Completely irreducible meet decompositions in lattices, with applications to Grothendieck categories and torsion theories (I)

by

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-Dedicated to the memory of Mark L. Teply (1942 - 2006)-

Abstract

The aim of this paper, consisting of two parts, is to investigate decompositions of elements in upper continuous modular lattices as intersections of (completely) irreducible elements. Thus, we extend from modules to such lattices the results of J. Fort [Math. Z. 103 (1967), 363-388] and consider a similar setting where irreducible submodules are replaced by completely irreducible submodules. We also extend from ideals to lattices some results of L. Fuchs, W. Heinzer, and B. Olberding [Trans. Amer. Math. Soc. 358 (2006), 3113-3131] concerning decompositions of ideals in arbitrary commutative rings as (irredundant) intersections of completely irreducible ideals. Applications of these results are given to Grothendieck categories and module categories equipped with a hereditary torsion theory.

Key Words: subdirectly irreducible poset, irreducible element, completely irreducible element, coirreducible (uniform) element, completely coirreducible element, upper continuous lattice, modular lattice, irreducible decomposition, irredundant irreducible decomposition, completely irreducible decomposition, irredundant completely irreducible decomposition, atomic lattice, weakly atomic lattice, strongly atomic lattice, semi-Artinian lattice, poset rich in subdirectly irreducibles, lattice rich in completely irreducibles, lattice rich in coirreducibles, lattice rich in completely coirreducibles, Krull dimension, dual Krull dimension, Gabriel dimension, Grothendieck category, torsion theory.

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Introduction

The aim of this paper is two-fold: first, to extend from modules to lattices the results of Fort [8] concerning modules rich in coirreducibles and to consider a similar setting where irreducible submodules are replaced by completely irreducible submodules, obtaining so the concept of lattice rich in completely coirreducibles, and second to extend from ideals to lattices some results of Fuchs, Heinzer, and Olberding [10] concerning decompositions of ideals in arbitrary commutative rings as (irredundant) intersections of completely irreducible ideals.

In doing so, we were inspired by the ideal or module theoretical situation, where the concept of essential ideal or submodule plays a key role. Though (meet) decompositions of elements in lattices have been extensively studied in the literature (see, e.g., Crawley and Dilworth [6], Erné [7], Walendziak [21], etc.) it is surprising that the concept of essential element was not yet involved.

The paper consists of 3 sections. Section 0 contains the basic definitions and facts, needed in the sequel, on subdirectly irreducible posets and completely irreducible elements in lattices.

In Section 1 we first extend from modules to upper continuous modular lattices the main result of Fort [8] concerning the characterization of modules $M_R$ rich in coirreducibles by means of irredundant irreducible decompositions of 0 in any submodule of $M$. Then, we consider a similar problem by replacing coirreducible submodules with completely coirreducible elements. It turns out that the lattices having this property, we called lattices rich in completely coirreducibles, are exactly the atomic lattices. Then, we extend from ideals to lattices some results of Fuchs, Heinzer, and B. Olberding [10] concerning decompositions of ideals in arbitrary commutative rings as (irredundant) intersections of completely irreducible ideals.

Section 2 contains applications of the obtained latticial results to Grothendieck categories and module categories equipped with a hereditary torsion theory.

0 Subdirectly irreducible posets and completely irreducible elements in lattices

In this section we present the basic terminology and results, needed in the sequel, on subdirectly irreducible posets and completely irreducible elements in lattices.

All posets considered in this paper are assumed to have a least element denoted by 0 and a last element denoted by 1. For a poset $P$ and elements $a \leq b$ in $P$ we write

$$b/a := [a, b] = \{ x \in P \mid a \leq x \leq b \},$$
$$[a, b[ := \{ x \in P \mid a \leq x < b \},$$
$$]a, b] := \{ x \in P \mid a < x \leq b \}.$$
An initial subinterval (resp. a quotient interval) of \( b/a \) is any interval \( c/a \) (resp. \( b/e \)) for some \( c \in b/a \). If \( x < y \) are elements of a poset \( P \) and there is no \( z \in P \) such that \( x < z < y \), then we say that \( x \) is covered by \( y \) (or \( y \) covers \( x \)), and we write \( x \prec y \) (or \( y > x \)). We say that the interval \( b/a \) is simple if \( a \prec b \).

An element \( a \in P \) is said to be an atom of \( P \) if the interval \( a/0 \) is simple, or equivalently, if \( 0 \prec a \). We denote by \( \mathcal{A}(P) \) the set, possibly empty, of all atoms of \( P \). A coatom of \( P \) is an element \( b \in P \) such that \( b \prec 1 \), i.e., a maximal element of \( P \setminus \{1\} \). We denote by \( \mathcal{A}^0(P) \) the set, possibly empty, of all coatoms of \( P \).

We denote by \( \mathcal{L} \) (resp. \( \mathcal{M}, \mathcal{C}, \mathcal{U} \)) the class of all lattices with 0 and 1 (resp. modular lattices with 0 and 1, complete lattices, upper continuous lattices). Throughout this paper a lattice will always mean a member of \( \mathcal{L} \), and \((\mathcal{L}, \leq, \wedge, \vee, 0, 1)\), or more simply, just \( L \), will always denote such a lattice. If \( L \in \mathcal{C} \), then for every subset \( S \) of \( L \) we denote \( \bigwedge S = \bigwedge_{x \in S} x \) and \( \bigvee S = \bigvee_{x \in S} x \). An element \( e \) of a lattice \( L \) is said to be essential in \( L \) if \( e \wedge x \neq 0 \) for each \( 0 \neq x \in L \). If \( L \in \mathcal{C} \), then the socle \( \text{Soc}(L) \) of \( L \) is the join of all atoms of \( L \). As in Năstăescu and Van Oystaeyen \[18\], a lattice \( L \) is said to be semi-Artinian if for any \( 1 \neq x \in L \), the lattice \( 1/x \) has at least an atom. As in Crawley and Dilworth \[6\], a poset \( P \) is said to be atomic (resp. strongly atomic, resp. weakly atomic) if for every \( 0 \neq x \in P \) there exists an atom \( a \in P \) such that \( a \leq x \) (resp. for every \( x < y \) in \( P \) the interval \( y/x \) contains an atom, resp. for every \( x < y \) there exist \( a, b \in P \) such that \( x \leq a \prec b \leq y \)). A lattice \( L \in \mathcal{U} \cap \mathcal{M} \) is strongly atomic if and only if it is semi-Artinian (see, e.g., Năstăescu and Van Oystaeyen \[18\] Proposition 1.9.3)]. As in Năstăescu and Van Oystaeyen \[18\], for a lattice \( L \in \mathcal{C} \) we define the radical \( r_L = \bigwedge_{b \in \mathcal{A}(L)} b \) of \( L \) as the meet of all coatoms of \( L \), putting \( r_L = 1 \) if \( L \) has no coatoms.

For all undefined notation and terminology on lattices, the reader is referred to Birkhoff \[5\], Crawley and Dilworth \[6\], Grätzer \[13\], and/or Stenström \[19\].
\[ \exists x_0 \in M, \forall N \in \text{Mod}_R, \forall g \in \text{Hom}_R(M, N) \text{ with } x_0 \notin \text{Ker}(g) \implies g \text{ is a monomorphism.} \]

To the best of our knowledge, the notion of cocyclic module appears for the first time in the literature in Fuchs [9, Section 3].

The next result provides various characterizations of cocyclic modules, that will naturally lead below (see Definition 0.2) to the most general concept of subdirectly irreducible poset. See Proposition 0.5 for an extension of these characterizations to complete lattices.

**Proposition 0.1.** The following statements are equivalent for a nonzero module \( M \).

1. \( M \) is cocyclic.
2. \( \bigcap_{0 \neq X \subseteq M} X \neq 0. \)
3. The poset \( \mathcal{L}(M) \setminus \{0\} \), ordered by inclusion, has a least element.
4. \( M \) has a simple essential socle.
5. \( M \) is subdirectly irreducible.

**Proof:** See Wisbauer [22, 14.8]. \( \square \)

Recall that a module \( M_R \) is called **subdirectly irreducible** if any representation of \( M \) as a **subdirect product** of other modules is trivial, i.e., for every family \((M_i)_{i \in I}\) of right \( R \)-modules and for every monomorphism \( \varepsilon : M \rightarrow \prod_{i \in I} M_i \) such that \( \pi_j \circ \varepsilon \) is an epimorphism \( \forall j \in I, \exists i \in I \) such that \( \pi_i \circ \varepsilon \) is an isomorphism, where \( \pi_j : \prod_{i \in I} M_i \rightarrow M_j, j \in I \), are the canonical projections.

Obviously, for any module \( M_R \) we have

\[ M = \sum_{C \in \mathcal{C}(M)} C \]

where \( \mathcal{C}(M) := \{ C \subseteq M \mid C \text{ is cyclic} \} \).

Dually, for any module \( M_R \) we have the following less obvious fact (see, e.g., Wisbauer [22, 14.9] or Remarks 0.15 (2))

\[ 0 = \bigcap_{X \in \mathcal{S}(M)} X \]

where \( \mathcal{S}(M) := \{ X \subseteq M \mid M/X \text{ is cocyclic} \} \).

In the sequel we present some definitions and results on subdirectly posets and lattices from Albu, Iosif, and Teply [2] along with some new ones.
Definitions 0.2. (Albu, Iosif, and Teply [2]). (a) A poset $P$ is said to be subdirectly irreducible (or colocal), abbreviated SI, if $P \neq \{0\}$ and the set $P \setminus \{0\}$ has a least element; i.e., there exists an element $0 \neq x_0 \in P$ such that $x_0 \leq x$ for every $0 \neq x \in P$.

(b) An element $s \in P$ is said to be a subdirectly irreducible element of $P$ if the interval $1/s$ is a subdirectly irreducible poset; in this case, the interval $[s, 1]$ has a (unique) least element, denoted by $s^*$ and called the cover of $s$.

(c) Dually, a poset $P$ is said to be local (or cosubdirectly irreducible) if $P \neq \{1\}$ and the set $P \setminus \{1\}$ has a last element; i.e., there exists an element $1 \neq x_1 \in P$ such that $x \leq x_1$ for every $1 \neq x \in P$. □

By Proposition 0.1, a module $M_R$ is subdirectly irreducible (resp. local) if and only if the lattice $\mathcal{L}(M_R)$ of all submodules of $M_R$ is subdirectly irreducible (resp. local).

For any poset $P$ we denote by $S(P)$ the set of all subdirectly irreducible elements of $P$; i.e.,

$$S(P) := \{ x \in P \mid 1/x \text{ is SI} \}.$$

For any module $M_R$ we set $S(M_R) := S(\mathcal{L}(M_R))$. Notice that this set has been denoted in Albu and Rizvi [4] by $\mathcal{C}^0(M_R)$.

Definitions 0.3. (a) A lattice $L$ is said to be coirreducible (or uniform) if $L \neq \{0\}$ and $x \wedge y \neq 0$ for any nonzero elements $x, y \in L$.

(b) A lattice $L$ is said to be completely coirreducible (or completely uniform) if $L \neq \{0\}$ and $\bigwedge_{i \in I} x_i \neq 0$ for any nonempty family $(x_i)_{i \in I}$ of nonzero elements $x_i \in L$, or shortly, if $\bigwedge S \neq 0$ for every $\emptyset \neq S \subseteq L \setminus \{0\}$.

(c) An element $x$ of a lattice $L$ is said to be meet irreducible, or just irreducible, if $x \neq 1$ and whenever $x = a \wedge b$ for $a, b \in L$, then $x = a$ or $x = b$.

(d) An element $x$ of a lattice $L$ is said to be completely meet irreducible, or just completely irreducible, abbreviated CI, if $x \neq 1$ and whenever $x = \bigwedge_{i \in I} a_i$ for a nonempty family $(a_i)_{i \in I}$ of elements of $L$, then $x = a_j$ for some $j \in I$, or shortly, if $x \neq 1$ and whenever $x = \bigwedge S$ for some $\emptyset \neq S \subseteq L$, then necessarily $x \in S$. □

Notice that sometimes in the literature the last element 1 of a lattice is considered to be (completely) irreducible. However, in this paper, any (completely) irreducible element of any lattice is always assumed to be $\neq 1$. Clearly, an element $x \in L$ is irreducible (resp. completely irreducible) if and only if the lattice $1/x$ is coirreducible (resp. completely coirreducible). For any lattice $L$ we denote by $\mathcal{I}(L)$ the set of all irreducible elements of $L$, and by $\mathcal{I}^c(L)$ the set of all completely irreducible elements of $L$. For any module $M_R$ we set $\mathcal{I}(M_R) := \mathcal{I}(\mathcal{L}(M_R))$ and $\mathcal{I}^c(M_R) := \mathcal{I}^c(\mathcal{L}(M_R))$.

Remarks 0.4. (1) The set $\mathcal{I}^c(L)$ may be empty; for instance, take as $L$ the interval $[0, 1]$ of real numbers. However, for any nonzero module $M_R$ we have $\mathcal{I}^c(M_R) \neq \emptyset$ (see, e.g., Albu and Rizvi [4] Lemma 0.2)].
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(2) Clearly, for any lattice $L$ we have $I^c(L) \subseteq I(L)$. In general, the inclusion $I^c(L) \subseteq I(L)$ may be strict. Indeed, if $L$ is the interval $[0, 1]$ of the set $\mathbb{R}$ of real numbers, then $I^c(L) = \emptyset$ and $I(L) = L$.

(3) The set $I(L)$ may be also empty (see Albu, Iosif, and Teply [2, Remarks 0.3 (3)]). However, for any nonzero module $M_R$ we have $I(M_R) \neq \emptyset$ by (1) and (2).

(4) If $L \in \mathcal{C}$, then clearly $s \in L$ is a subdirectly element of $L$ if and only if $s$ is completely irreducible, so $S(L) = I^c(L)$. In the sequel, for the term of subdirectly irreducible element of any lattice, we will always use the term of completely irreducible element. □

Proposition 0.5. The following statements are equivalent for a lattice $L \in \mathcal{C}$, $L \neq \{0\}$.

(1) $L$ is subdirectly irreducible.

(2) $\bigwedge_{x \in L \setminus \{0\}} x \neq 0$.

(3) $L$ is completely coirreducible.

(4) $L$ has an atom $a$ such that $a \leq x, \forall x \in L \setminus \{0\}$.

(5) $L$ has an atom $a$ that is essential in $L$.

(6) $L$ is coirreducible and $\text{Soc}(L) \neq 0$.

Proof: (1) $\implies$ (2): If $x_0$ is the least element of $L \setminus \{0\}$, then $0 \neq x_0 \leq \bigwedge_{x \in L \setminus \{0\}} x$.

(2) $\iff$ (3) is clear.

(3) $\implies$ (4): If $a := \bigwedge_{x \in L \setminus \{0\}} x$, then $a$ is an atom, and $a \leq x, \forall x \in L \setminus \{0\}$.

(4) $\implies$ (5): For any $x \in L \setminus \{0\}$, we have $a \leq x$, so $x \wedge a = a \neq 0$, i.e., $a$ is essential in $L$.

(5) $\implies$ (6): Let $x, y \in L \setminus \{0\}$. Then $a \wedge x \neq 0$ and $a \wedge y \neq 0$ since $a$ is essential in $L$, so $a \wedge x = a$ and $a \wedge y = a$ since $a$ is an atom of $L$. It follows that $a \leq x$ and $a \leq y$, which implies that $0 \neq a \leq x \wedge y$, i.e., $L$ is coirreducible. Clearly $a \leq \text{Soc}(L)$, so $\text{Soc}(L) \neq 0$.

(6) $\implies$ (1): Since $\text{Soc}(L) \neq 0$, $L$ has at least an atom, say $a$. We claim that $a$ is the single atom of $L$; indeed, if $a'$ is another atom of $L$, then $a \wedge a' \neq 0$ since $L$ is coirreducible, so $a \wedge a' = a = a'$. It follows that $\text{Soc}(L) = \{a\}$. For every $x \in L \setminus \{0\}$ we have $0 \neq x \wedge a \leq a$, so $x \wedge a = a$, and then $a \leq x$. This shows that $a$ is the least element of $L \setminus \{0\}$; hence $L$ is subdirectly irreducible. □
Corollary 0.6. If $L \in \mathcal{C}$ is a semi-Artinian lattice, in particular a strongly atomic lattice, then $\mathcal{T}(L) = \mathcal{T}^c(L)$.

Proof: If $x \in \mathcal{T}(L)$, then the lattice $1/x$ is coirreducible, and $\text{Soc}(1/x) \neq 0$ since $L$ is semi-Artinian lattice, so $1/x$ is a subdirectly irreducible lattice by Proposition 0.5, i.e., $x \in \mathcal{T}^c(L)$, as desired.

Proposition 0.7. The following statements are equivalent for $L \in \mathcal{C}$ and $1 \neq c \in L$.

(1) $c$ is completely irreducible in $L$.

(2) There exists $0 \neq x_0 \in L$ such that $c$ is maximal with respect to $x_0 \not\leq c$.

Proof: (1) $\implies$ (2): Since the lattice $L$ was assumed to be complete, we can consider its element $y_0 := \bigwedge_{x > c} x$. Then $y_0 > c$ since $c$ is CI, so $y_0 \not\leq c$. Now, $c$ is maximal with respect to $y_0 \not\leq c$, since for any $d > c$ we have $y_0 \leq d$ by the definition of $y_0$. Thus (2) holds with $x_0$ as $y_0$.

(2) $\implies$ (1): If we consider again the element $y_0 := \bigwedge_{x > c} x$ of $L$, then clearly $y_0 \geq c$. In order to prove that $c$ is CI, i.e., $1/c$ is a SI lattice, by Proposition 0.5, we have to show that $y_0 \neq c$. We have $x_0 \leq x$ for every $x > c$ by the maximality condition of $c$ in (2). This implies that $x_0 \leq y_0$. We cannot have $y_0 = c$ since then $x_0 \leq c$, which contradicts (2).

Observe that if $x_0 \neq 0$ is any compact element of $L \in \mathcal{C}$, then the set $L_0 = \{ x \in L \mid x_0 \not\leq x \}$ is inductive, so the Zorn’s Lemma can be applied to find a maximal element $c$ of $L_0$, which is completely irreducible by Proposition 0.7.

The concept of a lattice rich in subdirectly irreducibles, indispensable in the evaluation of the dual Krull dimension of lattices, has been introduced by Albu, Iosif, and Teply[2]. The definitions below are slight variations of it.

Definition 0.8. (a) A poset $P$ is said to be rich in subdirectly irreducibles, abbreviated RSI, if for every $a < b$ in $P$, the interval $b/a$ has a subdirectly irreducible quotient interval $b/c$, with $a \leq c < b$.

(b) A poset $P$ is said to be weakly rich in subdirectly irreducibles, abbreviated WRSI, if for every $a \neq 1$ in $P$, the interval $1/a$ has a subdirectly irreducible quotient interval $1/c$ with $a \leq c$.

(c) A lattice $L$ is called rich in completely irreducibles, abbreviated RCI (resp. weakly rich in completely irreducibles, abbreviated WRCI) if the poset $L$ is RSI (resp. WRSI).
Of course, any RSI poset is also WRSI but not conversely: indeed, the subset $[0, 1/2] \cup \{1\}$ of $\mathbb{R}$ is WRSI but not RSI. Observe that a poset $P$ is RSI if and only if the poset $b/0$ is WRSI for any $0 \neq b \in P$.

For the concepts of Krull dimension, dual Krull dimension, and Gabriel dimension of a lattice the reader is referred to Năstăsecu and Van Oystaeyen [13].

**Proposition 0.9.** (Albu, Iosif, and Teply [2, Proposition 1.2]). The following assertions are equivalent for $L \in \mathcal{U} \cap \mathcal{M}$.

1. $L$ is RCI.
2. For each $a < b$ in $L$ there exist $x < y$ in $b/a$ such that $y/x$ is simple, in other words, $L$ is weakly atomic.
3. For each $a < b$ in $L$ there exist $x < y$ in $b/a$ such that $y/x$ is compact.
4. For each $a < b$ in $L$ there exist $x < y$ in $b/a$ such that $y/x$ is compactly generated.
5. For each $a < b$ in $L$ there exist $x < y$ in $b/a$ such that $y/x$ has (dual) Krull dimension.
6. For each $a < b$ in $L$ there exist $x < y$ in $b/a$ such that $y/x$ has Gabriel dimension. □

**Corollary 0.10.** (Albu, Iosif, and Teply [2, Corollary 1.3]). Let $L \in \mathcal{U} \cap \mathcal{M}$. If $L$ has Gabriel dimension, then $L$ is RCI. In particular, if $L$ is Artinian, semi-Artinian, Noetherian, or has (dual) Krull dimension, then $L$ is RCI. □

**Corollary 0.11.** (Albu, Iosif, and Teply [2, Corollary 1.4]). Any compactly generated lattice $L \in \mathcal{U} \cap \mathcal{M}$ is RCI. □

**Remarks 0.12.** (1) The poset reduced to 0 is by definition RSI.

(2) Clearly, we can express equivalently the property of a poset $P$ being RSI as follows: $S(b/a) \neq \emptyset$ for every $a < b$ in $P$. Thus, if $P \neq \{0\}$ is SI, then $S(P) \neq \emptyset$, but not conversely; if $P = [0, 1/2] \cup \{1\} \subseteq \mathbb{R}$, then $S(P) = \{1/2\} \neq \emptyset$, but $P$ is not RSI since $S([0, 1/2]) = \emptyset$.

(3) For any module $M_R$, the lattice $\mathcal{L}(M)$ is RCI by Albu and Rizvi [1] Lemma 0.2 or by Corollary 0.11.

(4) Clearly, any strongly atomic lattice is RCI, but not conversely. Indeed, if $M_R$ is a module that is not semi-Artinian, then the lattice $\mathcal{L}(M)$ is RCI by (3) but is not semi-Artinian, or equivalently, not strongly atomic.

(5) Any Noetherian poset $P$ is RSI since for any $a < b$ in $P$, the interval $[a, b]$ contains a maximal element $c$, and so, the interval $b/c$ is simple, in particular SI.

(6) Clearly, any RSI poset is also weakly atomic, but not conversely. Indeed, let

$$L = \{ x_{n,k_n} \mid n \in \mathbb{N}^*, \; 1 \leq k_n \leq 2^n \} \cup \{0\} \cup \{1\}$$
be the lattice whose Hasse diagram is indicated below:

![Hasse diagram](image)

Then $L$ is an Artinian lattice, so also weakly atomic, which has no irreducible element, so no CI element too; thus, $L$ is not WRCI, and, a fortiori, not RCI. Observe that $L$ is neither upper continuous nor semi-modular.

(7) Examples involving torsion theories of lattices that are RCI are provided in Albu, Iosif, and Teply [2, Section 2].

(8) An example of an RCI upper continuous modular lattice that is not compactly generated is the following one given in Crawley and Dilworth [6] p. 16:
let \( \mathbb{N} = \mathbb{N} \cup \{ \infty \} \) be the chain of all natural numbers with a largest element \( \infty \) adjoined, and let \( L \) be the set of all functions \( f : \mathbb{N} \rightarrow \mathbb{N} \) such that \( f(n) = \infty \) for all but finitely many \( n \in \mathbb{N} \). With the partial order in \( L \) defined by \( f \leq g \iff f(n) \leq g(n), \forall n \in \mathbb{N} \), \( L \) becomes an upper continuous modular lattice having the following two properties: the interval \( 1/f \) of \( L \) is an atomic lattice for all \( f \neq o \), and \( o \) is the only compact element of \( L \), where \( o \) is the zero function, \( o(n) = 0 \) for all \( n \in \mathbb{N} \). So, \( L \) is RCI but not compactly generated. It would be interesting to find such an example based on lattices of type \( \text{Sat}_\tau(M) \) for a certain hereditary torsion theory \( \tau \) on the category \( \text{Mod-R} \) and a module \( M_R \). A possible candidate for that could be the example in Albu, Iosif, and Teply [2, Remarks 2.6 (3)].

(9) Examples from general topology of weakly or strongly atomic lattices that are upper continuous and distributive but not compactly generated are provided in Erné [7].

**Proposition 0.13.** (Albu, Iosif, and Teply [2, Lemma 1.6]). The following statements are equivalent for a nonzero lattice \( L \in U \cap M \).

1. The lattice \( L \) is RCI.
2. For every \( a < b \) in \( L \), one has \( a = \bigwedge_{x \in I^-(b/a)} x \).

In particular, if \( L \) is RCI, then \( 0 = \bigwedge_{x \in I^-(1)} x \). □

**Definition 0.14.** A lattice \( L \) is said to be with completely irreducible decomposition, abbreviated CID, if every \( 1 \neq a \in L \) can be written as a meet of a family of CI elements of \( L \), or equivalently \( a = \bigwedge_{x \in I^-(1/a)} x \). □

**Remarks 0.15.** (1) Consider the subset \( L = \{ 0 \} \cup [1/2, 1] \) of \( \mathbb{R} \). Then \( 0 \) is the only CI element of the lattice \( L \), but for every \( 0 < a < b \) in \( L \), the interval \([a, b]\) has no CI elements; in particular, the lattice \( L \) is not RCI. This example shows that a lattice \( L \in U \cap M \) may satisfy the property \( 0 = \bigwedge_{x \in I^-(1/a)} x \) of Proposition 0.13 without being necessarily RCI.

(2) By Remarks 0.12 (3) and Proposition 0.13, any proper submodule of any module \( M_R \) is an intersection of CI submodules of \( M \).

(3) Proposition 0.13 can be expressed by saying that a lattice \( L \) is RCI if and only if, for every \( 0 \neq b \in L \), the lattice \( b/0 \) is with CID. In particular, any RCI lattice is a lattice with CID. The converse may be not true. Indeed, consider the following example due to Erné [7]: let

\[
L := \{ (x, y) \mid x \in [0, 1], x + y \leq 1 \} \cup \{(1, 1)\},
\]

where \([0, 1]\) is the unit interval in the set \( \mathbb{R} \) of all real numbers. Then \( L \), ordered componentwise by the usual \( \leq \) relation, is a complete semi-modular lattice. Since the only covering pairs in \( L \) are \((x, 1-x) \prec (1, 1), x \in [0, 1]\), it follows that \( L \) is not weakly atomic; so it is not RCI too. However, every element of \( L \) can be
written as an irredundant intersection of at most two coatoms, so, \( L \) is a lattice with CID.

(4) Observe that if \( L \) is a lattice with CID, and \( a, b \in L \), then \( b \not\leq a \) if and only if there exists a CI element \( c \in L \) such that \( a \leq c \) and \( b \not\leq c \). Indeed, assume that for any CI element \( c \in L \) such that \( a \leq c \) we also have \( b \leq c \). Since \( L \) is with CID, we can write \( a = \bigwedge_{i \in I} c_i \) with CI irreducible elements \( c_i, i \in I \). Then \( b \leq c_i, \forall i \in I \), so \( b \leq \bigwedge_{i \in I} c_i = a \). This shows the nontrivial implication.

(5) In the next section we will discuss when a decomposition of an element of a lattice \( L \in U \cap M \) with CID as a meet of irreducible/completely irreducible elements is irredundant or unique. □

As in Walendziak [21], a lattice \( L \) is said to satisfy the condition \((P)\) (resp. \((M)\)) if for any two elements \( a \prec b \) in \( L \), the set \( P(a, b) = \{ x \in L \mid a = b \wedge x \} \) has a maximal element (resp. if for every \( c \in T^e(L) \) and for every \( a \in L \) with \( a \not\leq c \) one has \( c \wedge a \in T^e(a/(a \wedge c)) \)). Of course, the condition \((M)\) can be expressed equivalently as follows: for every \( c \in L \) with \( 1/c \) SI and for every \( a \in L \) with \( a \not\leq c \), the interval \( a/(a \wedge c) \) is SI. Clearly, any upper continuous lattice satisfies the condition \((P)\), and any modular lattice satisfies the condition \((M)\).

The existence of completely irreducible decompositions in complete lattices that are more general than upper continuous modular lattices is given by the following result.

**Proposition 0.16.** (Walendziak [21] Theorem 1, Theorem 2, Corollary 2). If \( L \in C \) satisfies the conditions \((P)\) and \((M)\), then \( 1 \not= a \in L \) has a decomposition into CI elements if and only if for every \( x > a \) in \( L \) there are \( u, v \in x/a \) with \( u \prec v \). In particular, a complete lattice satisfying the conditions \((P)\) and \((M)\) is a lattice with CID if and only if it is weakly atomic. □

We end this section with an extension of Proposition 0.13 to complete lattices satisfying the condition \((P)\).

**Proposition 0.17.** The following statements are equivalent for a nonzero lattice \( L \in C \) satisfying the condition \((P)\).

1. The lattice \( L \) is RCI.
2. For every \( a < b \) in \( L \), one has \( a = \bigwedge_{x \in T^e(b/a)} x \).

**Proof:** (1) \( \Rightarrow \) (2): Observe first that if a lattice \( L \) is RCI, then so is any of its intervals \( [x, y] \), and \( T^e(L) \neq \emptyset \iff L \neq \{0\} \). Therefore, it is sufficient to prove only that \( 0 = \bigwedge_{x \in T^e(L)} x \) if the lattice \( L \) is RCI. Set \( y = \bigwedge_{x \in T^e(L)} x \), and assume that \( y \neq 0 \). Since \( L \) is RCI, there exist \( a \prec b \) in \( y/0 \), and since \( L \) satisfies the condition \((P)\), there exists an element \( m \in P(a, b) := \{ x \in L \mid a = b \wedge x \} \).
which is maximal in $P(a,b)$, and then it is necessarily CI by Erné [7, Lemma]. So, $m \geq y \geq b$, and hence $a = m \land b = b$, which is a contradiction. Consequently, $y = 0$, and we are done.

(2) $\implies$ (1): Since $\bigwedge \emptyset = 1$, we have $T^c(b/a) \neq \emptyset$ for any $a < b$ in $L$, which says exactly that $L$ is RCI.

Observe that condition (2) in Proposition 0.17 expresses the fact that any element $a \in L$ has a decomposition in CI elements in any interval $[a, b]$ of $L$, $a < b$, and this does not require the condition $(M)$. 

1 Lattices rich in coirreducibles/completely coirreducibles, and irredundant meet decompositions

The aim of this section is two-fold. First we extend from modules to upper continuous modular lattices the main result of Fort [8] concerning the characterization of modules $M_R$ rich in coirreducibles by means of irredundant irreducible decompositions of 0 in any submodule of $M$. Then, we consider a similar problem by replacing coirreducible submodules with subdirectly irreducible submodules. It turns out that the lattices having this property, we called lattices rich in completely coirreducibles, are exactly the atomic lattices. Note that the existence of irredundant meet decompositions has been explored in the literature, also for lattices that are not necessarily modular or upper continuous (see, e.g., Crawley and Dilworth [6], Stern [20], Walendziak [21], etc.), but no connection with essential elements has been so far considered.

As in Fort [8], a module $M_R$ is said to be rich in coirreducibles, abbreviated RC (or rich in uniforms, abbreviated RU), if $M \neq 0$ and every of its nonzero submodules contains a coirreducible (or uniform) submodule. The next result characterizes RC modules.

**Theorem 1.1.** (Fort [8, Théorème 3]). The following statements are equivalent for a nonzero module $M_R$.

(1) $M$ is RC.

(2) $M$ is an essential extension of a direct sum of coirreducible submodules of $M$.

(3) The injective hull $E_R(M)$ of $M$ is an essential extension of a direct sum of indecomposable injective modules.

(4) $0$ has an irredundant irreducible decomposition in every nonzero submodule of $M$. 

Our first aim is to extend the characterization above from modules to upper continuous modular lattices, and then, to consider a similar problem, where irredundant irreducible decompositions are replaced by irredundant completely irreducible decompositions.

For brevity, we say that an element \( c \) of a lattice \( L \) is \textit{coirreducible} (or uniform) if the interval \( c/0 \) is coirreducible, that is, if \( c \neq 0 \) and \( x \wedge y \neq 0 \) for all \( x, y \in ]0, c[ \). We denote by \( \mathcal{C}(L) \) the set, possibly empty, of all coirreducible elements of \( L \). Similarly, we say that an element \( s \in L \) is \textit{completely coirreducible} (or \textit{cocompletely irreducible}, or \textit{cosubdirectly irreducible}) if the interval \( s/0 \) is subdirectly irreducible, that is, if \( s \neq 0 \) and \( \bigwedge_{i \in I} x_i \neq 0 \) for any nonempty family \( (x_i)_{i \in I} \) of elements with \( x_i \in ]0, s[ \), \( i \in I \). We denote by \( \mathcal{C}^c(L) \) the set, possibly empty, of all completely coirreducible elements of \( L \). Clearly \( \mathcal{A}(L) \subseteq \mathcal{C}^c(L) \subseteq \mathcal{C}(L) \). Observe that if \( 0 < x \leq y \) are elements in \( L \) and \( y \) is coirreducible (resp. completely coirreducible), then so is also \( x \). There exist nonzero modules \( M_R \) having no coirreducible submodules (see, e.g., Fort [3] Théorème 2), and for such \( M \) we have \( \mathcal{C}(L(M)) = \emptyset \).

\textbf{Definition 1.2.} A lattice \( L \) is said to be \textit{rich} in coirreducibles or rich in uniform, abbreviated RC or RU (resp. rich in completely coirreducibles or rich in completely uniforms, abbreviated RCC or RCU), if \( L \neq \{0\} \) and for any \( 0 \neq x \in L \) there exists \( c \in \mathcal{C}(L) \) (resp. \( c \in \mathcal{C}^c(L) \)) such that \( c \leq x \).

\textbf{Examples 1.3.} (1) Since any atom of a lattice is a coirreducible element, it follows that any atomic lattice is RCC. Conversely, since any subdirectly irreducible interval \( x/0 \) of any lattice \( L \) contains a (unique) atom, it follows that any RCC lattice is atomic. Consequently, a lattice is RCC if and only if it is atomic. In particular, any Artinian or semi-Artinian lattice is RCC.

(2) Any Noetherian lattice \( L \in \mathcal{M} \) is RC. To show that, observe first that any element \( 1 \neq x \in L \) can be written as a finite intersection of irreducible elements of \( L \) (see, e.g., Năstăsecu and Van Oystaeyen [18] Proposition 1.4.4). It follows that any element \( 1 \neq z \in L \) can be written as a finite irredundant intersection of irreducible elements of \( L \). Now, let \( 0 \neq x \in L \). Then \( x/0 \) is a Noetherian lattice, so we can write \( x = 0 \) as a finite irredundant intersection \( 0 = \bigwedge_{1 \leq i \leq n} x_i \) of irreducible elements of \( x/0 \). If \( n = 1 \), then \( 0 = x_1 \) is irreducible in \( x/0 \), so \( x \) is a coirreducible element. If \( n \geq 2 \), set \( y_1 := \bigwedge_{2 \leq i \leq n} x_i \). Since the decomposition \( 0 = \bigwedge_{1 \leq i \leq n} x_i \) is irredundant, we have \( y_1 \neq 0 \). But \( 0 = x_1 \wedge y_1 \), so \( y_1/0 = y_1/(x_1 \wedge y_1) \simeq (x_1 \vee y_1)/x_1 \subseteq x/x_1 \), and since the interval \( x/x_1 \) is coirreducible, it follows that so is also \( y_1/0 \), in other words, \( x \) contains the coirreducible element \( y_1 \), as desired.

(3) Let \( L \in \mathcal{U} \cap \mathcal{M} \). If \( L \) has Gabriel dimension \( g(L) \), in particular, if \( L \) is semi-Artinian or has (dual) Krull dimension, then \( L \) is RC. Indeed, let \( 0 \neq x \in L \). Then, the interval \( x/0 \) of \( L \) has Gabriel dimension \( g(x/0) > 0 \), hence it contains a Gabriel \( \gamma \)-simple interval \( c/0 \), \( 0 < c \leq x \), for some ordinal \( \beta \leq g(x/0) \) (see Năstăsecu and Van Oystaeyen [18] 3.4). We claim that \( c \) is
a coirreducible element of $L$, i.e., the interval $c/0$ is coirreducible. If not, then there would exist $a, b \in [0, c]$ such that $0 = a \wedge b$. Since $c/0$ is Gabriel $\gamma$-simple, we have $g(c/a) < \gamma$ and $g(c/b) < \gamma$. By modularity, we have $a/0 = a/(a \wedge b) \simeq (a \vee b)/b \subseteq c/b$, which implies that $g(a/0) \leq g(c/b) < \gamma$, hence $g(c/0) = \max\{g(c/a), g(a/0)\} < \gamma$, which is a contradiction. Consequently, for each $0 \neq x \in L$, there exists a $c \in L$, $0 < c \leq x$, such that $c$ is coirreducible, which means exactly that $L$ is RC, as desired.

(4) Clearly any RCC lattice is RC, but not conversely. Indeed, the lattice $\mathcal{L}(\mathbb{Z})$ of all subgroups of the Abelian group $\mathbb{Z}$ is RC by (2), but not RCC by (1) since the $\mathbb{Z}$-module $\mathbb{Z}$ does not contain any simple submodule.

In order to extend Theorem 1.1 from RC modules to RC lattices and also to consider the case of RCC lattices, we need the next lemmas, which are the latticial versions of the corresponding results used in its proof in Fort [8]. Note that their statements and proofs are given in a parallel manner, by replacing the word “irreducible” (resp. “coirreducible”) with “completely irreducible” (resp. “completely coirreducible”).

Recall that a subset $A$ of a lattice $L \in \mathcal{C}$ is said to be join independent, or just independent, if $0 \notin A$ and $a \wedge \bigvee (A \setminus \{a\}) = 0$ for all $a \in A$. If $L \in \mathcal{U}$, then $A \subseteq L$ is independent if and only if every finite subset of $A$ is independent. Alternatively, we say that a family $\{x_i\}_{i \in I}$ of elements of a lattice $L \in \mathcal{C}$ is independent if $x_i \neq 0$ and $x_i \wedge \bigvee_{j \in I \setminus \{i\}} x_j = 0$ for every $i \in I$, and in that case, necessarily $x_p \neq x_q$ for each $p \neq q$ in $I$; so, the two definitions of independence, using subsets or families of elements of $L$, are essentially the same.

**Lemma 1.4.** Let $L \in \mathcal{U} \cap \mathcal{M}$ and let $\{x_i\}_{i \in I}$ be an independent family of elements of $L$. If $y \in L$ is such that $y \wedge \bigvee_{i \in I} x_i \neq 0$, then there exist $i \in I$, $0 \neq x_i' \leq x_i$, and $0 \neq y' \leq y$ such that $x_i'/0 \simeq y'/0$.

**Proof:** Observe that the proof of Fort [8, Proposition 2] works not only for the lattice $\mathcal{L}(M_R)$ of all submodules of a module $M_R$, but also for any upper continuous modular lattice $L$.

It is known that for any upper continuous lattice $L$ and for any $\varnothing \neq B \subseteq L$, any independent subset $A$ of $B$ is contained in a maximal independent subset $S$ of $B$ (see e.g., Crawley and Dilworth [6, p.46]), and $\bigvee_{x \in S} x$ is called a maximal direct join of elements of $B$ and denoted by $\bigvee_{x \in S} x$. In particular, if $A = \varnothing$ and $B = \mathcal{C}(L)$ (resp. $B = \mathcal{C}(\ell)$, resp. $B = \mathcal{A}(L)$) then one obtains a maximal direct join of coirreducibles (resp. a maximal direct join of completely coirreducibles, resp. a maximal direct join of atoms) of $L$.

**Corollary 1.5.** Let $L \in \mathcal{U} \cap \mathcal{M}$, let $x \in L$, and let $m$ be a maximal direct join of coirreducibles (resp. a maximal direct join of completely coirreducibles) of $L$. Then $m \wedge x \neq 0$ if and only if there exists $y \in \mathcal{C}(L)$ (resp. $y \in \mathcal{C}(\ell)$) such that $y \leq x$.
Lemma 1.6. Let \( L \in \mathcal{U} \cap \mathcal{M} \). Then there exists \( t \in L \) which is maximal with respect to the property that there exists no \( y \in \mathcal{C}(L) \) (resp. \( y \in \mathcal{C}^c(L) \)) such that \( y \leq t \). If \( m \) is a maximal direct join of coirreducibles (resp. a maximal direct join of completely coirreducibles), then \( t \land m = 0 \) and \( t \lor m \) is an essential element of \( L \).

Proof: Let

\[ T := \{ u \in L \mid c \not\subseteq u, \forall c \in \mathcal{C}(L) \} \ (\text{resp. } T := \{ u \in L \mid c \not\subseteq u, \forall c \in \mathcal{C}^c(L) \}) \]

If \( \emptyset \neq D \subseteq T \) is a chain, then \( u_0 := \bigwedge_{d \in D} d \in T \), for otherwise \( u_0 \supseteq c \) for some \( c \in \mathcal{C}(L) \) (resp. \( c \in \mathcal{C}^c(L) \)), hence, by upper continuity, \( c = u_0 \land d = \bigwedge_{d \in D}(d \land c) = 0 ; \) indeed, if \( d \land c \neq 0 \) for some \( d \in D \), then \( d \supseteq d \land c \in \mathcal{C}(L) \) (resp. \( d \supseteq d \land c \in \mathcal{C}^c(L) \)), which is a contradiction. This shows that \( T \) is an inductive set, so, by Zorn’s Lemma, it has a maximal element, say \( t \), and \( t \land m = 0 \) by Corollary 1.5.

We are now going to show that \( t \lor m \) is an essential element of \( L \), i.e., if \( z \in L \) is such that \( (t \lor m) \land z = 0 \), then \( z = 0 \). We claim that \( (z \lor t) \land m = 0 \). Indeed, by modularity, we have

\[ (z \lor t) \land m \leq (z \lor t) \land (t \lor m) \leq ((z \land (t \lor m)) \lor t) = 0 \lor t = t. \]

On the other hand, \( (z \lor t) \land m \leq m \), so \( (z \lor t) \land m \leq t \land m = 0 \), which proves our claim.

By Corollary 1.5, \( z \lor t \) does not contain any coirreducible (resp. completely coirreducible) element of \( L \), so \( z \lor t \in T \). Since \( t \) is maximal in \( T \) it follows that \( t = z \lor t \). Then \( z \leq t \), and so \( 0 = (t \lor m) \land z = z \), as desired.

Lemma 1.7. Let \( L \in \mathcal{U} \cap \mathcal{M} \), and let \( x \leq y \). Then \( x \) is irreducible (resp. completely irreducible) in \( y/0 \) if and only if there exists an irreducible (resp. completely irreducible) element \( z \) in \( L \) such that \( x = z \land y \).

Proof: “\( \Rightarrow \)” Consider the set \( S := \{ u \in L \mid u \land y = x \} \). Since the lattice \( L \) is upper continuous, the set \( S \) is inductive, so, by Zorn’s Lemma, it has a maximal element, say \( z \). Assume that \( x \) is irreducible (resp. completely irreducible) in \( y/0 \), let \( I \) be a set of two elements (resp. arbitrary set), and let \( (z_i)_{i \in I} \) be a family of elements of \( L \) such that \( z = \bigwedge_{i \in I} z_i \). Then \( x = z \land y = \bigwedge_{i \in I}(z_i \land y) \).

By hypothesis, there exists \( j \in I \) such that \( x = z_j \land y \). Since \( z \) is maximal in \( S \), we deduce that \( z = z_j \), which proves that \( z \) is an irreducible (resp. completely irreducible) element in \( L \).

“\( \Leftarrow \)” Assume that \( z \) is an irreducible (resp. completely irreducible) element in \( L \). By modularity, we have \( y/x = y/(z \land y) \simeq (z \lor y)/z \subseteq 1/z \). Since the interval \( 1/z \) is coirreducible (resp. SI), so is also \( y/x \), in other words, \( x \) is irreducible (resp. completely irreducible) in \( y/0 \), as desired.
Definition 1.8. Let \( L \in \mathbb{C} \), and let \( x \in L \). An irreducible (meet) decomposition, abbreviated ID (resp. completely irreducible (meet) decomposition, abbreviated CID) of \( x \) in \( L \) is a family \( (x_i)_{i \in I} \) of irreducible (resp. completely irreducible) elements of \( L \) such that

\[
x = \bigwedge_{i \in I} x_i.
\]

We also say shortly that \( x = \bigwedge_{i \in I} x_i \) is an ID (resp. CID) of \( x \). The decomposition \( x = \bigwedge_{i \in I} x_i \) is said to be an irredundant irreducible decomposition, abbreviated IID (resp. irredundant completely irreducible decomposition, abbreviated ICID) if \( \bigwedge_{j \in I \setminus \{i\}} x_j \neq x_i \) for every \( i \in I \), in other words, none of the \( x_i \)'s can be omitted without changing the intersection.

Alternatively, using the notation \( \bigwedge C \) for \( \bigwedge_{c \in C} c \), we say that a representation \( x = \bigwedge A \) with \( \emptyset \neq A \subseteq I(L) \) (resp. \( \emptyset \neq A \subseteq I^c(L) \)) is an irreducible (meet) decomposition (resp. completely irreducible (meet) decomposition) of \( x \) in \( L \), which is called irredundant in case \( \bigwedge (A \setminus \{a\}) \neq x \) for every \( a \in A \).

\[\square\]

Lemma 1.9. Let \( x \) be an essential element of a lattice \( L \in \mathcal{U} \cap \mathcal{M} \). Then \( 0 = \bigwedge_{i \in I} x_i \) is an IID (resp. ICID) of 0 in \( x/0 \) if and only if there exists a family \( (y_i)_{i \in I} \) of elements of \( L \) such that \( x_i = x \land y_i, \forall i \in I \), and \( 0 = \bigwedge_{i \in I} y_i \) is an IID (resp. ICID) of 0 in \( L \).

Proof: “\( \Rightarrow \)” : Assume that \( 0 = \bigwedge_{i \in I} x_i \) is an IID (resp. ICID) of 0 in \( x/0 \). Then, by Lemma 1.7, there exists a family \( (y_i)_{i \in I} \) of irreducible (resp. completely irreducible) elements of \( L \) such that \( x_i = x \land y_i, \forall i \in I \). We claim that \( 0 = \bigwedge_{i \in I} y_i \) is an IID (resp. ICID) of 0 in \( L \). First, \( \bigwedge_{i \in I} y_i = 0 \) since \( (\bigwedge_{i \in I} y_i) \land x = \bigwedge_{i \in I} x_i = 0 \) and \( x \) is an essential element of \( L \). Second, the decomposition \( \bigwedge_{i \in I} y_i = 0 \) is irredundant, for otherwise, \( y_i \ni \bigwedge_{j \in I \setminus \{i\}} y_j \) for some \( i \in I \), hence \( x_i \ni x_j \), which is a contradiction.

“\( \Leftarrow \)” can be proved using similar arguments. \[\square\]

Lemma 1.10. The following statements are equivalent for a nonzero lattice \( L \in \mathcal{U} \cap \mathcal{M} \).

1. \( L \) is atomic.
2. \( L \) is RCC.
3. \( \text{Soc}(L) \) is essential in \( L \).

Proof: (1) \( \iff \) (2) has been shown in Examples 1.3 (1).

(1) \( \Rightarrow \) (3): First, we are going to show that \( \text{Soc}(L) \) coincides with any maximal direct join \( s \) of atoms of \( L \). Since, by definition, \( \text{Soc}(L) \) is the join of all atoms of \( L \), we clearly have \( s \leq \text{Soc}(L) \). Now, let \( a \in \mathcal{A}(L) \). Then \( a \land s \) is either 0 or \( a \). Since \( s \) is a maximal direct join of atoms, \( a \land s \neq 0 \), so
$a \land s = a$, i.e., $a \leq s$ for each $a \in \mathcal{A}(L)$, and then, $\text{Soc}(L) = \bigvee_{a \in \mathcal{A}(L)} a \leq s$, as desired. To show that $\text{Soc}(L)$ is essential in $L$, let $0 \neq x \in L$, and assume that $x \land \text{Soc}(L) = 0$. Let $a \in \mathcal{A}(L)$ with $a \leq x$. Then $a \lor \text{Soc}(L)$ is a direct join of atoms, which contradicts the fact that $\text{Soc}(L)$ is a maximal direct join of atoms of $L$.

(3) $\implies$ (1): Let $0 \neq x \in L$. Then $x \land \text{Soc}(L) \neq 0$, and since $\text{Soc}(L)$ is a direct union $\bigvee_{i \in I} a_i$ of atoms, by Lemma 1.4, there exist $i \in I$, $0 \neq a'_i \leq a_i$ and $0 \neq x' \leq x$ such that $a'_i/0 \simeq x'/0$. But $a'_i = a_i$, so $x'$ is an atom contained in $x$, as desired.

Remarks 1.11. (1) For the implication (1) $\implies$ (3) in Lemma 1.10 we needed the condition that $L$ is upper continuous, while for the opposite implication, we required that $L \in \mathcal{U} \cap \mathcal{M}$. Note that the equivalence (1) $\iff$ (2) is valid for any poset.

(2) An atomic lattice is not necessarily strongly atomic. Indeed, if $F$ is any field, $I$ is any infinite set, and $R$ is the product ring $F^I$ of $I$ copies of $R$, then $\text{Soc}(R)$ is essential in $R$, but $R$ is not a semi-Artinian ring, so the lattice $\mathcal{L}(R)$ of all ideals of $R$ is atomic but not strongly atomic. □

Lemma 1.12. (Grzeszczuk and Puczi/suppress lowski [15, Lemma 1.4]). Let $L \in \mathcal{U} \cap \mathcal{M}$, and let $X$ be an independent subset of $L$. Then, $(\bigvee A) \land (\bigvee B) = 0$ for any disjoint subsets $A, B$ of $X$. □

Lemma 1.13. Let $L \in \mathcal{U} \cap \mathcal{M}$, and let $X$ be an independent subset of $L$. Then, for any $n \in \mathbb{N}$, $n \geq 2$, and for any finite family $(A_i)_{1 \leq i \leq n}$ of subsets of $X$, we have

$$\bigcap_{1 \leq i \leq n} (\bigvee A_i) = \bigvee (\bigcap_{1 \leq i \leq n} A_i).$$

In particular, if $\bigcap_{1 \leq i \leq n} A_i = \emptyset$, then $\bigcap_{1 \leq i \leq n} (\bigvee A_i) = 0$.

Proof: We proceed by induction on $n$. Let $n = 2$. If $A_1 \cap A_2 = \emptyset$, then the result is exactly Lemma 1.12. So, we may assume that $A_1 \cap A_2 \neq \emptyset$, and then, we can partition $A_1$ as $A_1 = (A_1 \setminus A_2) \cup (A_1 \cap A_2)$. By modularity, we have

$$(\bigvee A_1) \land (\bigvee A_2) = (\bigvee A_2) \land ((\bigvee (A_1 \cap A_2)) \lor (\bigvee (A_1 \setminus A_2))) =$$

$$= ((\bigvee (A_1 \cap A_2)) \lor (\bigvee (A_1 \setminus A_2))) \lor ((\bigvee (A_1 \cap A_2)) \lor (\bigvee (A_1 \setminus A_2))) =$$

$$= \bigvee (A_1 \cap A_2) \lor (\bigvee (A_1 \setminus A_2)) = \bigvee (A_1 \cap A_2)$$

since $(\bigvee A_2) \land (\bigvee (A_1 \setminus A_2)) = 0$ by Lemma 1.12. If $n > 2$ and the result is true for $n - 1$, then by the inductive hypothesis and step $n = 2$, we have

$$\bigcap_{1 \leq i \leq n} (\bigvee A_i) = (\bigvee A_1) \land (\bigcap_{2 \leq i \leq n} (\bigvee A_i)) =$$
\[
= (\bigvee A_i) \bigwedge (\bigvee (\bigcap_{2 \leq i \leq n} A_i)) = \bigvee (\bigcap_{1 \leq i \leq n} A_i),
\]
which finishes the proof. \(\square\)

The next result shows that Lemma 1.13 also holds for infinite families of subsets of an independent set.

**Proposition 1.14.** Let \(L \in \mathcal{U} \cap \mathcal{M}\), and let \(X\) be an independent subset of \(L\). Then, for any family \((A_i)_{i \in I}\) of subsets of \(X\), we have

\[
\bigwedge_{i \in I} (\bigvee A_i) = \bigvee (\bigcap_{i \in I} A_i).
\]

In particular, if \(\bigcap_{i \in I} A_i = \emptyset\), then \(\bigwedge_{i \in I} (\bigvee A_i) = 0\).

**Proof:** Set \(A := \bigcup_{i \in I} A_i\), \(a_i := \bigvee A_i\), \(a := \bigwedge_{i \in I} a_i = \bigwedge_{i \in I} (\bigvee A_i)\), \(B_i := A \setminus A_i\), \(b_i := \bigvee B_i\), and \(b := \bigvee (\bigcap_{i \in I} A_i)\). Since \(\bigcap_{i \in I} A_i \subseteq A_j\) for each \(j \in I\), we have \(\bigvee (\bigcap_{i \in I} A_i) \leq \bigvee A_j\), and so \(\bigvee (\bigcap_{i \in I} A_i) \leq \bigwedge_{i \in I} (\bigvee A_i)\), i.e., \(b \leq a\). Thus, it remains to prove only that \(a \leq b\).

We have

\[
\bigwedge_{i \in I} b_i = \bigwedge_{i \in I} (\bigvee B_i) = \bigwedge_{i \in I} (\bigvee (A \setminus A_i)) = \bigwedge_{i \in I} (\bigcup_{i \in I} (A \setminus A_i)) = \bigwedge_{i \in I} (A \setminus A_i),
\]

so

\[
\bigvee A = (\bigvee (A \setminus \bigcap_{i \in I} A_i)) \vee (\bigvee (\bigcap_{i \in I} A_i)) = (\bigvee b_i) \vee b.
\]

Denote by \(\mathcal{F}(I)\) the poset, ordered by inclusion, of all finite nonempty subsets of \(I\), and for every \(F \in \mathcal{F}(I)\) set \(b_F := \bigvee_{i \in F} b_i\). Then \(\bigvee_{i \in I} b_i = \bigvee_{F \in \mathcal{F}(I)} b_F\), and so

\[
a = a \land (\bigvee A) = a \land (\bigvee (\bigvee b_i) \lor b) = a \land (\bigvee_{F \in \mathcal{F}(I)} (b_F \lor b)) = \bigvee_{F \in \mathcal{F}(I)} (a \land (b_F \lor b)),
\]

since \(L\) is upper continuous and \(\mathcal{F}(I)\) is a directed set.

Let \(F = \{i_1, \ldots, i_k\}\) be an arbitrary but fixed nonempty finite subset of \(I\). Then

\[
a = \bigwedge_{i \in I} a_i \leq a_{i_1} \land \ldots \land a_{i_k} = (\bigvee A_{i_1}) \land \ldots \land (\bigvee A_{i_k}) = \bigvee (A_{i_1} \cap \ldots \cap A_{i_k})
\]

by Lemma 1.13. On the other hand, we have

\[
b_F \lor b = ((\bigvee B_{i_1}) \lor \ldots \lor (\bigvee B_{i_k})) \lor b = (\bigvee (B_{i_1} \cup \ldots \cup B_{i_k})) \lor b = \bigvee (B_{i_1} \cup \ldots \cup B_{i_k}) \lor b
\]
\[
= \left( \bigvee (A \setminus (A_{i_1} \cap \ldots \cap A_{i_k})) \right) \lor b.
\]

We deduce that

\[
a \land (b_F \lor b) \leq \left( \bigvee (A_{i_1} \cap \ldots \cap A_{i_k}) \right) \land \left( \left( \bigvee (A \setminus (A_{i_1} \cap \ldots \cap A_{i_k})) \right) \lor b \right).
\]

For simplicity, denote \( A_F := A_{i_1} \cap \ldots \cap A_{i_k} \), and then the inequality above becomes

\[
a \land (b_F \lor b) \leq \left( \bigvee A_F \right) \land \left( \left( \bigvee (A \setminus A_F) \right) \lor b \right).
\]

We claim that

\[
\left( \bigvee A_F \right) \land \left( \left( \bigvee (A \setminus A_F) \right) \lor b \right) = b.
\]

Indeed \( b = \left( \bigvee (\bigcap_{i \in I} A_i) \right) \leq \bigvee A_F \), so, by modularity and by Lemma 1.12, we obtain

\[
\left( \bigvee A_F \right) \land \left( \left( \bigvee (A \setminus A_F) \right) \lor b \right) = \left( \left( \bigvee A_F \right) \land \left( \bigvee (A \setminus A_F) \right) \right) \lor b = 0 \lor b = b.
\]

This shows that \( a \land (b_F \lor b) \leq b \) for all \( F \in \mathcal{F}(I) \), which implies that

\[
a = \bigvee_{F \in \mathcal{F}(I)} (a \land (b_F \lor b)) \leq b,
\]

as desired. \( \square \)

If \((M_i)_{i \in I}\) is an independent family of submodules of a module \( M_R \), then clearly \( \bigcap_{i \in I} M'_i = 0 \), where \( M'_i := \sum_{j \in I \setminus \{i\}} M_j \) for all \( i \in I \). The next result, needed in the proof of Theorem 1.16, is a very particular case of Proposition 1.14, and shows that this simple fact on modules can be extended, with some effort, to any upper continuous modular lattice. We are very indebted to Patrick F. Smith for very helpful discussions about how this latticial extension can be proved.

**Corollary 1.15.** Let \( L \in \mathcal{U} \cap \mathcal{M} \), let \( I \) be an arbitrary nonempty set, let \((x_i)_{i \in I}\) be an independent family in \( L \), and for each \( i \in I \), set \( x'_i := \bigvee_{i \in I \setminus \{j\}} x_i \). Then \( 0 = \bigwedge_{i \in I} x'_i \). \( \square \)

As in Grzeszczuk and Pucziolski [Proposition 2] a basis of an arbitrary lattice \( L \) is a maximal independent subset of \( L \) consisting only of coirreducible elements of \( L \).

**Theorem 1.16.** The following statements are equivalent for a nonzero lattice \( L \in \mathcal{U} \cap \mathcal{M} \).

1. \( L \) is RC (resp. RCC).
2. There exists a join of an independent family of coirreducible (resp. completely coirreducible) elements of \( L \) that is essential in \( L \).
(3) \( L \) has a basis (resp. a basis consisting only of completely coirreducible elements).

(4) For every \( 0 \neq x \in L \) there exists a nonempty set \( I_x \) such that 0 can be written as an irredundant intersection

\[
0 = \bigwedge_{i \in I_x} x_i
\]

of irreducible elements (resp. completely irreducible elements) \( x_i \) in \( x/0 \), \( i \in I_x \).

Moreover, the equivalent conditions (1) – (4) for an RCC lattice can be reformulated as follows:

(1)' \( L \) is an atomic lattice.

(2)' \( \text{Soc}(L) \) is essential in \( L \).

(3)' \( L \) has a basis consisting only of atoms of \( L \).

(4)' For every \( 0 \neq x \in L \) there exists a nonempty set \( I_x \) such that 0 can be written as an irredundant intersection

\[
0 = \bigwedge_{i \in I_x} x_i
\]

of coatoms \( x_i \) in \( x/0 \), \( i \in I_x \), in other words, the radical \( r_{x/0} \) of \( x/0 \) is zero and an irredundant intersection of coatoms.

**Proof:** (1) \( \iff \) (2) follows immediately from Lemma 1.6.

(1) \( \Rightarrow \) (3): Let \( B \) be a maximal independent subset of \( C(L) \) (resp. \( C^c(L) \)), which exists by Zorn’s Lemma. We claim that \( B \) is also a maximal independent subset of \( L \). Indeed, if not, there exists an element \( 0 \neq y \in L \setminus B \) such that \( B \cup \{ y \} \) is independent, so \( y \wedge (\bigvee B) = 0 \). Since \( L \) is RC (resp. RCC), there exists a coirreducible (resp. completely coirreducible) element \( c \in L \) with \( c \leq y \). Then \( c \wedge (\bigvee B) \leq y \wedge (\bigvee B) = 0 \), so \( c \wedge (\bigvee B) = 0 \), and then, necessarily \( c \notin B \). This implies that \( B \cup \{ c \} \) is an independent subset of \( C(L) \) (resp. \( C^c(L) \)) strictly including \( B \), which contradicts the maximality of \( B \).

(3) \( \Rightarrow \) (1): Let \( B \) be a basis of \( L \) (resp. a basis of \( L \) consisting only of completely coirreducible elements). Then \( \bigvee B \) is a maximal direct join of coirreducibles (resp. completely coirreducibles) of \( L \). If \( L \) would not be RC (resp. RCC), then, by Lemma 1.6, there would exist a nonzero element \( t \in L \) such that \( t \wedge (\bigvee B) = 0 \), so \( B \cup \{ t \} \) would be an independent subset of \( L \) strictly containing \( B \), contradicting the maximality of \( B \).

(1) \( \Rightarrow \) (4): Let \( 0 \neq x \in L \). Since \( L \) is RC (resp. RCC), so is also the interval \( x/0 \). By the equivalence (1) \( \iff \) (2) applied for the lattice \( x/0 \), there
exists a direct join \( m = \bigvee_{i \in I} c_i \) of coirreducible (resp. completely coirreducible) elements \( c_i, i \in I \), in \( x/0 \) that is essential in \( x/0 \).

If \( I = \{i\} \) is a singleton then \( m = c_i \) is an essential coirreducible (resp. completely coirreducible) element in \( x/0 \), that is, \( 0 = 0 \) is an IID (resp. ICID) of \( 0 \) in \( m/0 \). By Lemma 1.9, this is also an IID (resp. ICID) of \( 0 \) in \( x/0 \).

If \( I \) has at least two elements, then for every \( j \in I \) we set

\[ c_j' := \bigvee_{i \in I \setminus \{j\}} c_i. \]

Since \( m/c_j' \simeq c_j/0 \), it follows that \( c_j' \) is an irreducible (resp. completely irreducible) element of \( m/0 \) for every \( j \in I \).

By Corollary 1.15, we have \( 0 = \bigwedge_{i \in I} c_i' \). We claim that \( \bigwedge_{i \in I \setminus \{j\}} c_i \not\simeq c_j' \) for all \( j \in I \). Indeed, if not, then, there exists \( j \in I \) such that \( \bigwedge_{i \in I \setminus \{j\}} c_i \leq c_j' \); hence \( c_j \leq \bigwedge_{i \in I \setminus \{j\}} c_i \leq c_j' \), and so \( 0 = c_j \land c_j' = c_j \), which is a contradiction.

This shows that \( 0 = \bigwedge_{i \in I} c_i' \) is an IID (resp. ICID) of \( 0 \) in \( m/0 \). Apply now Lemma 1.9 to see that this decomposition can be extended to an IID (resp. ICID) of \( 0 \) in \( x/0 \), as desired.

(4) \( \implies \) (1): Let \( 0 \neq x \in L \), and let \( 0 = \bigwedge_{i \in I} x_i \) be an IID (resp. ICID) of \( 0 \) in \( x/0 \). Assume that the set \( I \) has at least two elements. Then, for every \( i \in I \), set \( \pi_i := \bigwedge_{j \in I \setminus \{i\}} x_j \). By modularity, we have \( \pi_i \land x_i = \pi_i \land (x_i \lor \pi_i) \simeq (x_i \lor \pi_i)/x_i \leq x/\pi_i \). Since \( x_i \) is irreducible (resp. completely irreducible) in \( x/0 \), it follows that \( x/\pi_i \) is coirreducible (resp. completely coirreducible), and hence, so is also its initial subinterval \( (x_i \lor \pi_i)/x_i \). This implies that the interval \( \pi_i /0 \) is coirreducible (resp. completely coirreducible), i.e., \( \pi_i \) is coirreducible (resp. completely coirreducible) and \( \pi_i \leq x \), as desired. Observe that if \( I = \{i\} \) is a singleton, then \( x_i = 0 \), so \( 0 \) is irreducible (resp. completely irreducible) in \( x/0 \), in other words, \( x \) itself is coirreducible (resp. completely coirreducible).

(1') \( \iff \) (2') \( \iff \) (3') \( \iff \) (4)': Observe that instead of completely coirreducible elements in (2) and (3) we may take atoms; indeed, if \( (c_i)_{i \in I} \) is an independent set of completely coirreducible elements of \( L \) such that \( \bigvee_{i \in I} c_i \) is essential in \( L \), then \( a := \bigvee_{i \in I} a_i \) is also an essential element of \( L \), where, for every \( i \in I \), \( a_i \) is the unique essential atom in the subdirectly irreducible interval \( e_i/0 \). Then, the corresponding elements \( a_j' := \bigvee_{i \in I \setminus \{j\}} a_i \) are coatoms in \( a/0 \).

**Remarks 1.17.** (1) The equivalence \( 1) \iff 3) \) in Theorem 1.16, but only for coirreducibles, has been also established in Grzeszczuk and Pucziłowski [14] Proposition 2] for modular lattices \( L \) which are not necessarily upper continuous, using a completely different approach, namely the embedding of \( L \) into the modular upper continuous lattice \( Id(L) \) of all ideals of \( L \). According to Grzeszczuk and Pucziłowski [14] Theorem 1], any two bases of an RC modular lattice \( L \) have the same cardinality, called the Goldie dimension of \( L \).
One may ask whether condition (3) in Theorem 1.16 can be replaced by the following weaker one:

The element 0 has an irredundant decomposition \(0 = \bigwedge_{i \in I} x_i\) in irreducible elements (resp. completely irreducible elements) \(x_i\) in \(L\).

The answer is no for the case of completely irreducibles, as the following example, due to Fuchs, Heinzer, and Olberding [10, Example 2.4], shows. Let \(\{p_0, p_1, \ldots\}\) be the set of all positive prime numbers in \(\mathbb{Z}\), and consider the direct product ring \(T = \prod_{i \in \mathbb{N}} \mathbb{Z}/p_i\mathbb{Z}\). Let \(R\) be the subring of \(T\) generated by \((1, 1, \ldots), (p_0, p_1, p_2, \ldots),\) and \((0, \ldots, p_n, 0, \ldots)\) for each \(n \in \mathbb{N}\). The ideal 0 of \(R\) has an ICID in \(R\) and \(\text{Soc}(R) = \sum_{n \in \mathbb{N}} Rr_n\) is not an essential submodule of \(R\). So, by Theorem 1.16, the lattice \(L(R)\) of all ideals of \(R\) is not RCC, but 0 has an ICID in \(L(R)\).

For decompositions in irreducible elements we are looking for a module \(M_R\) that is not rich in coirreducibles such that 0 has an IID in \(M\). This is also an open problem mentioned in Fort [8, Théorème 1, Proposition 5] (see also Lemmas 1.6 and 1.9), the module \(M\) should be a direct sum of a module without any coirreducibles with one that is a maximal direct sum of coirreducibles.

The next results, extending some results of Fuchs, Heinzer, and Olberding [10] from ideals to lattices, investigate when a CI element in an ICID of a given element is relevant, i.e., cannot be omitted.

**Definition 1.18.** Let \(L\) be a lattice, and let \(a \leq c\) be elements of \(L\). We say that \(c\) is a relevant completely irreducible divisor, abbreviated an RCID, of \(a\) if \(a\) has a decomposition as an intersection of completely irreducibles elements of \(L\) in which \(c\) appears and is relevant, i.e., cannot be omitted.

**Proposition 1.19.** Let \(a \leq c\) be elements of a lattice \(L \in \mathcal{M}\) with CID such that \(c \in I^c(L)\). Then \(c\) is an RCID of \(a\) if and only if \(c\) is not an essential element in the interval \(1/a\).

**Proof:** “\(\Rightarrow\)” If \(c\) is an RCID of \(a\), then we can write \(a = c \wedge b\) where \(b\) is a meet of CI elements and \(c\) cannot be omitted in the intersection above, i.e., \(a = c \wedge b \neq b\), and so \(c < c \vee b\). Since the cover \(c^*\) of \(c\) is by definition the least element of the interval \([c, 1]\) of \(L\) and \(c \vee b \in [c, 1]\), it follows that \(c^* \leq c \vee b\).

Using now the fact that \(L\) is a modular lattice, we deduce that

\[(c \vee (c^* \wedge b))/c \simeq (c^* \wedge b)/(c \wedge (c^* \wedge b)) = (c^* \wedge b)/(c \wedge b) = (c^* \wedge b)/a.\]

On the other hand, again by modularity, we have \(c \vee (c^* \wedge b) = c^* \wedge (c \vee b) = c^*\). Thus \((c^* \wedge b)/a \simeq c^*/c\), and so, the interval \((c^* \wedge b)/a\) is simple, in particular \(c^* \wedge b \neq a\). Since \(c \wedge (c^* \wedge b) = c \wedge b = a\), we deduce that \(c\) is not essential in \(1/a\), as desired.
“⇐” : If \( c \) is not essential in \( 1/a \), then there exists \( d \in ]a, 1] \) with \( c \wedge d = a \). Since \( L \) is a lattice with CID by hypothesis, we can write \( d \) as an intersection \( d = \bigwedge_{j \in J} d_j \) of CI elements \( d_j, j \in J \). Now, observe that \( c \) is relevant in the intersection \( a = c \wedge (\bigwedge_{j \in J} d_j) \) since \( a \neq d \), and we are done.

\[ \square \]

**Corollary 1.20.** Let \( L \in \mathcal{M} \) be a lattice with CID, and let \( a \leq c \) in \( L \). If \( c \in \mathcal{I}^c(L) \), then the cover \( c^* \) of \( c \) is an essential element in the interval \( 1/a \).

**Proof:** If \( c \) is not an RCID of \( a \), then \( c \) is essential in \( 1/a \) by Proposition 1.19, so \( c^* \) is also essential in \( 1/a \).

Assume now that \( c \) is an RCID of \( a \). Then we can write \( a = c \wedge b \), where \( b \) is a meet of CI elements and \( c \wedge b \neq b \).

If \( a = c \), then \( c^* \) is essential in \( 1/c = 1/a \). So, we may assume that \( a < c \). In order to show that \( c^* \) is essential in \( 1/a \) we have to prove that \( x \wedge c^* \neq a \) for every \( x > a \).

If \( x \leq c \), then \( x \wedge c^* = x \neq a \). If \( x \not\leq c \), then \( c < c \vee x \), so \( c^* \not\leq c \vee x \). It follows that

\[ c \neq (c \vee x) \wedge c^* = c \vee (x \wedge c^*) , \]

and hence \( x \wedge c^* \neq a \), as desired.

\[ \square \]

**Corollary 1.21.** Let \( L \in \mathcal{U} \cap \mathcal{M} \) be a lattice with CID, and let \( 1 \neq a \in L \). Then there exists an RCID of \( a \) if and only if \( \text{Soc}(1/a) \neq a \).

**Proof:** “⇒” : If \( c \geq a \) is an RCID of \( a \), then we can write \( a = c \wedge b \), where \( b \) is a meet of CI elements and \( c \wedge b \neq b \).

If \( c = a \), then \( \text{Soc}(1/a) = \text{Soc}(1/c) = c^* \succ a \). If \( c > a \), then \( b > a \), and as in the proof of the implication “⇒” in Proposition 1.19, we have \( (c^* \wedge b)/a \cong c^*/c \), and so, the interval \( (c^* \wedge b)/a \) is simple. Therefore \( a \prec (c^* \wedge b) \leq \text{Soc}(1/a) \), as desired.

“⇐” : If \( \text{Soc}(1/a) \neq a \), let \( s \succ a \). Since \( s \not\leq a \), the set

\[ C := \{ c | c \geq a, s \not\leq c \} \]

is not empty. Now observe that \( c \in C \iff s \wedge c = a \) because \( s \succ a \). Using now the upper continuity of \( L \), we deduce that the set \( C \) is inductive, so it has a maximal element \( c_0 \) by Zorn’s Lemma. By Proposition 0.7, it follows that \( c_0 \) is CI in \( 1/a \), so also in \( L \). Since \( L \) is a lattice with CID, we can write \( s \) as a meet of CI elements, and \( c_0 \) is clearly relevant in the intersection \( a = c_0 \wedge s \), which finishes the proof.

\[ \square \]

**Lemma 1.22.** Let \( L \in \mathcal{U} \cap \mathcal{M} \), and let \( x = \bigwedge_{i \in I} x_i \) be an ICID of \( x \neq 1 \). For every \( i \in I \) let \( x_i := \bigwedge_{j \in I \setminus \{i\}} x_i \). Then, the following statements hold.
For each \( i \in I \), the interval \( \pi_i/x \) is subdirectly irreducible with \( u_i := x_i^* \wedge \pi_i \in [x, \pi_i] \) the unique atom covering \( x \).

(2) In the intersection \( x = \bigwedge_{i \in I} x_i \) no \( x_i \) can be replaced by a larger element of \( L \) and still preserving the intersection equal to \( x \).

(3) The family \( (u_i)_{i \in I} \) is independent in \( 1/x \) and \( \bigvee_{i \in I} u_i \leq \text{Soc}(1/x) \).

**Proof:**

(1) Let \( i \) be a fixed element of \( I \). As in the proof of the implication \( (4) \implies (1) \) in Theorem 1.16, by modularity, we have \( \pi_i/x = \pi_i/(x_i \wedge \pi_i) \cong (x_i \vee \pi_i)/x_i \subseteq 1/x_i \). Since \( x_i \) is CI, it follows that \( 1/x_i \) is SI, and hence, so is also its initial subinterval \( (x_i \vee \pi_i)/x_i \). This implies that the interval \( x_i^* \wedge \pi_i \) is SI.

(2) Let \( i \in I \) and \( y_i \in L \) with \( x_i < y_i \). Then \( x_i^* \leq y_i \) since \( 1/x_i \) is SI, so

\[
x = x_i \wedge \pi_i < u_i = x_i^* \wedge \pi_i \leq y_i \wedge \pi_i,
\]

as desired.

(3) We have \( u_i = x_i^* \wedge \pi_i \leq \pi_i \leq x_i \) for every \( j \in I \setminus \{i\} \), and so,

\[
x \leq u_i \wedge \left( \bigvee_{j \in I \setminus \{i\}} u_j \right) \leq \pi_i \wedge x_i = x,
\]

i.e., \( x = u_i \wedge (\bigvee_{j \in I \setminus \{i\}} u_j) \) for every \( i \in I \). This shows that the family \( (u_i)_{i \in I} \)

is independent in \( 1/x \) and \( \text{Soc}(1/x) = \bigvee_{i \in I} u_i \).

**Proposition 1.23.** Let \( L \in \mathcal{U} \cap \mathcal{M} \). Then \( L \) is semi-Artinian if and only if every \( x \in L \) has an ICID in \( L \).

**Proof:** \( \leftarrow \rightarrow \): If \( L \) is semi-Artinian, then so is the interval \( 1/x \) for every \( 1 \neq x \in L \). It follows that \( 1/x \) is a RCC lattice. By Theorem 1.16, \( x \) has an ICID \( x = \bigwedge_{i \in I} x_i \) in \( 1/x \), with all \( x_i, i \in I \), CI in \( 1/x \), so also CI in \( L \).

\( \leftarrow \rightarrow \): If every \( 1 \neq x \in L \) has an ICID in \( L \), then the interval \( 1/x \) has at least an atom by Lemma 1.22, so, by definition, \( L \) is semi-Artinian.

Note that the condition that \( L \) is modular in Proposition 1.23 can be weakened to \( "L \text{ satisfies condition } (M)" \), cf. Walendziak [21, Theorem 4].

**Corollary 1.24.** (Crawley and Dilworth [6, 6.3]). If every element of a modular compactly generated lattice \( L \) has an ICID in \( L \), then \( L \) is strongly atomic.
Proof: Since any compactly generated lattice is upper continuous (see, e.g., Crawley and Dilworth [6, 2.3]), we can apply Proposition 1.23 to conclude that $L$ is semi-Artinian, i.e., strongly atomic.

Remarks 1.25. (1) Any Noetherian lattice $L \in C$ is a lattice with CID. To show that, we proceed as in the proof of Năstăsescu and Van Oystaeyen [18, Proposition 1.4.4]. Assume that $L$ is not with CID. Then, the set $C$ of all elements $1 \neq x \in L$ that cannot be written as intersections of CI elements of $L$ is nonempty, so $C$ has a maximal element, say $m$. Clearly $m$ is not CI, so we can write $m = \bigwedge_{i \in I} a_i$ with $a_i > m$ for all $i \in I$. Thus $a_i \in L \setminus C$ for all $i \in I$, and then, each $a_i$ can be written as an intersection of CI elements. This implies that $m$ can be written as an intersection of CI elements, which is a contradiction.

(2) A Noetherian lattice $L \in U \cap M$ is not necessarily a lattice with ICID, as the following example shows. Let $L$ be the lattice $L(Z)$ of all subgroups of the Abelian group $Z$. Then $L$ is a Noetherian lattice which is not with ICID since the zero ideal of $Z$ has no ICID in $L$ by Lemma 1.22.

We end this section by mentioning the following two results on the uniqueness and replacement property of completely irreducible decompositions in lattices, that extend the corresponding results of Heinzer and Olberding [16] and Fuchs, Heinzer, and Olberding [10].

Theorem 1.26. (Crawley and Dilworth [6, Theorems 7.1 and 7.2]).

(1) If $L$ is a complete distributive lattice, then any element $1 \neq a \in L$ has at most one ICID.

(2) If $L$ is a complete modular lattice, and $1 \neq a \in L$ has two CID decompositions $a = \bigwedge C = \bigwedge C'$, then the Kurosh - Ore replacement property holds: for every $c \in C$ there exists $c' \in C'$ such that $a = c' \wedge \bigwedge (C \setminus \{c\})$; moreover this resulting decomposition is irredundant if the decomposition $a = \bigwedge C$ is irredundant.

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