\textbf{D-stable $C^*$-algebras, the ideal property and real rank zero}

by

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Abstract

Let $D$ be a strongly self-absorbing, $K_1$-injective $C^*$-algebra (e.g., the Jiang-Su algebra $Z$ and $O_\infty$). We characterize, in particular, when $A \otimes D$ has the ideal property, where $A$ is a separable, purely infinite $C^*$-algebra.

Answering a natural question, we prove that there is a separable, nuclear $C^*$-algebra $B$ such that $\text{RR}(B) = \text{RR}(B \otimes Z) = \text{sr}(B) = \text{sr}(B \otimes Z) = 1$ and $\text{Prim}(B)$ has two elements (in particular, $\text{Prim}(B)$ has a basis consisting of compact-open sets) but $B \otimes Z$ does not have the ideal property. We also study some (permanence) properties of large classes of separable, $D$-stable $C^*$-algebras with the ideal property. For "many" separable $C^*$-algebras $C$ we characterize when $\text{RR}(C \otimes Z) = 0$.

Key Words: $C^*$-algebra, minimal tensor product of $C^*$-algebras, ideal property, strongly self-absorbing, the Jiang-Su algebra, weakly purely infinite, purely infinite, strongly purely infinite, exact $C^*$-algebra, nuclear $C^*$-algebra, primitive ideal spectrum, real rank, stable rank, $K$-theory groups, $K_1$-injective, $D$-stable, $C(X)$-algebra, $K_0$-liftable.

\textbf{2000 Mathematics Subject Classification:} Primary 46L05, Secondary 46L06.

1 Introduction

A unital and separable $C^*$-algebra $D \not\cong \mathbb{C}$ is \textit{strongly self-absorbing} if there is a *-isomorphism $D \cong D \otimes D$ which is approximately unitarily equivalent to the inclusion map $D \to D \otimes D$, $d \mapsto d \otimes 1_D$ ([37]). It is known that strongly self-absorbing $C^*$-algebras are simple and nuclear; moreover, they are either purely infinite or stably finite. The only known examples of strongly self-absorbing $C^*$-algebras are the UHF algebras of infinite type (i.e., every prime number that occurs in the respective supernatural number occurs with infinite multiplicity), the Cuntz algebras $O_2$ and $O_\infty$, the Jiang-Su algebra $Z$ ([19]) and the tensor
products of $O_{\infty}$ with UHF algebras of infinite type (see [37]). All these examples are $K_1$-injective, where a unital $C^*$-algebra $A$ is said to be $K_1$-injective if the canonical homomorphism $U(A)/U_0(A) \to K_1(A)$ is injective. For a strongly self-absorbing $C^*$-algebra $D$, we say that a second $C^*$-algebra $A$ is $D$-stable if $A \otimes D$ is *-isomorphic to $A$. Recently, a lot of interesting results about $D$-stable $C^*$-algebras have been proved by several authors (see, e.g., the excellent survey paper [11]).

Elliott’s classification program for separable, nuclear $C^*$-algebras by discrete invariants including the $K$-theory is one of the most successful research directions in Operator Algebras ([9]; see also [34]). While it is clear that not all the separable, nuclear $C^*$-algebras can be classified, very large classes of $C^*$-algebras are known to be classifiable (see e.g. [34]). The Jiang-Su algebra $Z$ is a unital, separable, simple, stably finite, projectionless, nuclear, infinite dimensional $C^*$-algebra with the same Elliott invariant as the complex numbers, $C$ (in particular, $K_0(Z) = Z$, $K_1(Z) = 0$ and $Z$ has a unique tracial state) ([19]). It became clear that $Z$-stability is an important regularity property for nuclear $C^*$-algebras ([11]). Jiang and Su showed in [19] that $Z \cong Z \otimes Z \cong \bigotimes_{n=1}^{\infty} Z$ and that $A$ is $Z$-stable if $A$ is a unital, simple, infinite dimensional $AF$ algebra or if $A$ is a Kirchberg algebra. Toms and Winter proved in [38] that the separable, approximately divisible $C^*$-algebras are $Z$-stable, generalizing, in particular, the latter result. Based on a result of Gong, Jiang and Su in [16], it follows that the Elliott invariant of an arbitrary simple, unital $C^*$-algebra $A$ and of $A \otimes Z$ are isomorphic if $K_0(A)$ is weakly unperforated. "Most" of the classifiable $C^*$-algebras are known to be $Z$-stable. On the other hand, there are recent strong results by Winter (and other authors) showing that a $C^*$-algebra is classifiable if it is $Z$-stable and satisfies also some extra conditions. All known counterexamples to Elliott’s classification conjecture fail to be $Z$-stable.

The real rank (denoted by $RR(.)$) is an invariant for $C^*$-algebras that can be seen as a non-commutative notion of dimension and it was introduced by Brown and Pedersen in [4]. A $C^*$-algebra $A$ is said to have real rank zero (written $RR(A) = 0$) if its unitization $\tilde{A} := A + C \cdot 1$ has the property that the set of all invertible, selfadjoint elements of $\tilde{A}$ is dense in the set of all selfadjoint elements of $A$ ([4]). Note that many interesting $C^*$-algebras have real rank zero and that Elliott’s conjecture has been verified for large classes of $C^*$-algebras of real rank zero (see, e.g., [34]).

A $C^*$-algebra $A$ is said to have the ideal property if each of its ideals is generated (as an ideal) by its projections (in this paper, by an ideal we mean a closed, two-sided ideal). Every simple $C^*$-algebra with an approximate unit of projections and every $C^*$-algebra of real rank zero has the ideal property. The ideal property has been studied extensively by us (alone or in collaboration), for example in [28-32] and [14-15]. The ideal property is important in Elliott’s classification program. In [36], K. Stevens classified by a $K$-theoretical invariant a certain class of (non-simple) $AF$ algebras with the ideal property; this result was recently generalized in [18]. In [28] we classified the $AH$ algebras with the ideal property and
with slow dimension growth up to a shape equivalence and we gave several characterizations of when an arbitrary $AH$ algebra has the ideal property. In [14-15], jointly with Gong, Jiang and Li, we proved a reduction theorem saying that every $AH$ algebra with the ideal property and with the dimensions of the local spectra uniformly bounded can be written as an $AH$ algebra with the ideal property with (special) local spectra of dimensions $\leq 3$. This result generalizes similar and strong reduction theorems for real rank zero $AH$ algebras proved by Dadarlat ([7]) and Gong ([12])-and also for simple $AH$ algebras proved by Gong ([13])—which have been major steps in the classification of the corresponding classes of $AH$ algebras. Also, in [28-29], we proved several nonstable $K$-theoretical results for a large class of $C^*$-algebras with the ideal property. Indeed, if $A$ is an $AH$ algebra with the ideal property and with slow dimension growth, we proved in [28] that $A$ has stable rank one ($\text{sr}(A) = 1$) (that means, in the unital case, that the set of the invertible elements in $A$ is dense in $A$ (see [44] for more on the stable rank)), that $K_0(A)$ is weakly unperforated in the sense of Elliott and is also a Riesz group ([28-29]) and that the strict comparability of the projections in $A$ is determined by the tracial states of $A$, when $A$ is unital ([28]). Also, jointly with Rørdam, we proved in [31] that the ideal property is not preserved by taking minimal tensor products (even in the separable case). The purely infinite $C^*$-algebras have been introduced by Kirchberg and Rørdam in [25], extending the definition in the simple case given by Cuntz [6]. A $C^*$-algebra $A$ is said to be purely infinite if $A$ has no characters (or, equivalently, no non-zero abelian quotients) and if for every $a, b \in A^+$ such that $a$ belongs to the ideal of $A$ generated by $b$, it follows that there is a sequence $(x_n)$ of elements in $A$ such that $\|a - x_n^*bx_n\| \to 0$ as $n \to \infty$ ([25]). The study of purely infinite $C^*$-algebras was motivated by Kirchberg’s classification of the separable, nuclear, $\mathcal{O}_\infty$-stable $C^*$-algebras up to stable isomorphism by an ideal related $KK$-theory. Kirchberg and Rørdam introduced in [26] two more important notions of being purely infinite: weakly purely infinite and strongly purely infinite (we refer to [26] for their definitions). It was observed in [26] that in general we have: strongly purely infinite $\Rightarrow$ purely infinite $\Rightarrow$ weakly purely infinite. Jointly with Rørdam, we characterized when a separable, purely infinite $C^*$-algebra has real rank zero and also when it has the ideal property ([32]).

If $B$ is an arbitrary $C^*$-algebra, its primitive ideal spectrum, denoted $\text{Prim}(B)$, is the set of all the primitive ideals in $B$ (i.e., the kernels of the irreducible representations) equipped with the Jacobson topology. In Section 2 we obtain, in particular, several characterizations of the ideal property for $A \otimes \mathcal{D}$, where $A$ is a separable, purely infinite $C^*$-algebra and $\mathcal{D}$ is a separable, unital, strongly self-absorbing $C^*$-algebra (Corollary 2.5) (for a more general result see Proposition 2.1). One of these characterizations is that $\text{Prim}(A)$ has a basis consisting of compact-open sets. Answering a natural question suggested by this result (Question 2.7), we prove that there is a separable, nuclear $C^*$-algebra $B$ such that $\text{RR}(B) = \text{RR}(B \otimes \mathbb{Z}) = \text{sr}(B) = \text{sr}(B \otimes \mathbb{Z}) = 1$ and $\text{Prim}(B)$ has two elements (in particular, $\text{Prim}(B)$ has a basis consisting of compact-open sets) but $B \otimes \mathbb{Z}$
Cornel Pasnicu does not have the ideal property (Theorem 2.9).

In Section 3 we study some (permanence) properties of large classes of separable, $D$-stable $C^*$-algebras with the ideal property, where $D$ is a strongly self-absorbing, $K_1$-injective $C^*$-algebra. Let $C(D)$ be the class of all the separable, purely infinite $C^*$-algebras with the ideal property that are $D$-stable (Definition 3.1). We prove that $C(D)$ is stable under taking hereditary sub-$C^*$-algebras (in particular ideals), stable isomorphisms, quotients, countable inductive limits and extensions but not under taking minimal tensor products (see Proposition 3.2).

Also, we show that if $A$ is a $C(X)$-algebra, where $X$ is a compact, Hausdorff topological space, and if $A$ locally belongs to $C(D)$, then $A$ belongs to $C(D)$ (see Proposition 3.2). If $C_{nuc}(D) := C(D) \bigcap N$, where $N$ denotes the class of all the nuclear $C^*$-algebras (see Definition 3.6), then $C_{nuc}(D)$ has all the above mentioned properties of $C(D)$, except that $C_{nuc}(D)$ is stable under taking tensor products (see Proposition 3.7). If $C$ is the class of all the separable, purely infinite, nuclear $C^*$-algebras with the ideal property (see Definition 3.6), then we also show that $C = C_{nuc}(O_\infty) = C_{nuc}(Z)$ and that $C_{nuc}(D) = C \otimes D$, where $C \otimes D := \{A \otimes D : A \in C\}$ (see Proposition 3.7).

In Section 4, we prove that for a separable $C^*$-algebra $A$, $RR(A \otimes Z) = 0 \Rightarrow RR(A \otimes O_\infty) = 0$ (Proposition 4.1) and we observe that $RR(A \otimes O_\infty) = 0 \Rightarrow RR(A \otimes Z) = 0$ (Remark 4.3). However, we prove that for every separable, weakly purely infinite $C^*$-algebra $A$, the following four properties are equivalent: (1) $RR(A \otimes Z) = 0$; (2) $RR(A \otimes O_\infty) = 0$; (3) $Prim(A)$ has a basis consisting of compact-open sets and $A$ is $K_0$-liftable; (4) $A \otimes Z$ has the ideal property and $A$ is $K_0$-liftable (Theorem 4.4). Recall that a $C^*$-algebra $B$ is said to be $K_0$-liftable if for every pair of ideals $I \subseteq J$ in $B$ the natural map $K_0(J) \rightarrow K_0(J/I)$ is surjective ([32, Definition 3.1]). Based on [32], we characterize when a separable, weakly purely infinite ($Z$-stable) $C^*$-algebra has real rank zero (Proposition 4.6 and Corollary 4.7).

If $A$ is a $C^*$-algebra, the fact that $I$ is an ideal of $A$ will be denoted $I \triangleleft A$. The symbol $\otimes$ will mean the minimal tensor product of $C^*$-algebras. We shall denote by $K$ the $C^*$-algebra of the compact operators on $\ell^2(\mathbb{N})$. If $X$ is a locally compact, Hausdorff space, $C_0(X)$ will denote the $C^*$-algebra of the continuous functions on $X$ with values in $\mathbb{C}$ and which vanish at infinity.

2 The ideal property

We are interested in characterizing when a $C^*$-algebra of the form $A \otimes D$ has the ideal property, where $A$ is a separable $C^*$-algebra and $D$ is a strongly self-absorbing $C^*$-algebra (specifically, $Z$). A particular answer follows from Corollary 2.5 below, which is a consequence of the following result:

**Proposition 2.1.** Let $A$ be a separable, purely infinite $C^*$-algebra and let $B$ be a separable, simple, exact $C^*$-algebra with the ideal property. Then, the following are equivalent:
(1) $A \otimes B$ has the ideal property;
(2) $A$ has the ideal property;
(3) $A$ has the ideal property and $A \otimes B$ is strongly purely infinite.

The proof of the above proposition will use [32], [31] and, among other things, the following results of Blanchard-Kirchberg and Kirchberg:

**Theorem 2.2** ([2]). If $A$ and $B$ are separable $C^*$-algebras and either $A$ or $B$ is exact, then $\text{Prim}(A \otimes B)$ and $\text{Prim}(A) \times \text{Prim}(B)$ are homeomorphic.

**Theorem 2.3** ([22]). If $A$ and $B$ are $C^*$-algebras such that one of $A$ or $B$ is exact and the other one is strongly purely infinite, then $A \otimes B$ is strongly purely infinite.

**Proof of Proposition 2.1.** (1) $\Rightarrow$ (2). Assume now that $A \otimes B$ has the ideal property. Since $A$ and $B$ are separable and $B$ is exact, Theorem 2.2 implies that $\text{Prim}(A \otimes B)$ is homeomorphic to $\text{Prim}(A)$ (since $B$ being simple, $\text{Prim}(B)$ consists of only one element). But since $A \otimes B$ has the ideal property, it follows by [3, page 76] that $\text{Prim}(A \otimes B)$ has a basis consisting of compact-open sets, and hence, by the above fact, so does $\text{Prim}(A)$. Using this, the fact that $A$ is separable and purely infinite and [32, Proposition 2.11] we conclude that $A$ has the ideal property.

(2) $\Rightarrow$ (3). Assume that (2) is true. Since $A$ is purely infinite and it has the ideal property, it follows from [32, Proposition 2.14] that $A$ is strongly purely infinite. Now, since $B$ is exact, it follows from Theorem 2.3 that $A \otimes B$ is strongly purely infinite.

(3) $\Rightarrow$ (1). Assume that (3) is true. Since both $C^*$-algebras $A$ and $B$ have the ideal property and $B$ is exact, then by [31, Corollary 1.3] (which is based on a result of Kirchberg in [21]) it follows that (1) is true.

**Remark 2.4.** Observe that since $A$ is separable and purely infinite, it follows by [32, Proposition 2.11] that the condition (2), and therefore the conditions (1), (2) and (3) in Proposition 2.1 are equivalent to:

(4) $\text{Prim}(A)$ has a basis consisting of compact-open sets.

Note also that the equivalence between the conditions (1) and (2) in Proposition 2.1 and the condition (4) above was proved in [32, Corollary 4.3(ii)] in the particular case when $B = O_2$.

**Corollary 2.5.** Let $A$ be a separable, purely infinite $C^*$-algebra and let $D$ be a separable, unital, strongly self-absorbing $C^*$-algebra. Then, the following are equivalent:

(1) $A \otimes D$ has the ideal property;
(2) $A$ has the ideal property;
(3) $A$ has the ideal property and $A \otimes D$ is strongly purely infinite.
Proof. It follows from Proposition 2.1 since $D$ is separable, simple, unital and nuclear by [37] (based on results in [8] and [24]) (and every nuclear $C^*$-algebra is exact).

Let $A$ be a separable $C^*$-algebra. Consider the following conditions:
(a) $A \otimes \mathbb{Z}$ has the ideal property;
(b) $A$ has the ideal property;
(c) Prim($A$) has a basis consisting of compact-open sets.

Question 2.6: Is the equivalence (a) $\iff$ (b) true?

Question 2.7: Is the equivalence (a) $\iff$ (c) true?

Remark 2.8. Clearly, (b) $\iff$ (c), even if Prim($A$) has one element, i.e. $A$ is simple. Indeed, one can take $A$ to be a non-zero, separable, simple, nonunital, projectionless $C^*$-algebra. (One could also show that (b) $\iff$ (c) observing that the class $C_3$ of the separable $C^*$-algebras satisfying (c) is closed under stable isomorphism while the class $C_2$ of the separable $C^*$-algebras satisfying (b) is not ([30]), or, one could observe that $C_3$ is closed under extensions ([32, Corollary 4.4 (i)]) while $C_2$ is not ([29]).

Theorem 2.9. There is a separable, nuclear $C^*$-algebra $A$ such that $\mathrm{RR}(A) = \mathrm{RR}(A \otimes \mathbb{Z}) = \mathrm{sr}(A) = \mathrm{sr}(A \otimes \mathbb{Z}) = 1$ and Prim($A$) has two elements (in particular, Prim($A$) has a basis consisting of compact-open sets) but $A \otimes \mathbb{Z}$ does not have the ideal property (and, implicitly, $A$ does not have the ideal property).

Proof. Let $D$ be a Bunce-Deddens algebra [5]. We showed in the proof of [29, Theorem 5.1], jointly with Dadarlat, that there is an extension of $C^*$-algebras:

\[
0 \to D \otimes \mathcal{K} \overset{i}{\to} A \overset{\pi}{\to} \mathcal{C} \to 0 \tag{2.1}
\]

such that $A$ does not have the ideal property, the corresponding exponential map $\delta_0 : K_0(\mathcal{C}) \to K_1(D \otimes \mathcal{K})$ is injective and $\mathrm{RR}(A) = \mathrm{sr}(A) = 1$. Since $\mathbb{Z}$ is exact (being nuclear), tensoring the extension (2.1) with $\mathbb{Z}$, we obtain the following short exact sequence of $C^*$-algebras:

\[
0 \to D \otimes \mathcal{K} \otimes \mathbb{Z} \overset{\otimes id}{\to} A \otimes \mathbb{Z} \overset{\pi \otimes id}{\to} \mathcal{C} \otimes \mathbb{Z} \cong \mathbb{Z} \to 0 \tag{2.2}
\]

We have a commutative diagram:

\[
\begin{array}{ccc}
0 & \to & D \otimes \mathcal{K} \overset{i}{\to} A \overset{\pi}{\to} \mathcal{C} \to 0 \\
\downarrow & & \downarrow \\
0 & \to & D \otimes \mathcal{K} \otimes \mathbb{Z} \overset{\otimes id}{\to} A \otimes \mathbb{Z} \overset{\pi \otimes id}{\to} \mathcal{C} \otimes \mathbb{Z} \cong \mathbb{Z} \to 0
\end{array}
\]

where the vertical maps are of the form $x \mapsto x \otimes 1$. Using the naturality of the exponential map, we get a commutative diagram at the level of $K_0$ groups.
involving the associated exponential maps $\delta_0$ and $\delta'_0$, of the form:

$$
\begin{array}{c}
K_0(\mathbb{C}) \xrightarrow{\delta_0} K_1(D \otimes K) \\
\downarrow \\
K_0(\mathbb{Z}) \xrightarrow{\delta'_0} K_1(D \otimes K \otimes \mathbb{Z})
\end{array}
$$

where the vertical maps are group isomorphisms (use [19, Lemma 2.11] and the fact that $D \otimes K$ has a countable, increasing, approximate unit of projections (because $D$ has such an approximate unit of projections)). Then, since $\delta_0$ is injective, it follows that $\delta'_0$ is also injective. Since also $(A \otimes \mathbb{Z})/(\iota \otimes \text{id}) \cong Z$ is non-zero and stably finite, it follows from one of our joint results with Dadarlat (see [29, Lemma 5.2]) that $A \otimes \mathbb{Z}$ is not generated (as an ideal of $A \otimes \mathbb{Z}$) by its projections. Hence, $A \otimes \mathbb{Z}$ does not have the ideal property. The short exact sequence (2.1) implies that $A$ is separable and nuclear (as an extension of the separable and nuclear $C^*$-algebras $D \otimes K$ and $\mathbb{C}$) and $\text{Prim}(A)$ has two elements (since, by (2.1), $A$ is an extension of two simple $C^*$-algebras). Note that the separable $C^*$-algebra $D$ is approximately divisible by [10], since $D$ is a simple, unital, infinite dimensional inductive limit of a sequence of circle algebras, and hence, by [38, Theorem 2.3] we have that $D \cong D \otimes \mathbb{Z}$. Therefore, $D \otimes K \otimes \mathbb{Z} \cong D \otimes K$ and since $sr(D \otimes K) = 1$ ($D \otimes K$ is an inductive limit of circle algebras that have stable rank one and now one can use [33]), we deduce that $sr(D \otimes K \otimes \mathbb{Z}) = 1$. Also, $sr(\mathbb{Z}) = 1$ (use e.g. [35, Theorem 6.7]). But the index map $\delta_1 : K_1(\mathbb{Z}) = 0 \rightarrow K_0(D \otimes K \otimes \mathbb{Z})$ associated to the short exact sequence (2.2) is zero. These things imply that $sr(A \otimes \mathbb{Z}) = 1$ (see e.g. [27]). But since $A \otimes \mathbb{Z}$ does not have the ideal property (as noticed before), it follows that $\text{RR}(A \otimes \mathbb{Z}) \neq 0$ (the same conclusion can be obtained using the fact that by (2.2), $(A \otimes \mathbb{Z})/(\iota \otimes \text{id}) \cong Z$ is not of real rank zero (see [4])). By [4, Proposition 1.2] it follows that $\text{RR}(A \otimes \mathbb{Z}) \leq 2 \text{sr}(A \otimes \mathbb{Z}) - 1 = 1$, and in conclusion, $\text{RR}(A \otimes \mathbb{Z}) = 1$. (Finally, note that if $B$ is a $C^*$-algebra such that $B \otimes \mathbb{Z}$ does not have the ideal property, then by [31, Corollary 1.3] it follows that $B$ does not have the ideal property).

**Remark 2.10.** The above theorem proves that the answer to Question 2.7 is negative even if $A$ is nuclear, $\text{RR}(A) = \text{RR}(A \otimes \mathbb{Z}) = \text{sr}(A) = \text{sr}(A \otimes \mathbb{Z}) = 1$ and $\text{Prim}(A)$ has two elements.

3 Permanence properties

We will study some (permanence) properties of large classes of separable, $D$-stable $C^*$-algebras with the ideal property, where $D$ is a strongly self-absorbing, $K_1$-injective $C^*$-algebra.
Definition 3.1. Let $D$ be a strongly self-absorbing, $K_1$-injective $C^*$-algebra. Let $C(D)$ be the class of all the separable, purely infinite $C^*$-algebras with the ideal property that are $D$-stable.

Recall from [20] that if $X$ is a compact, Hausdorff topological space, then a $C(X)$-algebra is a $C^*$-algebra $A$ endowed with a unital *-homomorphism from $C(X)$ in the center $Z(M(A))$ of the multiplier $C^*$-algebra $M(A)$ of $A$. If $F \subseteq X$ is a closed subset we denote by $A_F$ the quotient of $A$ by the ideal $C_0(X \setminus F)A$.

Proposition 3.2. Let $D$ be as in Definition 3.1.

(1) $C(D)$ is stable under taking hereditary sub-$C^*$-algebras (in particular ideals), stable isomorphisms, quotients, countable inductive limits and extensions but not under taking minimal tensor products.

(2) Let $A$ be a $C(X)$-algebra, where $X$ is a compact, Hausdorff topological space. Then, if every $x \in X$ has a compact neighborhood $V_x \subseteq X$ such that $A_{V_x} \in C(D)$, then $A \in C(D)$.

The proof of the above proposition will need the following three preliminary results:

Proposition 3.3. Let $D$ be a strongly self-absorbing $C^*$-algebra. Then, the class of all the separable, purely infinite, $D$-stable $C^*$-algebras with the ideal property is not stable under taking minimal tensor products.

Proof. Let $A$ and $C$ be the separable $C^*$-algebras from [32, Proposition 4.5]. Then, $A \otimes O_2$ and $C \otimes O_2$ have the ideal property and $(A \otimes O_2) \otimes (C \otimes O_2) \cong A \otimes C \otimes O_2$ does not have the ideal property (by [32, Proposition 4.5]). Define $E := A \otimes O_2 \otimes D$ and $F := C \otimes O_2 \otimes D$. Then $E$ and $F$ are separable, $D$-stable, purely infinite $C^*$-algebras (since $M \otimes O_2$ is purely infinite for any $C^*$-algebra $M$, by [25, Proposition 4.5]). Note also that $E$ and $F$ have the ideal property by [31, Corollary 1.3], since, as noticed above, $D$ is simple, unital and nuclear. On the other hand, $E \otimes F \cong (A \otimes C \otimes O_2) \otimes D$ does not have the ideal property, by Corollary 2.5 (because $A \otimes C \otimes O_2$ is separable, purely infinite (again by [25, Proposition 4.5]) and it does not have the ideal property). The proof is over.

Proposition 3.4. Suppose that we have a pullback diagram of $C^*$-algebras:

$$
\begin{array}{ccc}
C & \xrightarrow{\pi_1} & A_1 \\
\pi_2 & \downarrow & \varphi_1 \\
A_2 & \xrightarrow{\varphi_2} & B
\end{array}
$$

where $C$ is the pullback of $(A_1, A_2)$ along $(\varphi_1, \varphi_2)$ and at least one of the maps $\varphi_1, \varphi_2$ is surjective.

(1) If $A_1$ and $A_2$ are separable, purely infinite and with the ideal property, then so is $C$. 

(2) If $A_1$ and $A_2$ are nuclear, then so is $C$.

**Proof.** (1). It is inspired by the proof of [17, Proposition 4.9]. Let us assume that $\varphi_2$ is surjective. Therefore, $\pi_1$ is also surjective and we have the following short exact sequence of $C^*$-algebras:

$$0 \longrightarrow \ker(\pi_1) \xrightarrow{\iota} C \xrightarrow{\pi_1} A_1 \longrightarrow 0 \tag{3.1}$$

where $\iota$ is the inclusion. We may identify:

$$C = \{(a_1, a_2) \in A_1 \oplus A_2 : \varphi_1(a_1) = \varphi_2(a_2)\} \subseteq A_1 \oplus A_2.$$

Under this identification, we clearly have:

$$\ker(\pi_1) = \{(0, a) : \varphi_2(a) = 0\} = 0 \oplus \ker(\varphi_2).$$

Since the separability, the ideal property and the property of being purely infinite pass to ideals (for the last assertion see [25, Proposition 4.3]) and since $A_2$ is separable, purely infinite and with the ideal property, it follows that $\ker(\varphi_2)$ ($\llcorner A_2$) is separable, purely infinite and with the ideal property and hence, so is $\ker(\pi_1)$. Finally, since the class of the separable, purely infinite $C^*$-algebras with the ideal property is closed under extensions (by [32, Corollary 4.4 (ii)]), it follows from (3.1) that $C$ is separable, purely infinite and with the ideal property.

(2). It is similar with the above proof of (1) and uses the well known facts that ideals and extensions of nuclear $C^*$-algebras are also nuclear.

**Corollary 3.5.** Let $X$ be a compact, Hausdorff topological space and let $A$ be a $C(X)$-algebra. Suppose that each $x \in X$ has a compact neighborhood $V_x \subseteq X$ such that $A_{V_x}$ is separable, purely infinite and with the ideal property (respectively, $A_{V_x}$ is nuclear). Then, $A$ is separable, purely infinite and with the ideal property (respectively, $A$ is nuclear).

**Proof.** It follows from the above Proposition 3.4 as in the proof of [17, Proposition 4.11].
(2). It follows from the proof of [17, Proposition 4.11] that $A$ is $\mathcal{D}$-stable. Now, to end the proof, we use also the above Corollary 3.5.

**Definition 3.6.** Let $\mathcal{D}$ and $\mathcal{C}(\mathcal{D})$ be as in Definition 3.1. Let $\mathcal{N}$ denote the class of all the nuclear $C^*$-algebras. Define $\mathcal{C}_{nuc}(\mathcal{D}) := \mathcal{C}(\mathcal{D}) \cap \mathcal{N}$ and define $\mathcal{C}$ to be the class of all the separable, purely infinite, nuclear $C^*$-algebras with the ideal property.

**Proposition 3.7.** Let $\mathcal{D}$ be as in Definition 3.6.

1. $\mathcal{C} = \mathcal{C}_{nuc}(\mathcal{O}_\infty) = \mathcal{C}_{nuc}(\mathcal{Z})$;
2. $\mathcal{C}_{nuc}(\mathcal{D})$ has the same properties as $\mathcal{C}(\mathcal{D})$ in Proposition 3.2, except that it is stable under taking tensor products.
3. $\mathcal{C}_{nuc}(\mathcal{D}) = \mathcal{C} \otimes \mathcal{D}$, where $\mathcal{C} \otimes \mathcal{D} := \{ A \otimes \mathcal{D} : A \in \mathcal{C} \}$.

**Proof.** (1). We will prove first that:

$$ \mathcal{C} \subseteq \mathcal{C}_{nuc}(\mathcal{O}_\infty) \quad (3.2) $$

Indeed, let $A \in \mathcal{C}$, that is let $A$ be a separable, purely infinite, nuclear $C^*$-algebra with the ideal property. Since $A$ is purely infinite and with the ideal property, it follows by [32, Proposition 2.14] that $A$ is strongly purely infinite. Since the property of being strongly purely infinite is preserved by stable isomorphisms (by [26, Proposition 5.11 (iii)]), one obtains the $A \otimes \mathcal{K}$ is strongly purely infinite. Then, [26, Theorem 8.6] implies that the separable, nuclear, strongly purely infinite, stable $C^*$-algebra $A \otimes \mathcal{K}$ is $\mathcal{O}_\infty$-stable. Using now [37, Corollary 3.2], we can conclude that $A$ is $\mathcal{O}_\infty$-stable. Hence, (3.2) is proved.

Observe also that since $\mathcal{O}_\infty$ is $\mathcal{Z}$-stable (by [19, Corollary 2.13]), it follows that the following inclusion is true:

$$ \mathcal{C}_{nuc}(\mathcal{O}_\infty) \subseteq \mathcal{C}_{nuc}(\mathcal{Z}) \quad (3.3) $$

Finally observe that (3.2), (3.3) and the obvious inclusion $\mathcal{C}_{nuc}(\mathcal{Z}) \subseteq \mathcal{C}$ imply that (1) is true.

(2). It is known that $\mathcal{N}$ is stable under taking hereditary sub-$C^*$-algebras, stable isomorphisms, quotients, inductive limits, extensions and tensor products. Now, the proof of (2) ends using Proposition 3.2, Corollary 3.5 and [32, Proposition 4.6].

(3). Observe that $\mathcal{C}_{nuc}(\mathcal{D}) \subseteq \mathcal{C} \otimes \mathcal{D}$ is obvious (since any element of $\mathcal{C}_{nuc}(\mathcal{D})$ is $\mathcal{D}$-stable and $\mathcal{C}_{nuc}(\mathcal{D}) \subseteq \mathcal{C}$). Conversely, let us prove now that $\mathcal{C} \otimes \mathcal{D} \subseteq \mathcal{C}_{nuc}(\mathcal{D})$. For this, let $A \in \mathcal{C}$. Since $A$ and $\mathcal{D}$ have the ideal property ($\mathcal{D}$ is simple and unital), $A$ is purely infinite and $\mathcal{D}$ is exact (since $\mathcal{D}$ is nuclear), [32, Proposition 4.6] implies that $A \otimes \mathcal{D}$ is purely infinite and with the ideal property. This ends the proof of the inclusion $\mathcal{C} \otimes \mathcal{D} \subseteq \mathcal{C}_{nuc}(\mathcal{D})$, since the fact that $A \otimes \mathcal{D}$ is separable, $\mathcal{D}$-stable and nuclear is obvious.
4 Real rank zero

We will characterize when $\text{RR}(A \otimes \mathcal{Z}) = 0$, for "many" separable $C^*$-algebras $A$. We begin with the following:

**Proposition 4.1.** Let $A$ be a separable $C^*$-algebra. If $\text{RR}(A \otimes \mathcal{Z}) = 0$, then $\text{RR}(A \otimes \mathcal{O}_\infty) = 0$.

The proof of the above result uses, among other things, the following:

**Lemma 4.2.** Let $A$ be a $C^*$-algebra. Then, the following are equivalent:

1. $A$ is $K_0$-liftable;
2. $A \otimes \mathcal{Z}$ is $K_0$-liftable.

**Proof.** It is inspired by the proof of [32, Lemma 3.4]. Observe first that since $\mathcal{Z}$ is nuclear and simple, any ideal of $A \otimes \mathcal{Z}$ is of the form $I \otimes \mathcal{Z}$, where $I \triangleleft A$ (by [1], see also [2, Proposition 2.16]). Also, if $I \triangleleft A$, $J \triangleleft A$ such that $I \subseteq J$, then $(J \otimes \mathcal{Z})/(I \otimes \mathcal{Z}) \cong (J/I) \otimes \mathcal{Z}$, since $\mathcal{Z}$ is exact (being nuclear).

Now, let $I \triangleleft A$, $J \triangleleft A$ such that $I \subseteq J$. Consider the commutative diagram:

$$
\begin{array}{ccc}
J & \rightarrow & J/I \\
\downarrow & & \downarrow \\
J \otimes \mathcal{Z} & \rightarrow & (J/I) \otimes \mathcal{Z}
\end{array}
$$

where the vertical maps are of the form $x \mapsto x \otimes 1$ and the horizontal maps are the canonical surjections. Note that the vertical maps above induce isomorphisms at the level of $K_0$. This follows from the fact that if $D$ is an arbitrary $C^*$-algebra, then the canonical embedding of $D$ into $D \otimes \mathcal{Z}$ induces a group isomorphism at the level of $K_0$; in the unital case this was proved in [16, Corollary 1.3] and in the general case one needs to adjoin a unit and to use a standard argument involving the unital case and the six-term exact sequence in $K$-theory (notice that $K_1(\mathcal{Z}) = 0$). It is now obvious that $A$ is $K_0$-liftable if and only if $A \otimes \mathcal{Z}$ is $K_0$-liftable. Hence, (1) $\Leftrightarrow$ (2).

**Proof of Proposition 4.1.** Assume that $\text{RR}(A \otimes \mathcal{Z}) = 0$. Then, $A \otimes \mathcal{Z}$ has the ideal property and, by [3, page 76] it follows that $\text{Prim}(A \otimes \mathcal{Z})$ has a basis consisting of compact-open sets. Since $\mathcal{Z}$ is simple and exact, by Theorem 2.2 it follows that $\text{Prim}(A)$ has a basis consisting of compact-open sets ($\mathcal{Z}$ being simple, $\text{Prim}(\mathcal{Z})$ has only one element). Now, by [4] any $C^*$-algebra of real rank zero is $K_0$-liftable. Hence, $A \otimes \mathcal{Z}$ is $K_0$-liftable, and, by Lemma 4.2, it follows that $A$ is $K_0$-liftable. Now, since $A$ is separable, [32, Corollary 4.3 (i)] implies that $\text{RR}(A \otimes \mathcal{O}_\infty) = 0$.

**Remark 4.3.** Observe that the converse of the implication in Proposition 4.1 is not true. Indeed, if $A = \mathbb{C}$, then $\text{RR}(A \otimes \mathcal{O}_\infty) = \text{RR}(\mathcal{O}_\infty) = 0$ by [39], since $\mathcal{O}_\infty$.
is purely infinite and simple ([34]), while \( \text{RR}(A \otimes Z) = \text{RR}(Z) \neq 0 \).

We show that in spite of Remark 4.3, there are "many" separable \( C^* \)-algebras \( A \) for which \( \text{RR}(A \otimes Z) = 0 \) \( \iff \text{RR}(A \otimes O_\infty) = 0 \):

**Theorem 4.4.** Let \( A \) be a separable, weakly purely infinite \( C^* \)-algebra. Then, the following are equivalent:

1. \( \text{RR}(A \otimes Z) = 0 \);
2. \( \text{RR}(A \otimes O_\infty) = 0 \);
3. \( \text{Prim}(A) \) has a basis consisting of compact-open sets and \( A \) is \( K_0 \)-liftable;
4. \( A \otimes Z \) has the ideal property and \( A \) is \( K_0 \)-liftable.

**Proof.** Observe first that since \( A \) is weakly purely infinite, [26, Theorem 4.8 (ii)] implies that \( A \) is traceless, and hence, by [23, Corollary 3.12] it follows that \( A \otimes Z \) is strongly purely infinite. Therefore, by [26, Proposition 5.4], it follows that \( A \otimes Z \) is purely infinite.

(1) \( \Rightarrow \) (2) follows from Proposition 4.1.

(2) \( \Rightarrow \) (3) follows from [32, Corollary 4.3 (i)].

Using the fact that the separable \( C^* \)-algebra \( A \otimes Z \) is purely infinite, (3) \( \Rightarrow \) (1) follows from [32, Theorem 4.2], Lemma 4.2 and Theorem 2.2 (\( \text{Prim}(A \otimes Z) \) is homeomorphic to \( \text{Prim}(A) \), since \( Z \) is exact and simple).

Since, as proved above, the separable \( C^* \)-algebra \( A \otimes Z \) is purely infinite, (3) \( \iff \) (4) follows from [32, Proposition 2.11] and Theorem 2.2.

**Remark 4.5.** The proof of Theorem 4.4 shows in fact that if \( A \) is a separable \( C^* \)-algebra such that \( A \otimes Z \) is purely infinite, then the conditions (1), (2), (3) and (4) in Theorem 4.4 are equivalent.

Finally, let us observe that using [32], one could easily characterize when a separable, weakly purely infinite (\( Z \)-stable) \( C^* \)-algebra has real rank zero:

**Proposition 4.6.** Let \( A \) be a separable, weakly purely infinite \( C^* \)-algebra. Then, the following are equivalent:

1. \( \text{RR}(A) = 0 \);
2. \( A \) has the ideal property and \( A \) is \( K_0 \)-liftable.

**Proof.** (1) \( \Rightarrow \) (2). This is true for every \( C^* \)-algebra \( A \), since any \( C^* \)-algebra of real rank zero has the ideal property and and it is also \( K_0 \)-liftable (by [4]).

(2) \( \Rightarrow \) (1). Assume now that \( A \) has the ideal property and \( A \) is \( K_0 \)-liftable. Since \( A \) is weakly purely infinite and with the ideal property, it follows from [32, Proposition 2.14] that \( A \) is purely infinite. Also, since \( A \) has the ideal property, then by a general argument ([3, page 76]) it follows that \( \text{Prim}(A) \) has a basis consisting of compact-open sets. Finally, [32, Theorem 4.2] implies that \( \text{RR}(A) = 0 \).

**Corollary 4.7.** Let \( A \) be a separable \( C^* \)-algebra and assume that \( A \otimes Z \) is a
weakly purely infinite $C^*$-algebra. Then, the following are equivalent:

1. $\text{RR}(A \otimes \mathbb{Z}) = 0$;
2. $A \otimes \mathbb{Z}$ has the ideal property and $A$ is $K_0$-liftable.

**Proof.** It follows from Proposition 4.2 and Proposition 4.6.

**References**


Received: 04.11.2008.

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