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On Geometric Simple Connectivity

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Abstract

We will give a very quick panoramic view of some more or less recent research, mostly by the second author (V.P.), in the fields of low-dimensional topology and geometric group theory.

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A Review

The notion of the ***geometric simple connectivity*** (***GSC***) stems from smooth manifold theory. A smooth manifold M, which is possibly non compact with non empty boundary, is said to be GSC if it admits Morse functions f without critical points of index $\lambda = 1$. Here, when M is non compact one asks that f be PROPER and, anyway, the $f|\partial M$ has to be taken into account too. There is also a more combinatorial version, in terms of handlebody decompositions, the condition being then that the 1-handles and 2-handles should be in cancelling position; here, of course, all the 1-handles are involved but only a subset of the 2-handles. This second definition makes also sense outside of the smooth realm, for cell-complexes. For the connections between GSC and the mere $\pi_1 = 0$ condition we refer here to the brief note [49] by the last two authors (V.P. and C.T.). More recently, there has been here the big celebrated breakthrough by Grisha

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Perelman, namely his proof of the 3-dimensional Poincaré Conjecture, and this is completely equivalent to the following implication

$$\{\pi_1 = 0\} \Longrightarrow \text{GSC, for closed 3-manifolds.}$$

See here [20], [21], [22], [1], [2], [13], [16] and [17]. The Perelman-Hamilton Ricci flow approach (which, as such, has no connection with GSC) has yielded not only the Poincaré Conjecture, but also the full Thurston Geometrization Conjecture for closed 3-manifolds.

The second author (V.P.) has also developed, completely independently of Perelman and Hamilton, his own approach to the 3-dimensional Poincaré Conjecture, based on GSC for smooth 4-manifolds mainly non compact ones with non empty boundary. This approach, an outline of which can be found in [39], [42], and see also [40], [41], consists of three successive steps which we will very briefly review below. One will notice that GSC occurs preeminently in all three of them. Notice also that, according to [49], the connection between GSC and mere simple connectivity stays certainly mysterious and problematic when the dimension is four, at least in the DIFF context, or when non compact manifolds with non empty boundaries are concerned. Both of these aspects are present in the approach to the Poincaré Conjecture by the second author (V.P.), while the second aspect, albeit in very high dimensions, occurs in the work (also by V.P.) in group theory, described later in this section.

We finally point out that there is, also, another related notion which is the counterpart, in the polyhedral category, of the GSC of open manifolds and which, in general, is slightly weaker, namely the ***weak GSC*** condition, developed by the first author (D.O.) in collaboration with L. Funar in [7], [8] and [18]. We will say more about it towards the end of this paper.

STEP I. We start here by introducing another weakening of the GSC concept, following a suggestion by Barry Mazur. An open smooth manifold X^n will be said to be ***GSC at long distance*** if for every compact $K \subset X^n$ we can sandwich a compact smooth GSC submanifold $K \subset M^n \subset X^n$.

Next, to any smooth compact bounded 4-manifold M^4 we can associate, *canonically*, the following smooth open 4-manifold

$$X^4(M^4) \overset{\text{def}}{=} \text{int}(M^4 \# \infty \# (S^2 \times D^2)). \tag{1}$$

Here is the first, purely 4-dimensional, result, which is totally unrelated to dimension three; full proofs are given in [37] and a relatively detailed outline is to be found in [38].

Theorem 1. *If M^4 is GSC at long distance, with a boundary ∂M^4 which is a* homology 3*-sphere, then $X^4(M^4)$ is GSC.*

This is one of the ingredients for the first step, the final result of which is the following

Theorem 2. *For any homotopy 3-ball Δ^3, the smooth open 4-manifold $X^4(\Delta^3 \times I)$ is GSC.*

Modulo theorem 1, the full proof of theorem 2 is contained in [28] to [32].

Step II. This intermediary step was only achieved more recently, much later than the III to come below. Both for it and for theorem 1, severe criticism by M. Freedman, D. Gabai and F. Quinn of earlier flawed versions, have been essential. The main result here is the following

Theorem 3. *If $X^4(\Delta^3 \times I)$ is GSC, then so is $\Delta^3 \times I$ itself.*

An outline of the proof is to be found in [42], but the long handwritten manuscript with full details still waits to be typed.

Step III. Once we know that $\Delta^3 \times I$ is GSC, then we also get the coherence theorem, for which we will refer here to [9] and [37]; we will describe it later in this paper. But then, once we have coherence, we can also apply the so-called strange compactification procedure ([35], [39], [9] and more about it will be said below) and get the following final result, the conclusion of which **is** the 3-dimensional Poincaré Conjecture

Theorem 4. *If $\Delta^3 \times I$ is GSC, then $\Delta^3 = B^3$.*

An outline of the proof of this theorem can be found in the papers [9], [34], [35], [39], [41] and then, complete details in [33].

In the sketchy 3-stage outline above we had side by side two kinds of issues: show that some specific object is GSC and then, deduce from this some desirable geometric result. This kind of duality will be present throughout the rest of this present short paper.

The 3-stage approach outlined above is essentially unrelated to 3-dimensional topology and so one cannot expect much from it concerning the full geometrization conjecture. But then, it has other kinds of fall-outs and/or ramifications, to which we will turn next. These are totally outside the world of 3-manifolds. To begin with, let us mention a category of smooth non-compact four-manifolds with non-empty boundary, which play a preeminent role in Steps I and III above. These are the ***sort of links*** $\boldsymbol{V^4}$, defined by the following prescriptions

$$\text{int}V^4 = \mathbb{R}^4_{\text{standard}}, \quad \partial V^4 = \sum_1^{\alpha \leq \infty} S^1_i \times \text{int}D^2_i. \tag{2}$$

where the null-framing is meant in the last formula. Classical links trivially generate such V^4's (with $\alpha < \infty$) and these V^4's will be called ***smoothly tame***. But then, Casson Handles [12] are also examples of sort of links (with $\alpha = 1$) and M. Freedman has proved (see [5] and [6]) that they are ***topologically tame***. Moreover, by combining Freedman and Donaldson [4], in particular because exotic $\mathbb{R}^4$'s **do** exists, one can show that at least some of them have to be ***smoothly wild***. Issues of triviality and of (smooth) tameness for various specially constructed sort of links are among the big items in the 4-dimensional approach to the Poincaré Conjecture described above, see here also [35] and [41]. Notice that the distinction smoothly tame versus topologically tame is foreign to classical knot theory. But then, there is here the idea that just like this classical knot theory is related, via the quantum invariants, to the entangled quantum many-particle systems, so our sort of links should be connected to (not necessarily topological) quantum field theory.

In the discussion above, concerning the topological tameness versus the smooth tameness for the "sort of links" V^4 (2), as well as in the discussion which will follow next, concerning the 4-dimensional smooth Schoenflies problem, we touch upon the issue of the deep, uncharted precipice which exists exactly in dimension four between the categories DIFF and TOP. Remember that in dimension three or less, the two categories are equivalent, while in dimensions strictly larger than four, the difference between the two is completely accounted for by discrete invariants, by characteristic classes of sorts. But, in dimension four, this is beyond the power of algebraic topology or of any kind of discrete invariants, and the whole issue is, as yet, shrouded in deep mystery. One may legitimately suspect that basic issues concerning the nature of space and time are being touched upon here, too.

We move now to an important fall-out of step II and this concerns the 4-dimensional DIFF Schoenflies problem. Some historical details concerning this problem can be found in [41]; we just remind here the reader that a smooth 4-dimensional ***Schoenflies ball*** is a smooth compact submanifold $\Delta^4 \subset S^4$, with $\partial \Delta^4 = S^3$. Please do not confuse the Schoenflies ball Δ^4 with the much more general M^4 which occurs in theorem 1.

One of the results of Barry Mazur's celebrated work on the Schoenflies problem [14] (and see here [44] too) is the following diffeomorphism

$$\Delta^4 - \{\text{a boundary point}\} \overset{\text{DIFF}}{=} B^4 - \{\text{a boundary point}\}, \tag{3}$$

which implies, of course, that $\Delta^4 \overset{\text{TOP}}{=} B^4$; but, beyond the (3) above, Δ^4 is one of the big mysteries of 4-dimensional differential topology. So, here comes our next result

Theorem 5. *Δ^4 is GSC.*

One can find the outline of the proof in [42], but the full detailed version is still waiting to be typed. The very general idea for theorem 5 is that, very much like the premise $X^4(\Delta^3 \times I)) \in \text{GSC}$ is used in the context of theorem 3 to show that $\Delta^3 \times I \in \text{GSC}$, so we can use now Barry's (3) in order to draw a similar conclusion for the Schoenflies ball Δ^4.

Now, once we know that Δ^4 is GSC, the next aim should be to deduce from this that Δ^4 is standard, i.e. that $\Delta^4 \overset{\text{DIFF}}{=} B^4$. This is the subject of current research of the second author (V.P.), together with David Gabai.

We do not want to enter into the discussion of this current activity here, we will just mention a typical post-GSC result, which might eventually come handy in the present context. This is an old theorem of David Gabai, solving an even older conjecture of the second author (V.P.), namely the following result:

Let W^4 be a compact smooth 4-manifold having a handlebody decomposition with exactly one handle for the indices $\lambda = 0, \lambda = 2$ and $\lambda = 3$. Assume also that the homology of W^4 is the same as the homology of a point. Then $W^4 \overset{\text{DIFF}}{=} B^4$.

In [10] this is a corollary of the following so-called "Po's conjecture", which is also proved in [10]. Start with the following abstract 2-dimensional stratified object (Σ_n, S^1). Inside an oriented S^2 consider $2n+1$ disjoined disks. Then delete the interiors of the disks in question and glue the $2n+1$ remaining circles to a unique copy of S^1, using the normal orientation for $n+1$ of them, and the opposite one for the remaining n. With this, D. Gabai proves the following:

If Σ_n is embedded in S^3, then the (S^3, S^1) is unknotted.

In [10] one can also find a more general conjecture than the post-GSC 4-dimensional result quoted above, but this remains beyond our present state of the art, and as mysterious as ever.

The next fall-out concerns geometric group theory. Abstracting from earlier work by A. Casson [11], by the second author (V.P.), see here [25] to [27], and also by others, S. Brick and M. Mihalik have introduced the notion ***QSF*** (= ***quasi-simply-filtered***) for finitely presented groups Γ (see [3] and [51].) We start by defining QSF for locally compact spaces X, in the simplicial category. We will say that X is QSF if for any compact $k \subset X$ there is some (abstract) compact simply connected K endowed both with an inclusion $k \overset{i}{\hookrightarrow} K$ and with a map f into X, creating a commutative diagram

$$\begin{tikzcd} k \arrow[rr, hook] \arrow[dr, hook, "i"'] & & X \\ & K \arrow[ur, "f"'] & \end{tikzcd}$$

which is such that $ik \cap \{\text{double points of } f\} = \emptyset$, something which is reminiscent of the classical Dehn-Papakyriakopoulos lemma, in 3-dimensional topology, of course (see [23]).

In [3] it is shown, among other things, that

i) Consider two ***presentations*** P_1, P_2 for the same finitely presented group Γ, i.e. finite simplicial complexes with $\pi_1 = \Gamma$. We have then $\widetilde{P}_1 \in$ QSF IFF $\widetilde{P}_2 \in$ QSF, which makes QSF be a bona fide group theoretical notion.

ii) For $\Gamma = \pi_1 M^3$ the QSF implies $\pi_1^\infty \Gamma = 0$, where $\pi_1^\infty \Gamma \overset{\text{def}}{=} \pi_1^\infty \widetilde{M}^3$.

The next item is a corollary of Perelman's work on the geometrization of 3-manifolds.

iii) All $\Gamma = \pi_1 M^3$ are QSF.

See the next section for more details about the various group-theoretical ramifications of QSF. But here comes the next result by the second author (V.P.), to be compared to iii) above.

Theorem 6*. *All *finitely presented groups* Γ *are QSF.*

This is, of course, totally independent from iii) above which only applies to fundamental groups of closed 3-manifolds, while theorem 6 claims the same kind of result for **all** finitely presented groups. Let us say that iii) appears as a very special case of theorem 6. Of course, theorem 6 implies then also that for all closed M^3's we have $\pi_1^\infty \widetilde{M}^3 = 0$, but this is about as far as we can get with this technology in the direction of the geometrization conjecture for 3-manifolds.

For an outline of the proof of theorem 6 we refer to [43]. The preprint [45] is a first instalment of the detailed proof. The rest is to follow later, when typing will be available. This being said, the three successive papers which will eventually present the complete proof of theorem 6 (the first one being [45]) are a very massive bulk of unconventional technology. Hence partial results, showing that various special classes of groups are QSF, proved by more mundane, shorter and lighter technology, remain valid and interesting (e.g. [3], [8] and [18]).

The reason for talking about theorem 6 here is that it is strongly connected with GCS in more than one way. To begin with, the first author (D.O.) and Louis Funar have shown that Γ is QSF IFF there is some smooth compact manifold M with $\pi_1 M = \Gamma$, such that $\widetilde{M} \in$ GSC (see [8] and [18]). But unlike in i) above, this is presentation-dependent.

In his approach to theorem 6, the second author (V.P.) works with presentations of Γ which are ***singular*** 3-manifolds $M^3(\Gamma)$. The occurrence of 3-dimensional objects here, does not imply, by any means, a connection with 3-manifolds; these remain beyond our scope. It is actually just a technical ingredient. These, necessarily singular, presentations of Γ allow to take full advantage of the richness of the double point structures for maps from dimension two to dimension three when, later on, 2-dimensional REpresentations for Γ, as defined below, will be considered; we will come back to this later. Anyway, in view of the work of the first author (D.O.) and Louis Funar, it would suffice now to show that for some smooth high dimensional thickening $\Theta^N(M^3(\Gamma))$ the $\widetilde{\Theta}^N \in$ GSC. But life is not that easy, and what one actually manages to achieve, is just another smooth high-dimensional simply-connected manifold $S_u(\Gamma)$ (never mind the reason for this notation here), related to the $\Theta^N(M^3(\Gamma))^\sim$ and similarly endowed with a free action of Γ but, unfortunately, the $S_u(\Gamma)/\Gamma$ is NON-COMPACT. Very roughly speaking, one gets S_u from $\widetilde{\Theta}^N$ by changing each of the infinitely many compact fundamental domain of $\Theta^N(M^3(\Gamma))^\sim$ into some foamy non compact object, like turning solid compact ice-cubes into some infinitely complicated beer-foam. The main hard step in the proof of theorem 6 is actually getting this kind of $S_u(\Gamma)$ to be GSC. Afterwards, getting from this rather unusual object to

$\Gamma \in \mathrm{GSC}$, although not a trivial step, is easier.

But the GSC notion is present from the very beginning, on the road to theorem 6. The first basic concept is now the notion of (inverse) ***representation*** for Γ, an arrow looking like $\{\to \Gamma\}$ rather than the more mundane $\{\Gamma \to\}$'s. Also, please do not confuse "representation" of Γ, like defined below, and "presentation" of Γ like for instance $M^3(\Gamma)$.

By definition, a ***representation for*** Γ is a diagram

$$X \xrightarrow{f} \widetilde{M}^3(\Gamma) \tag{4}$$

where

a) X is a **not** necessarily locally-finite simplicial complex of dimension ≤ 3 and f a non degenerate simplicial map. We call dim X the dimension of the representation; the meaningful cases being of course dimensions 2 and 3.

b) $X \in \mathrm{GSC}$.

c) One can ***zip*** (4), in the sense that the smallest equivalence relation on X which is compatible with f and which kills all the non-immersive points, we call them the ***mortal singularities***, kills all the double points, see here [24], for instance, and then [45] too. These mortal singularities pertain to f while, completely independently of them, we also have the singularities of $\widetilde{M}^3(\Gamma)$ itself which, by definition, are ***immortal***.

d) One will also ask that f be "essentially surjective". For dim $X = 3$ this means that the image of f is dense, i.e. $\overline{fX} = \widetilde{M}^3(\Gamma)$. If dim $X = 2$, it means that $\widetilde{M}^3(\Gamma) - \overline{\mathrm{Im} f}$ consists of a union of (possibly infinitely many) small cells of dimensions two and three.

There is nothing group theoretical, yet at least, concerning this notion which stems from the very beginning of the second author's (V.P.) approach to the Poincaré Conjecture; see here in particular [28] and then also [29] to [35]. Actually, the structure of the definition above is such that any object which is being "represented" comes automatically with $\pi_1 = 0$; and in all the papers just mentioned it was homotopy 3-spheres which were being represented. Then in [48] the last two authors (V.P. and C.T.) have represented wild open simply-connected 3-manifolds (the Whitehead manifold Wh^3). Chaotic behavior in the guise of Julia sets occurs then, something which in the context of the representations necessary for theorem 6 would be mortally disastrous. Next, the second author (V.P.) alone, or the last two authors (V.P. and C.T.), have represented universal covering spaces of closed 3-manifolds, in [26], [27], [36], [46], [47], but these papers are by now largely superseded by Perelman's works (see the iii) above), and so is [11] too.

With all this, here is the first step in the proof of theorem 6, namely the following

Theorem 7. *For any finitely presented group Γ there exists a representation (4) with the following additional features*

*i) The representation is 3-dimensional (i.e. dim $X = 3$) and the X is **locally finite**.*

*ii) There is a free action $\Gamma \times X \longrightarrow X$, such that f is **equivariant** (for the standard action $\Gamma \times \widetilde{M}^3(\Gamma) \longrightarrow \widetilde{M}^3(\Gamma)$), i.e. $f(gx) = gf(x)$, $\forall x \in X$, $g \in \Gamma$.*

*iii) There is a **uniform bound** N such that for any double point $(x, y) \in X \times X$ of f there is a **zipping strategy** of length $\leq N$.*

The complete detailed proof is contained in [45], a paper which relies heavily on the work [50] of the last two authors (V.P. and C.T.).

Although written originally in the context of smooth 3-manifolds, this paper [50] seems to be actually tailor-made for singular 3-manifolds too, i.e. for presentations of completely general finitely presented groups Γ. As somebody said once in a different context, papers may sometimes be more clever than their authors. The work [36] should still be a useful reference for understanding what is going on in all this framework.

The context of the theorems 6 and 7 above was a good occasion for introducing the topic of representations, like the one in (4). But now we want to discuss these representations in more depth and also in a more general context which will concern the second author's (V.P.) work, both on the Poincaré Conjecture and on geometric group theory.

We will be concerned with a representation of some simply connected 3-manifold, which might be singular like $\widetilde{M}^3(\Gamma)$ in (4), but possibly smooth too. Also it might be non-compact or compact. For this object Y^3 we will consider a representation

$$X \xrightarrow{f} Y^3, \tag{5}$$

with the features a), b), c), d) already listed in connection with (4) and, if Y^3 is compact, then the X might be compact too. But we will anyway assume now that dim $X = 2$ and, as it will be argued below, this **is** the most interesting case. We will also need to look into the set of double points of f, namely the

$$X \supset M_2(f) \stackrel{\text{def}}{=} \{\text{the set of those } x \in X \text{ s.t. card } f^{-1}f(x) > 1\}. \tag{6}$$

The point we want to make is that, in a situation like the one we are considering now, with a generic map from a 2-dimensional space to a 3-dimensional one with not too nasty singularities, then there is a wealth of structures connected with $M_2(f)$, which can be put to use.

There are two special cases of (5) to be looked into now, corresponding respectively to the second author's (V.P.) work on the 3-dimensional Poincaré Conjecture, or on group theory. These two cases of (5) are

(7-I) $Y^3 =$ a smooth homotopy 3-sphere Σ^3 to which the homotopy 3-ball $\Delta^3 = \Sigma^3 - \mathrm{int}B^3$ is attached, and X is a compact 2-dimensional GSC complex;

(7-II) Our group theoretical (4) for which the X is assumed now to be a non compact but locally finite 2-dimensional complex (it turns out that, with some work, such local finiteness can always be achieved).

In both contexts (7-I) and (7-II), it may be assumed, without loss of generality, that all the mortal singularities (i.e. the non-immersive points of f (5)) $x \in X$ are of the following type, which we call ***undrawable***. The $f(x)$ belongs to the smooth part of Y^3, and, in the neighborhood of x, the X consists of two copies of $\mathbb{R}^2$, call them A and B. The $f|A$, $f|B$ inject and fA, fB cut through each other transversally along a line $L = (-\infty, \infty)$. The A and B are already glued together at the source, along the half-line $\frac{1}{2}L = (-\infty, 0]$, ending at $x = 0$. The $(0, \infty)$ consists entirely of double points, and this should also indicate how the zipping starts. Figure 1 in [28] can serve as an illustration for all this, and a little thought should make it clear too why we call the singular x "undrawable".

From here on, we will specialize for a while to (7-I). By definition, an abstract ***desingularization*** for $X \xrightarrow{f} \Sigma^3$ is a 2-valued function

(8) {the set of all local branches A_x, B_x for all the mortal, undrawable singularities $x \in X$ } $\xrightarrow{\phi}$ {the set with two elements s, n}

such that for each individual x we should always have $\phi(A_x) \neq \phi(B_x)$, i.e. if $\phi(A_x) = s$ (or n) then $\phi(B_x) = n$ (or s).

Now, our $X \xrightarrow{f} \Sigma^3$ is a representation, hence it can be zipped starting from the finite set Sing $(f) \subset X$. In more concrete terms, this complete zipping up of X means that the quotient-space projection $X \longrightarrow fX$ can be factorized (actually in many different ways), as

(9) $$X \xrightarrow{f_1} X_1 \xrightarrow{f_2} fX \subset \Sigma^3,$$

where the following two things happen

(9-1) The first piece f_1 is a precisely *ordered* sequence of elementary zipping moves which are all homotopy equivalences (we will say they are "acyclic"). This is not hard to describe explicitly; it is very clearly displayed in the figures 2 to 5 of [28] or 2.3 of [10], and we will not say more about it here.

(9-2) The f_2 is a collection of independent steps commuting with each other, localized in disjointed open neighborhood of X_1. Typically, each such neighborhood is the union of two copies of $\mathbb{R}^2$, call them now C and D, such that the following happens.
The $f_1|C, f_1|D$ inject, the two planes f_1C, f_1D cut through each other transversally along a line $(-\infty, \infty)$; at the source X_1, C and D are glued along $(-\infty, -1]$ and $[1, \infty)$, with undrawable singularities x_1, x_2 at ± 1. The zipping is finished by continuing to glue f_1C and f_1D together along $[-1, +1]$, killing x_1, x_2 as singularities, in the process.

The whole factorization (9), with the order of operations inside f_1, will be called a ***zipping strategy***, for (7-I).

Assume now that, for our representation (7-I), both a desingularization (8) and a zipping strategy (9) are given. It is not hard to see that the initially given desingularization propagates canonically along the ordered sequence of acyclic moves of f_1, inducing a desingularization ϕ_1 for

$$X_1 \xrightarrow{f_2} \Sigma^3. \tag{10}$$

Two things should be stressed here, before we go on. To begin with, ϕ_1 is sensitive to the precise order inside f_1, hence our name "strategy". Then, (10) is **not** a representation since, although simply-connected, X_1 fails, generally speaking, to be GSC.

For each $X_1 \supset C \cup D \xrightarrow{f_2} \Sigma^3$, the desingularization ϕ_1 smears the letters s, n on the branches at x_1, x_2. When one ends the zipping, with x_1 and x_2 meeting in a head-on collision, then these two (abstract) desingularizations might match (the ***coherent*** case) or mismatch (the ***non-coherent*** case) and this occurs, generally speaking, at each individual $C \cup D \subset X_1$, independently of each other.

At this point, we are able to state a result which lives in the very middle of the second author's (V.P.) approach to the Poincaré Conjecture, dividing it into two halves of comparable difficulties, namely the following

Coherence Theorem 8. *For every homotopy 3-sphere Σ^3 there is a representation $X \xrightarrow{f} \Sigma^3$, a desingularitation ϕ and a strategy, such that with the induced desingularization ϕ_1, all the movements in f_2 are* COHERENT.

This may look like a purely combinatorial statement concerning 2-complexes, but this is deceptive. To begin with, there is a fairly easy argument (for which we refer to [9] or to [37]), showing that the COHERENCE THEOREM 8 is completely equivalent to the statement that $\Delta^3 \times I \in$ GSC, which is the conclusion of theorem 3. This **is** the first author's (V.P.) road to the COHERENCE THEOREM, namely to get it as corollary of theorem 3, *after* the steps I and II, from the beginning of the paper, are already achieved.

Actually, the whole issue of COHERENCE is a question in DIFF 4-dimensional topology. Our abstract desingularizations are actually recipes both for thickening

X in dimension four and for desingularizing it, in the sense of the algebraic geometers (see here [29] and [9]). What COHERENCE really means, is being able to realize geometrically the zipping, in dimension four, staying GSC. The obstruction to COHERENCE is quite subtle. On the one hand it is ***unstable***, in the sense that one kills it as soon as one goes to dimension five or more (by an appropriate product with [0,1], ...). On the other hand, it has the following "abelianization", which is ***stable***. Consider the double points at the level of $X \times X$ or $X \times X \big/ \mathbb{Z}/2\mathbb{Z}$ instead of X. Call them now $M^2(f)$ and $M^2(f) \big/ \mathbb{Z}/2\mathbb{Z}$ respectively, coming with a map $M^2(f) \longrightarrow M_2(f)$ which desingularizes $M_2(f)$ and with an obvious $\mathbb{Z}/2\mathbb{Z}$ principal fibration

$$M^2(f) \longrightarrow M^2(f) \big/ \mathbb{Z}/2\mathbb{Z}.$$

COHERENCE implies, among other things, that this fibration is trivial (not a 2-sheeted covering), but the converse is false.

Of course, the work of Perelman (see [20] to [22] or [2] and [17]) immediately implies the theorem 8 too, but what we are talking about here is a **different** approach to the Poincaré Conjecture. And then, also, the 4-dimensional DIFF technology for the proof of theorem 3 (via which the COHERENCE THEOREM is gotten here) can be adapted so as to yield the purely 4-dimensional theorem 5 (see here [42]). Our aim here was to discuss the complexity of $M_2(f)$ in the context of (7-I), but we will still add a few words about what goes on, once we are in the possession of theorem 8. So we give a very brief bird's eye view of Step III in the approach to the Poincaré Conjecture.

Starting from the COHERENCE THEOREM for Σ^3, one can develop an *infinite process* achieving the following two items

1. A first, finite initial truncation of the process generates a smoothly tame sort of link V^4 $\big($see (2)$\big)$, from which one can get the smooth compact 4-manifold $(\Delta^3 \times I) \# n \# (S^2 \times D^2)$, for some large n.

2. Being smoothly tame, our V^4 naturally compactifies to B^4, but the infinite process generates a second, so-called ***strange compactification***, from which one can read that $\pi_1^\infty V^4$ is a free group. All this implies that $\Sigma^3 = S^3$.

We move now from (7-I) to (7-II). This time, our technology will make use of differential topology in higher dimensions, certainly higher than four, and we do not care about COHERENCE any longer. But there is now another problem: for a generic finitely presented group Γ we cannot guarantee that the subset $M_2(f) \subset X$ is **closed**. This is a very serious pathology which has to be faced when one wants to prove theorem 6, which would be a relatively simple matter, without the pathology in question.

Here is an illustration of what goes on, locally. Consider a smooth open neighborhood $U \subset X$. Generically, $M_2(f) \cap U$ will be an infinite family of parallel lines. Like one does for foliations or for laminations, we will cut $M_2(f) \cap U$ with a tight compact transversal τ, and next look at $\tau \cap M_2(f)$. Of course, if for all possible (U, τ)'s this would be finite, then $M_2(f)$ would be closed. But, generically, it is not and so one starts worrying about the accumulation pattern of $M_2(f) \cap \tau \subset \tau$. If this would always be cantorian, like in [48], then this would kill our approach to theorem 6. Fortunately, one can show that for any Γ there is a 2-dimensional representation like in (7-II), having the features i), ii), iii) from theorem 7, and also the next one below

(iv) For any compact transversal $\tau, M_2(f) \cap \tau$ accumulates on a finite set.

Short of $M_2(f) \cap \tau$ itself being finite, this is the best we can hope for. Also, this is the beginning of the proof of theorem 6, about which we will not say more here.

So, how does a non-closed $M_2(f)$ arise at all? Well, generically, for our representations (4), the X is an infinite sheaf of paths, each being a thickened version of $[0, \infty)$, which via f explores every nook and hook of $\widetilde{M}^3(\Gamma)$, a bit like in the context of the celebrated Feynman's path-integral. In this exploration a same compact spot in $\widetilde{M}^3(\Gamma)$ might be visited infinitely many times. This kind of phenomenon was called "Whitehead's nightmare" in [36] and it is this nightmare which produces non-closed $M_2(f)$'s.

At this point let us introduce the following definition, for an a priori arbitrary group Γ

(11) We will call a 2-dimensional representation like in (7-II) ***easy*** if $M_2(f)$ is closed. A group Γ which admits at least one such representation is called ***easy*** too, otherwise it will be called ***difficult***. So the "easy groups" are exactly those which manage to avoid Whitehead's nightmare.

Here is a list of known facts, concerning these notions

(11-1) It is fairly easy to show that if Γ has an easy representation, then it is QSF. The proof uses only the very basics of the technology in [29].

(11-2) From [26] it is not hard to extract a proof that all the hyperbolic groups, in the sense of Gromov, admit easy representations. Very many other well-known and well-studied classical groups admit easy representations and, to the best of our knowledge, nobody has ever encountered a group which is difficult, in the sense above.

(11-3) When $\Gamma = \pi_1 M^3$, where M^3 is a closed 3-manifold, then, making use of the full strength of Perelman's work, one can show that these Γ's have easy representations too. So, 3-manifold groups are all easy, in the sense above, but this is a result which is about as difficult to prove as any.

So, what then next? To begin with, there is a conjecture by the second author (V.P.) which we will explain now (and see here [45] too). This time we will be concerned with 3-dimensional locally finite representations

$$X^3 \xrightarrow{f} \widetilde{M}^3(\Gamma), \tag{12}$$

with X^3 an infinite union of fundamental domains Δ and $\widetilde{M}^3(\Gamma)$ an infinite union of fundamental domains δ. From the group Γ comes a Gromov word-length $\|\delta\|$, which is well-defined up to quasi-isometry.

Conjecture 9. *For each finitely presented group Γ there is a representation (12) and a function $\mathbb{Z}_+ \xrightarrow{\mu} \mathbb{Z}_+$ such that for each δ there are at most $\mu(\|\delta\|)$ fundamental domains $\Delta \subset X^3$ coming with $f\Delta \cap \delta \neq \emptyset$. (No Whitehead nightmares!).*

The second author (V.P.) is currently working on this conjecture. The approach relies heavily both on the results of theorem 6, and also on the techniques of the proof of the theorem in question. Differential topology is used again, but not just the high-dimensional manifolds involved in the proof of theorem 6. Now, manifolds modelled on the Hilbert cube play a big role too. Their virtue is that they combine infinite dimensionality and compactness, all in one. When handlebody structures and surgery are concerned, this may become a big technical help.

If true, as it is hoped, conjecture 9 implies easily that each Γ has an easy 2-dimensional representation. This means an easy implication going from conjecture 9 to theorem 6, a deceptive statement indeed, since the proof of the conjecture is supposed to rely heavily on theorem 6.

Finally, concerning the function μ which occurs in the statement of the conjecture 9, its asymptotic properties might be a sort of new invariant for discrete groups, worth investigating.

This is as much as we want to say concerning (7-II) and, noticing that all the preceding discussions starting with (5) have used a mixture of low-dimensional topology, double points $M_2(f)$ and group theory, it is hard not to look back into times long bygone when, albeit in a very different format, such mixtures had already occurred.

In the very early fifties, when both topology and group theory were still in their infancy, there was the strange and somewhat tragic figure of Chrystos Papakyriakopoulos. By some very clever arguments concerning the double points of maps of surfaces into 3-manifolds, he managed to prove three theorems, for 3-manifolds, which at that time may have been the best there was in low-dimensional topology (the curious reader may look here into the "nécrologie" which the second author (V.P.) wrote for Papa, at the demand of the Greek Mathematical Society [23]). The most important of these results was the so-called "Sphere Theorem". Then, John Stallings looked in great depth at what was actually going on in this theorem and, almost twenty years later, came with his own celebrated theorem on groups with infinitely many ends. Again, in its days, this may have been the best geometric group theory there was. And then things went on ...

This is about as much as we wanted to tell here concerning representations and double points structures. To end in a more philosophical vein, one might muse on why we should focus on GSC rather that, let us say, on "no handles of index $\lambda = 2$, or $\lambda = 3$, or ..."? Two reasons, at least, come here to our minds. To begin with, while the $\lambda = 1$ issue belongs to the non-commutative realm, the $\lambda \geq 2$ is essentially bound to the commutative one. Next, as the present paper might have already made it abundantly clear, not only is GSC related to some of the most hotly burning issues in low-dimensional topology but it looms big in group theory too.

Those of us who are more historically minded might also remember here Smale's initial grab at the high-dimensional h-cobordism Theorem, Stallings's reaction to it and then Smale's final courageous answer; some of this story is told in [41].

Tameness conditions for discrete groups

As already said, a concept which is strongly connected with the GSC is the ***weak geometric simple connectivity*** (or ***WGSC***) (see [7], [8] and [18]). The WGSC property for polyhedra can be viewed as the piecewise-linear counterpart of the geometric simple connectivity of open manifolds. Specifically, a polyhedron is WGSC if it admits an exhaustion by compact connected and simply connected polyhedra.

This condition should also be compared with the QSF, which basically amounts to finding an exhaustion "approximable" by finite simply connected complexes. But the WGSC is more flexible than the GSC, and enables us to work within the realm of polyhedra and hence (discrete) groups. Of course, on the other side, the GSC is much closer to collapsibility, something which goes far beyond the simple-connectivity.

However, it is possible to show that the WGSC and the GSC are equivalent for non-compact manifolds of dimension different from 4 (under the additional irreducibility assumption for dimension 3): one needs to use a classical result of Wall asserting that the GSC is equivalent, in dimension at least 5 and in the compact case, to the simple connectivity (for more details see [7]).

A good "extension" of the GSC to non-simply connected spaces, which is suitable for applications to 3-manifolds, is the ***Tucker property*** (see [52]), which we recall here. The non-compact PL space X is ***Tucker*** if, for any finite subcomplex K of X, the fundamental group of each component of $X - K$ is finitely generated.

This definition was motivated by Tucker's work on 3-manifolds with boundary (see [52]). In the realm of manifolds with boundary the relevant tameness condition that somehow replace the simple connectivity at infinity is the following notion: a non-compact manifold is a ***missing boundary*** manifold if it is obtained from a compact manifold with boundary by removing a closed subset of its boundary. In [52] it is shown that a $\mathbb{P}^2$-irreducible connected 3-manifold is a missing boundary 3-manifold if and only if it is Tucker (where a manifold is $\mathbb{P}^2$-irreducible if it is irreducible and does not contain 2-sided projective planes properly embedded).

Notice also that there is a big difference between the GSC and the Tucker condition: while the GSC is the property of having no extra 1-handles, the Tucker property is related (more or less) to the fact that some handlebody decomposition needs only finitely many 1-handles, without any control on their number.

Furthermore, it turns out that the Tucker property can also be formulated as a group theoretical property for coverings and, actually, it is equivalent, for discrete groups, to a combinatorial condition of (metric) complexes: the ***tame 1-combability*** (see [15]).

Group combings were essential ingredients in Thurston's attempt, inspired from the theory of automatic groups, to abstract finiteness properties of fundamental groups of negatively curved manifolds, while *tame* 1-combings of groups were considered by M. Mihalik and S. Tschantz in [15], as higher dimensional analogs of usual combings. We don't want to say more on these interesting topics here, and we refer to [15] and [8] for details.

What we just want to notice now is that all these notions may have different flavors in (geometric) group theory. In this setting there are, in fact, three "levels" of group properties. We have *combinatorial* invariants, which are well-defined group theoretical notions for (discrete) groups (i.e. presentation independent); there are *geometric* invariants in the sense of M. Gromov, namely those properties that are invariant under the (metric) notion of quasi-isometry; and finally one has the topological properties imported from the realm of infinite complexes by means of the following recipe.

Say that a finitely presented group has a certain property A if the universal covering of SOME (and not necessarily all) finite complex with this fundamental group has the required property.

It is exactly in this context that we can speak about the GSC, or the WGSC, for finitely presented groups.

All this being said, we can come back to our properties and be more precise in which sense they are slightly different. Actually, we have that the Tucker property is a geometric invariant [15] (and also a proper homotopy invariant [19]). The QSF is a combinatorial property for groups [3]: if one universal covering of a finite complex with given fundamental group is QSF then all such universal coverings are QSF. While this is not anymore true for the WGSC condition: there are examples of presentations of a WGSC group which lead to non WGSC complexes (see [8]).

In order to compare such different notions of groups, the second author (D.O.) with Louis Funar introduced in [8] an equivalence relation for these topological conditions defined for groups.

Say that two topological properties A and B are ***almost-equivalent*** for finitely presented groups if a finitely presented group has A if and only if it has B.

One of the results of [8] by the first author (D.O.) and Louis Funar says that **all** these properties define the same class of groups. More precisely

Theorem 10. *The GSC, QSF, WGSC, the tame 1-combability and the Tucker property are almost-equivalent topological properties of finitely presented groups.*

This, in particular, implies the following useful corollary

Corollary 11. *The group Γ is QSF if and only if the universal covering $\widetilde{M}^n$ of any compact manifold M^n with $\pi_1(M) = \Gamma$ and dimension $n \geq 5$ is GSC.*

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