} \) converge in \( L^2(\Omega) \) and have the same limit.

iv) Let \( u \) be a generalized solution to \( \Delta_b u = f \). If \( u \in D(\Delta_b) \cap D(T^* T) \) then \( u \) is a classical solution.

**Proof:** The proof of (i)-(iii) in Theorem 3 is similar to that of Theorem 4.3 in [6], p. 247. Statement (iv) is an analog of Proposition 4.3 in [6], p. 248, and follows from Theorem 2 above (while (i)-(iii) are independent statements). Let \( u \in S \). Then (by (32) and the Cauchy-Schwartz inequality)

\[ \mathcal{F}(u) = (\Delta_b u, Ku)_{L^2(\Omega)} - 2 \text{Re} (Ku, f)_{L^2(\Omega)} \geq \]
\[ \geq \frac{1}{\beta^2} \| Ku \|_{L^2(\Omega)}^2 - 2\| Ku \|_{L^2(\Omega)} \| f \|_{L^2(\Omega)} \geq -\beta^2 \| f \|_{L^2(\Omega)}. \]

Statement (ii) follows (by (31)) from

\[ t\mathcal{F}(g) + (1-t)\mathcal{F}(h) - \mathcal{F}[tg + (1-t)h] = \\
= t(1-t) (\Delta_b (g - h), K (g - h))_{L^2(\Omega)} \geq \\
\geq \alpha^2 t(1-t) \| g - h \|_{L^2(\Omega)}^2. \]

We recall that \( \{u_\nu\}_{\nu \geq 1} \subset S \) is a minimizing sequence for \( \mathcal{F} : S \to \mathbb{R} \) if \( \mathcal{F}(u_\nu) \to a = \inf_{u \in S} \mathcal{F}(u) \) as \( \nu \to \infty \). Let \( t = 1/2 \) and \( g = u_\mu, h = u_\nu \) in (33) so that to obtain

\[ \frac{\alpha^2}{2} \| u_\mu - u_\nu \|_{L^2(\Omega)}^2 \leq \mathcal{F}(u_\mu) + \mathcal{F}(u_\nu) - 2 \mathcal{F}
\]

\[ \leq \mathcal{F}(u_\mu) + \mathcal{F}(u_\nu) - 2a \]

hence \( \{u_\nu\}_{\nu \geq 1} \) is a Cauchy sequence in \( L^2(\Omega) \). Given two minimizing sequences \( \{g_\nu\}_{\nu \geq 1} \) and \( \{h_\nu\}_{\nu \geq 1} \) with the limits \( g_0, h_0 \in L^2(\Omega) \) we set \( t = 1/2 \) and \( g = g_\nu, h = h_\nu \) in (33) so that to obtain

\[ \frac{\alpha^2}{2} \| g_\nu - h_\nu \|_{L^2(\Omega)}^2 \leq \mathcal{F}(g_\nu) + \mathcal{F}(h_\nu) - 2a \to 0, \quad \nu \to \infty, \]

where from \( g_0 = h_0 \).

It remains that we prove statement (iv) in Theorem 3. We recall that a generalized solution to \( \Delta_b u = f \) is the limit in \( L^2(\Omega) \) of a minimizing sequence for the functional \( \mathcal{F} : S \to \mathbb{R} \).

The functional \( \mathcal{F} : S \to \mathbb{R} \) is \( G \)-differentiable and its Gateaux differential is given by

\[ (\mathcal{V}\mathcal{F})(u)h = 2 \text{Re} \int_{\Omega} (\Delta_b u - f) K(h) dv \]

for any \( u, h \in S \). By the mean value formula (cf. e.g. Theorem 7.7 in [6], p. 322) for any \( u, v \in S \) there is \( 0 < \tau < 1 \) such that

\[ \mathcal{F}(v) - \mathcal{F}(u) = (\mathcal{V}\mathcal{F})[u + \tau(v-u)](v-u) = \\
= 2 \text{Re} (\Delta_b[u + \tau(v-u)] - f, K(v-u))_{L^2(\Omega)} \geq \\
\geq 2 \text{Re} (\Delta_b u - f, K(v-u))_{L^2(\Omega)} + \frac{2\tau}{\beta^2} \| K(v-u) \|_{L^2(\Omega)}^2 \]

as a consequence of (32). We conclude that

\[ \mathcal{F}(v) - \mathcal{F}(u) \geq 2 \text{Re} (\Delta_b u - f, K(v-u))_{L^2(\Omega)} \]

(34)

for any \( u, v \in S \). An analog of the inequality (34) is claimed in [6], p. 248, as a mere consequence of the convexity of \( \mathcal{F} \) (the proof depends however on (32)). Let
\{u_\nu\}_{\nu \geq 1} be a minimizing sequence for \(F\) on \(S\). Then \(\{K(u_\nu)\}_{\nu \geq 1}\) is convergent in \(L^2(\Omega)\). Indeed for any \(0 \leq t \leq 1\) and any \(g, h \in S\) (by (32))

\[
tF(g) + (1-t)f(h) - F \left[ (1-t)h \right] = t(1-t) \left( \Delta_b (g-h), K(g-h) \right)_{L^2(\Omega)} \geq \frac{1}{\beta^2} t(1-t) \|K(g-h)\|_{L^2(\Omega)}^2.
\]

Let us set \(t = 1/2\) and \(g = u_\nu, h = u_\mu\) so that to obtain

\[
\frac{1}{2\beta^2} \|K(u_\nu - u_\mu)\|_{L^2(\Omega)}^2 \leq F(u_\nu) + 2F\left(\frac{u_\nu + u_\mu}{2}\right) \leq F(u_\nu) + F(u_\mu) - 2a
\]

hence \(\{K(u_\nu)\}_{\nu \geq 1}\) is a Cauchy sequence in \(L^2(\Omega)\). Let now \(u_0\) be a generalized solution to \(\Delta_b u = f\) belonging to \(S\). Let \(\{u_\nu\}_{\nu \geq 1}\) be a minimizing sequence for \(F\) on \(S\). Then (by part (iii) in Theorem 3) \(u_\nu \to u_0\) in \(L^2(\Omega)\). On the other hand if \(g = \lim_{\nu \to \infty} K(u_\nu) \in L^2(\Omega)\) then \(K(u_\nu - u_0) \to g - K(u_0)\) as \(\nu \to \infty\). As \(K\) is a preclosed operator it must be that \(g - K(u_0) = 0\) hence \(K(u_\nu - u_0) \to 0\) as \(\nu \to \infty\). Going back to (34) (for \(v = u_\nu\) and \(u = u_0\))

\[
F(u_\nu) - F(u_0) \geq 2 \Re (\Delta_b u_0 - f, K(u_\nu - u_0))
\]

hence for \(\nu \to \infty\) we get \(a \geq F(u_0)\). On the other hand (as \(u_0 \in S\)) \(a \leq F(u_0)\). We conclude that \(u_0\) is a minimum point for \(S\) and then (by Theorem 2) \(\Delta_b u_0 = f\).

Let \(K = X + Y\) where \(\{X, Y\}\) are given by (13). Let \(\Omega \subset \mathbb{H}_1\) be a bounded domain with \(C^2\) boundary. We close this section with an example of an operator \(B_p : D_{xy}(B_p) \subset L^2(\Omega) \to L^2(\Omega)\) arising on \(\mathbb{H}_1\) and satisfying the inequalities

\[
(B_p u, K u)_{L^2(\Omega)} \geq \alpha \|u\|_{L^2(\Omega)}^2, \quad \|K u\|_{L^2(\Omega)}^2 \leq \beta^2 (B_p u, K u)_{L^2(\Omega)},
\]

for some constants \(\alpha > 0, \beta > 0\), and any \(u \in D_{xy}(B_p)\). Let \(p \in \mathbb{Z}, p \geq 2\), and let us set

\[
B_p u = \sum_{i=0}^{p-1} \sum_{j=0}^{p} (-1)^{p-1-i} X^{p-1-i} Y^i (a_{ij} Y^j X^{p-j} u) + a(Xu + Yu),
\]

where \(a \in C^0(\Omega), a \geq 0\), and \(a_{ij} \in C^{p-1}(\Omega)\) for \(1 \leq i \leq p - 1\) and \(0 \leq j \leq p\). Let \(\mathcal{D}(B_2)\) consist of all \(u \in C^3(\Omega)\) such that \(Xu = Y u = 0\) on \(\partial \Omega\) while if \(p \geq 3\) the domain \(\mathcal{D}(B_p)\) of \(B_p\) consists of all \(u \in C^{2p-1}(\Omega)\) such that

\[X^{k+1} u = 0, \quad X^k Y u = 0, \quad 0 \leq k \leq p - 2 - i, \quad 0 \leq i \leq p - 2,\]
Sublaplacians on CR manifolds

\[
Y^\ell X u = 0, \quad Y^{\ell+1} u = 0, \quad 0 \leq \ell \leq p - 2,
\]
\[
Y^\ell X^{p-i} u = 0, \quad Y^\ell X^{p-1-i} Y u = 0, \quad 0 \leq \ell \leq i - 1, \quad 1 \leq i \leq p - 2,
\]
on \partial \Omega. We shall establish

**Theorem 4.** If \( D(K) = W^{1, 2}_{(X, Y)}(\Omega) \) then \( K : D(K) \subset L^2(\Omega) \to L^2(\Omega) \) is a preclosed operator. Let us set

\[
b_{ij} = \begin{cases} 
a_{0j}, & \text{if } i = 0, \\
a_{i-1,j} + a_{ij}, & \text{if } 1 \leq i \leq p - 1, \\
a_{p-1,j}, & \text{if } i = p,
\end{cases}
\]

and assume that

\[
\sum_{i,j=0}^p b_{ij} \xi_i \xi_j \geq C^2 \left( \sum_{i=0}^p \binom{p}{i} \xi_i \right)^2 \quad \text{on } \Omega \tag{37}
\]

for some constant \( C > 0 \) and any \((\xi_0, \ldots, \xi_p) \in \mathbb{R}^{p+1}\). Let \( D \subset C \) be a bounded domain with \( C^2 \) boundary and \( \Omega = D \times (0, c), \ c > 0 \). Then the inequalities (35) hold for any \( u \in D_{xy}(B_p) = D(B_p) \cap C^{2p-1}(\partial D) \).

If \( a_{ij} \in C^{p-1}(D) \) the restriction of \( B_p \) to \( D_{xy}(B_p) \) is a weakly elliptic differential operator (in the sense of [6], p. 241). It should be observed that the weak ellipticity requirement (37) is similar to the \( X \)-ellipticity condition in [9] (i.e. one requires that \( \sum_{i,j} b_{ij} \xi_i \xi_j \geq C^2 (X, \xi) \) with \( X = \binom{p}{i} \partial / \partial x^i \)).

**Lemma 5.** Let \( L \in \{X, Y\} \). If \( L^i g = 0 \) on \( \partial \Omega \) for any \( 0 \leq i \leq m - 1 \) then

\[
\int_\Omega L^m(f)g\, dv = (-1)^m \int_\Omega f L^m(g)\, dv.
\]

Let us take the \( L^2 \) inner product of (36) with \( Ku, \ u \in D(B_p) \), and take into account Lemma 5 in order to integrate by parts. We obtain

\[
(B_p u, \ Ku)_{L^2(\Omega)} = \int_\Omega a(Ku)^2\, dv + \\
+ \sum_{j=0}^p \int_\Omega (Y^j X^{p-j} u) \sum_{i=0}^{p-1} a_{ij} (Y^i X^{p-1-i} u + Y^i X^{p-1-i} Y u)\, dv.
\]

On the other hand \([X, Y] = -4T\) and \([X, T] = [Y, T] = 0\) (where \( T = \partial / \partial t \)) hence

\[
X^m Y = Y X^m - 4m X^{m-1} T, \quad m \geq 1.
\]
Consequently
\[
(B_p u, Ku)_{L^2(\Omega)} = \sum_{i,j=0}^{p} \int_{\Omega} b_{ij} (Y^i X^{p-i} u) (Y^j X^{p-j} u) \, dv + \quad (38)
\]
\[
+ \int_{\Omega} a (Ku)^2 \, dv - 4 \sum_{j=0}^{p-2} \sum_{i=0}^{p-2} (p-i-1) \int_{\Omega} a_{ij} (Y^i X^{p-i} u) Y^j X^{p-j-2} Tu \, dv.
\]
If \( u \in \mathcal{D}_{xy}(B_p) \) then (by (37))
\[
(B_p u, Ku)_{L^2(\Omega)} \geq C^2 \int_{\Omega} (K^p u)^2 \, dv.
\]
It should be observed that the restriction of \( K \) to \( \mathcal{D}_{xy}(B_p) \) is the first order operator \( \delta = \partial/\partial x + \partial/\partial y \) in [6], p. 242. Then (by Proposition 4.1 in [6], p. 239)
\[
\int_{\Omega} (Ku)^2 \, dv \geq c \gamma^2 \int_{D} u^2 \, dx \, dy
\]
for some constant \( \gamma > 0 \). An iterative argument now leads to (35). Theorem 4 is proved.

The choice \( p = 2 \) and \( a_{00} = a_{10} = a_{02} = a_{12} = 1, \ a_{01} = a_{11} = 0 \) leads to the third order operator \( B_2 u = K \Delta_b u + aKu \). However \( B_2 \) is not weakly elliptic and integration by parts as in the proof of Theorem 4 gives but \( (B_2 u, Ku)_{L^2(\Omega)} = \int_{\Omega} a(Ku)^2 \, dv - (\Delta_b u, K^2 u)_{L^2(\Omega)} \) for any \( u \in \mathcal{D}(B_2) \).

References


Received: 12.11.2008.