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An inequality between depth and Stanley depth

by
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To Professor S. Ianuş on the occasion of his 70th Birthday

Abstract

We show that Stanley's Conjecture holds for square free monomial ideals in five variables, that is the Stanley depth of a square free monomial ideal in five variables is greater or equal with its depth.

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Introduction

Let $S = K[x_1, \dots, x_n]$ be a polynomial ring in n variables over a field K and M a finitely generated multigraded (i.e. $\mathbb{Z}^n$-graded) S-module. Given $m \in M$ a homogeneous element in M and $Z \subseteq \{x_1, \dots, x_n\}$, let $mK[Z] \subset M$ be the linear K-subspace of all elements of the form mf, $f \in K[Z]$. This subspace is called Stanley space of dimension $|Z|$, if $mK[Z]$ is a free $K[Z]$-module. A Stanley decomposition of M is a presentation of the K-vector space M as a finite direct sum of Stanley spaces $\mathcal{D}: \ M = \bigoplus_{i=1}^r m_i K[Z_i]$. Set $\operatorname{sdepth} \mathcal{D} = \min\{|Z_i| : i = 1, \dots, r\}$. The number

$$\operatorname{sdepth}(M) := \max\{\operatorname{sdepth}(\mathcal{D}) : \ \mathcal{D} \text{ is a Stanley decomposition of } M\}$$

is called Stanley depth of M. R. Stanley [9, Conjecture 5.1] gave the following conjecture.

Stanley's Conjecture $\operatorname{sdepth}(M) \geq \operatorname{depth}(M)$ for all finitely generated $\mathbb{Z}^n$-graded S-modules M.

Our Theorem 1.4, completely based on [6], shows that the above conjecture holds when $\dim_S \ M \leq 2$. If $n \leq 5$ Stanley's Conjecture holds for all cyclic S-modules by [1] and [6, Theorem 4.3].

It is the purpose of our paper to study Stanley's Conjecture on monomial square free ideals of S, that is:

Weak Conjecture Let $I \subset S$ be a monomial square free ideal. Then $\operatorname{sdepth}_S \ I \geq \operatorname{depth}_S \ I$.

Our Theorem 2.8 gives a kind of inductive step in proving the above conjecture, which is settled for $n \leq 5$ in our Theorem 2.11. Note that the above conjecture says in fact that $\operatorname{sdepth}_S \ I \geq 1 + \operatorname{depth}_S \ S/I$ for any monomial square free ideal I of S. This remind us a question raised in [8], saying that $\operatorname{sdepth}_S \ I \geq 1 + \operatorname{sdepth}_S \ S/I$ for any monomial ideal I of S. This question is harder since there exist

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few known properties of Stanley depth (see [3], [7], [4], [8]), which is not the case of the usual depth (see [2], [10]). A positive answer of this question in the frame of monomial square free ideals would state the Weak Conjecture as follows:

$$\text{sdepth}_S\ I \geq 1 + \text{sdepth}_S\ S/I \geq 1 + \text{depth}_S\ S/I = \text{depth}_S\ I,$$

the second inequality being a consequence of [6, Theorem 4.3], or of our Theorem 1.4.

1 Some inequalities on depth and Stanley depth

Let $S = K[x_1, \dots, x_n]$ be a polynomial ring over a field K, $I \subset S$ a monomial ideal. A. Rauf stated in [8] the following result:

Proposition 1.1. $\text{depth}_S\ S/(I, x_n) \geq \text{depth}_S\ S/I - 1$.

It is worth to mention that this result holds only in monomial frame and we will use in the proof of our Lemma 2.6. Next we present two easy lemmas necessary in the next section:

Lemma 1.2. *Let $I \subset J$, $I \neq J$ be some monomial ideals of $S' = K[x_1, \dots, x_{n-1}]$. Then*

$$\text{sdepth}_S\ JS/x_nIS \geq \min\{\text{sdepth}_S\ JS/IS, \text{sdepth}_{S'}\ I\}.$$

Proof: From the filtration $x_nIS \subset IS \subset JS$ we get an isomorphism of linear K-spaces $JS/x_nIS \cong JS/IS \oplus IS/x_nIS$. It follows that

$$\text{sdepth}_S\ JS/x_nIS \geq \min\{\text{sdepth}_S\ JS/IS, \text{sdepth}_S\ IS/x_nIS\}.$$

To end note that the inclusion $I \subset IS$ induces an isomorphism of linear K-spaces $I \cong IS/x_nIS$, which shows that $\text{sdepth}_{S'}\ I = \text{sdepth}_S\ IS/x_nIS$. □

Lemma 1.3. *Let $I \subset J$, $I \neq J$ be some monomial ideals of $S' = K[x_1, \dots, x_{n-1}]$ and $T = (I + x_nJ)S$. Then*

1.
$$\text{sdepth}\ T \geq \min\{\text{sdepth}_{S'}\ I, \text{sdepth}_S\ JS\},$$

2.
$$\text{sdepth}\ T \geq \min\{\text{sdepth}_S\ JS/IS, \text{sdepth}_S\ IS\}.$$

Proof: Note that $T = I \oplus x_nJS$ as linear K-spaces and so (1) holds. On the other hand the filtration $0 \subset IS \subset T$ induces an isomorphism of linear K-spaces $T \cong IS \oplus T/IS$ and so

$$\text{sdepth}\ T \geq \min\{\text{sdepth}_S\ T/IS, \text{sdepth}_S\ IS\}.$$

Note that the multiplication by x_n induces an isomorphism of linear K-spaces $JS/IS \cong T/IS$, which shows that $\text{sdepth}_S\ T/IS = \text{sdepth}_S\ JS/IS$. Thus (2) holds too. □

An important tool in the next section is the following result, which unifies some results from [6].

Theorem 1.4. *Let U, V be some monomial ideals of S such that $U \subset V$, $U \neq V$. If* $\dim_S\ V/U \leq 2$ *then* $\text{sdepth}_S\ V/U \geq \text{depth}_S\ V/U$.

Proof: If V/U is a Cohen-Macaulay S-module of dimension 2 then it is enough to apply [6, Theorem 3.3]. If $\dim_S\ V/U = 2$ but $\text{depth}_S\ V/U = 1$ then the result follows from [6, Theorem 3.10]. If $\dim_S\ V/U \leq 1$ then we may apply [5, Corollary 2.2]. □

Corollary 1.5. *Let $S = K[x_1, x_2, x_3]$, $I \subset J$, $0 \neq I \neq J$ be two monomial ideals. Then* $\text{sdepth}_S\ J/I \geq \text{depth}_S\ J/I$.

For the proof note that $\text{depth}_S\ J/I \leq \dim_S\ S/I \leq 2$ and apply Theorem 1.4.

2 A hard inequality

Let $S' = K[x_1, \dots, x_{n-1}]$ be a polynomial ring in $n-1$ variables over a field K, $S = S'[x_n]$ and $U, V \subset S'$, $U \subset V$ two homogeneous ideals. We want to study the depth of the ideal $W = (U + x_n V)S$ of S. Actually every monomial square free ideal T of S has this form because then $(T : x_n)$ is generated by an ideal $V \subset S'$ and $T = (U + x_n V)S$ for $U = T \cap S'$.

Lemma 2.1. *Suppose that $U \neq V$ and $\operatorname{depth}_{S'} S'/U = \operatorname{depth}_{S'} S'/V = \operatorname{depth}_{S'} V/U$. Then $\operatorname{depth}_S S/W = \operatorname{depth}_{S'} S'/U$.*

Proof: Set $r = \operatorname{depth}_{S'} S'/U$ and choose a sequence $f_1, \dots, f_r$ of homogeneous elements of $m_{n-1} = (x_1, \dots, x_{n-1}) \subset S'$, which is regular on S'/U, S'/V and V/U simultaneously. Set $\bar{U} = (U, f_1, \dots, f_r)$, $\bar{V} = (V, f_1, \dots, f_r)$. Then tensorizing by $S'/(f_1, \dots, f_r)$ the exact sequence

$$0 \to V/U \to S'/U \to S'/V \to 0$$

we get the exact sequence

$$0 \to V/U \otimes_{S'} S'/(f_1, \dots, f_r) \to S'/\bar{U} \to S'/\bar{V} \to 0$$

and so $\bar{V}/\bar{U} \cong V/U \otimes_{S'} S'/(f_1, \dots, f_r)$ has depth 0.

Note that $f_1, \dots, f_r$ is regular also on S/W and taking $\bar{W} = W + (f_1, \dots, f_r)S$ we get $\operatorname{depth}_S S/W = \operatorname{depth}_S S/\bar{W} + r$. Thus passing from U, V, W to $\bar{U}, \bar{V}, \bar{W}$ we may reduce the problem to the case $r = 0$.

If $r = 0$ then there exists an element $v \in V \setminus U$ such that $(U : v) = m_{n-1}$. Thus the non-zero element of S/W induced by v is annihilated by m_{n-1} and x_n because $v \in V$. Hence $\operatorname{depth}_S S/W = 0$. □

Example 2.2. Let $n = 4$, $V = (x_1, x_2)$, $U = V \cap (x_1, x_3)$ be ideals of $S' = K[x_1, x_2, x_3]$ and $W = (U + x_4 V)S$. Then $\{x_3 - x_2\}$ is a maximal regular sequence on V/U and on S/W as well. Thus $\operatorname{depth}_{S'} V/U = \operatorname{depth}_{S'} S'/U = \operatorname{depth}_{S'} S'/V = \operatorname{depth}_S S/W = 1$.

Lemma 2.3. *Let $I, J \subset S'$, $I \subset J$, $I \neq J$ be two monomial ideals, $T = (I + x_n J)S$ such that*

1. $\operatorname{depth}_{S'} S'/I = \operatorname{depth}_S S/T - 1$,
2. $\operatorname{sdepth}_{S'} I \geq 1 + \operatorname{depth}_{S'} S'/I$,
3. $\operatorname{sdepth}_{S'} J/I \geq \operatorname{depth}_{S'} J/I$.

Then $\operatorname{sdepth}_S T \geq 1 + \operatorname{depth}_S S/T$.

Proof: By Lemma 1.3 we have

$$\operatorname{sdepth}_S T \geq 1 + \min\{\operatorname{sdepth}_{S'} I, \operatorname{sdepth}_{S'} J/I\} \geq 1 + \min\{1 + \operatorname{depth}_{S'} S'/I, \operatorname{depth}_{S'} J/I\}$$

using (3), (2) and [3, Lemma 3.6]. Note that in the following exact sequence

$$0 \to S/JS = S/(T : x_n) \xrightarrow{x_n} S/T \to S/(T, x_n) \cong S'/I \to 0$$

we have $\operatorname{depth}_S S/JS = \operatorname{depth}_{S'} S'/I + 1$ because of (1) and the Depth Lemma [10, Lemma 1.3.9]. Thus $\operatorname{depth}_{S'} S'/I = \operatorname{depth}_{S'} S'/J$. As $\operatorname{depth}_{S'} S'/I \neq \operatorname{depth}_S S/T$ we get $\operatorname{depth}_{S'} S'/I \neq \operatorname{depth}_{S'} J/I$ by Lemma 2.1. But $\operatorname{depth}_{S'} J/I \geq \operatorname{depth}_{S'} S'/I$ because of the Depth Lemma applied to the following exact sequence

$$0 \to J/I \to S'/I \to S'/J \to 0.$$

It follows that $\operatorname{depth}_{S'} J/I \geq 1 + \operatorname{depth}_{S'} S'/I$ and so

$$\operatorname{sdepth}_S T \geq 2 + \operatorname{depth}_{S'} S'/I = 1 + \operatorname{depth}_S S/T.$$

□

Remark 2.4. The above lemma introduces the difficult hypothesis (3) and one can hope that it is not necessary at least for square free monomial ideals. It seems this is not the case as shows somehow the next example.

Example 2.5. Let $n = 4$, $J = (x_1x_3, x_2)$, $I = (x_1x_2, x_1x_3)$ be ideals of $S' = K[x_1, x_2, x_3]$ and $T = (I + x_4J)S = (x_1, x_2) \cap (x_2, x_3) \cap (x_1, x_4)$. Then $\{x_4 - x_2, x_3 - x_1\}$ is a maximal regular sequence on S/T. Thus $\text{depth}_S\ S/T = 2$, $\text{depth}_{S'}\ S'/I = \text{depth}_{S'}\ S'/J = 1$.

Lemma 2.6. *Let $I, J \subset S'$, $I \subset J$, $I \neq J$ be two monomial ideals, $T = (I + x_nJ)S$ such that*

1. $\text{depth}_{S'}\ S'/I \neq \text{depth}_S\ S/T - 1$,
2. $\text{sdepth}_{S'}\ I \geq 1 + \text{depth}_{S'}\ S'/I, \quad \text{sdepth}_{S'}\ J \geq 1 + \text{depth}_{S'}\ S'/J.$

Then $\text{sdepth}_S\ T \geq 1 + \text{depth}_S\ S/T$.

Proof: By Lemma 1.3 we have

$$\text{sdepth}_S T \geq \min\{\text{sdepth}_{S'} I, 1 + \text{sdepth}_{S'} J\} \geq 1 + \min\{\text{depth}_{S'} S'/I, 1 + \text{depth}_{S'} S'/J\}$$

using (2). Applying Proposition 1.1 we get $\text{depth}_{S'} S'/I = \text{depth}_S S/(T, x_n) \geq \text{depth}_S S/T - 1$, the inequality being strict because of (1). We have the following exact sequence

$$0 \to S/JS = S/(T : x_n) \xrightarrow{x_n} S/T \to S/(T, x_n) \cong S'/I \to 0.$$

If $\text{depth}_{S'} S'/I > \text{depth}_S S/T$ then $\text{depth}_S S/JS = \text{depth}_S S/T$ by Depth Lemma and so

$$\text{sdepth}_S T \geq 1 + \min\{\text{depth}_{S'} S'/I, \text{depth}_S S/JS\} = 1 + \text{depth}_S S/T.$$

If $\text{depth}_{S'} S'/I = \text{depth}_S S/T$ then $\text{depth}_S S/JS \geq \text{depth}_{S'} S'/I$ again by Depth Lemma and thus

$$\text{sdepth}_S T \geq 1 + \text{depth}_{S'} S'/I = 1 + \text{depth}_S S/T.$$

□

Example 2.7. Let $n = 5$, $J = (x_1, x_2, x_3)$, $I = (x_1, x_2) \cap (x_3, x_4)$ be ideals of $S' = K[x_1, \ldots, x_4]$ and $T = (I + x_5J)S$. Then $\{x_4, x_3 - x_1\}$ is a maximal regular sequence on J/I and so $\text{depth}_{S'}\ J/I = 2 > 1 = \text{depth}_{S'}\ S'/I = \text{depth}_{S'}\ S'/J = \text{depth}_S\ S/T$.

Theorem 2.8. *Suppose that the Stanley's conjecture holds for factors V/U of monomial square free ideals, $U, V \subset S' = K[x_1, \ldots, x_{n-1}]$, $U \subset V$, that is* $\text{sdepth}_{S'}\ V/U \geq \text{depth}_{S'}\ V/U$. *Then the Weak Conjecture holds for monomial square free ideals of $S = K[x_1, \ldots, x_n]$.*

Proof: Let $r \leq n$ be a positive integer and $T \subset S_r = K[x_1, \ldots, x_r]$ a monomial square free ideal. By induction on r we show that $\text{sdepth}_{S_r}\ T \geq 1 + \text{depth}_{S_r}\ S_r/T$, the case $r = 1$ being trivial. Clearly, $(T : x_r)$ is generated by a monomial square free ideal $J \subset S_{r-1}$ containing $I = T \cap S_{r-1}$. By induction hypothesis we have $\text{sdepth}_{S_{r-1}}\ I \geq 1 + \text{depth}_{S_{r-1}}\ S_{r-1}/I$, $\text{sdepth}_{S_{r-1}}\ J \geq 1 + \text{depth}_{S_{r-1}}\ S_{r-1}/J$. If $I = J$ then $T = IS$, x_r is regular on S_r/T and we have

$$\text{sdepth}_{S_r}\ T = 1 + \text{sdepth}_{S_{r-1}}\ I \geq 2 + \text{depth}_{S_{r-1}}\ S_{r-1}/I = 1 + \text{depth}_{S_r}\ S_r/T,$$

using [3, Lemma 3.6]. Now suppose that $I \neq J$. If $\text{depth}_{S_{r-1}}\ S_{r-1}/I \neq \text{depth}_{S_r}\ S_r/T - 1$, then it is enough to apply Lemma 2.6. If $\text{depth}_{S_{r-1}}\ S_{r-1}/I = \text{depth}_{S_r}\ S_r/T - 1$, then apply Lemma 2.3. □

Corollary 2.9. *The Weak Conjecture holds in $S = K[x_1, \ldots, x_4]$.*

Proof: It is enough to apply Lemmas 2.3, 2.6 after we show that for monomial square free ideals $I, J \subset S' = K[x_1, \dots, x_3]$, $I \subset J$, $I \neq J$, $T = (I + x_4 J)S$ with $\operatorname{depth}_{S'}\ S'/I = \operatorname{depth}_S\ S/T - 1$, we have $\operatorname{sdepth}_{S'}\ J/I \geq \operatorname{depth}_{S'}\ J/I$. But then $I \neq 0$ because otherwise $\operatorname{depth}_S\ S/T \leq 3 = \operatorname{depth}_{S'}\ S'/I$, which is false. Thus $\dim_{S'}\ J/I \leq 2$ and we may apply Corollary 1.5. □

Lemma 2.10. *Let $I, J \subset S = K[x_1, \dots, x_4]$, $I \subset J$, $0 \neq I \neq J$ be two monomial square free ideals with* $\operatorname{depth}_S\ S/J = \operatorname{depth}_S\ S/I < \operatorname{depth}_S\ J/I$. *Then* $\operatorname{sdepth}_S\ J/I \geq \operatorname{depth}_S\ J/I$.

Proof: Note that $\operatorname{depth}_S\ S/I < \operatorname{depth}_S\ J/I \leq \dim_S\ S/I \leq 3$. If $\dim_S\ J/I \leq 2$ then we may apply Theorem 1.4. Otherwise, there exists a prime ideal of dimension 3 in $\operatorname{Ass}_S\ S/I$, let us say (x_4), which is not in $\operatorname{Ass}_S\ S/J$. Then x_4 is regular on S/J and I has the form $I = J \cap U \cap (x_4)$, where x_4 is regular on S/U. Note that $x_4 J \not\subset U$, because otherwise we have $\operatorname{depth}_S\ J/I = \operatorname{depth}_S\ J/x_4 J = \operatorname{depth}_S\ J - 1 = \operatorname{depth}_S\ S/J$, which is false. It follows $I = x_4 J \cap U$ and $(J : x_4) = J$. In the following exact sequence

$$0 \to J/(U \cap J) \xrightarrow{x_4} J/I \to J/(I, x_4 J) \to 0$$

we have $J/(I, x_4 J) \cong J/x_4 J$ and so

$$\operatorname{depth}_S\ J/(I, x_4 J) = \operatorname{depth}_S\ J - 1 = \operatorname{depth}_S\ S/J < \operatorname{depth}_S\ J/I.$$

By Depth Lemma we get $\operatorname{depth}_S\ J/(U \cap J) = 1 + \operatorname{depth}_S\ S/J$.

If $\operatorname{depth}_S\ J/I > \operatorname{depth}_S\ J/(U \cap J)$ then $\operatorname{depth}_S\ J/I \geq 2 + \operatorname{depth}_S\ S/J \geq 3$ because x_4 is regular on S/J. It follows that J/I is a Cohen-Macaulay S-module of dimension 3 and so all height one prime ideals of $\operatorname{Ass}_S\ S/I$, must be in $\operatorname{Ass}_S\ S/J$. Thus I has the form $I = J \cap (v)$ for some square free monomial $v \in S$, which is regular on S/J. Consequently, $I = vJ$ and we get

$$\operatorname{depth}_S\ J/I = \operatorname{depth}_S\ J/vJ = \operatorname{depth}_S\ J - 1 = \operatorname{depth}_S\ S/J,$$

which is false.

Thus $\operatorname{depth}_S\ J/I = \operatorname{depth}_S\ J/(U \cap J)$. Applying Depth Lemma in the following exact sequence

$$0 \to (U \cap J)/I \to J/I \to J/(U \cap J) \to 0$$

we get that $\operatorname{depth}_S\ (U \cap J)/I \geq \operatorname{depth}_S\ J/I$. But

$$\operatorname{depth}_S\ (U \cap J)/I = \operatorname{depth}_S\ (U \cap J)/x_4(U \cap J) = \operatorname{depth}_S\ (U \cap J) - 1 = \operatorname{depth}_S\ S/(U \cap J)$$

because x_4 is regular on $S/(U \cap J)$. Hence $\operatorname{depth}_S\ S/(U \cap J) \geq \operatorname{depth}_S\ J/I$.

Since x_4 is regular on S/U, S/J we see that U, J are generated by some ideals U', J' respectively of $S' = K[x_1, x_2, x_3]$ and by Corollaries 1.5, 2.9 we have $\operatorname{sdepth}_{S'}\ J'/U' \geq \operatorname{depth}_{S'}\ J'/U'$, $\operatorname{sdepth}_{S'}\ (U' \cap J') \geq \operatorname{depth}_{S'}\ (U' \cap J')$. By Lemma 1.2 we have

$$\begin{gathered}\operatorname{sdepth}_S\ J/I \geq \min\{\operatorname{sdepth}_S\ J/(U \cap J), \operatorname{sdepth}_{S'}\ (U' \cap J')\} \geq \\ \min\{1 + \operatorname{sdepth}_{S'}\ J'/(J' \cap U'), \operatorname{depth}_{S'}\ (U' \cap J')\}\} \geq \\ \min\{1 + \operatorname{depth}_{S'}\ J'/(J' \cap U'), \operatorname{depth}_S\ (U \cap J) - 1\} = \\ \min\{\operatorname{depth}_S\ J/(U \cap J), \operatorname{depth}_S\ S/(U \cap J)\} = \operatorname{depth}_S\ J/I\end{gathered}$$

using [3, Lemma 3.6]. □

Theorem 2.11. *The Weak Conjecture holds in $S = K[x_1, \dots, x_5]$.*

For the proof note that Lemma 2.10 gives what is necessary in the proof of Theorem 2.8 to pass from S_4 to S_5.

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