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On an irreducibility criterion of Perron for multivariate polynomials

by
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Dedicated to the memory of Laurenţiu Panaitopol (1940-2008) on the occasion of his 70th anniversary

Abstract

We combine a method of L. Panaitopol with some techniques for nonarchimedean absolute values to provide a new proof for an irreducibility criterion of Perron for multivariate polynomials over an arbitrary field.

Key Words: Irreducibility, nonarchimedean absolute value.
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1 Introduction

A famous irreducibility criterion that requires no information on the canonical decomposition of the coefficients of an integer polynomial is the following result of Perron [5].

Theorem A (Perron) *Let* $F(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0 \in \mathbb{Z}[X]$, *with* $a_0 \neq 0$. *If* $|a_{n-1}| > 1 + |a_{n-2}| + \cdots + |a_1| + |a_0|$, *then* F *is irreducible in* $\mathbb{Z}[X]$.

Using some techniques that require the study of sheets of a Riemann surface, Perron also stated in [5], in a slightly modified form, the following analogous irreducibility criterion for polynomials in two variables over an arbitrary field.

Theorem B (Perron) *Let* K *be a field,* $F(X,Y) = a_n(X)Y^n + \cdots + a_1(X)Y + a_0 \in K[X,Y]$, *with* $a_0, \ldots, a_{n-1} \in K[X]$, $a_n \in K$, $a_0a_n \neq 0$. *If* $\deg a_{n-1} > \max\{\deg a_0, \deg a_1, \ldots, \deg a_{n-2}\}$, *then* F *is irreducible over* $K(X)$.

As an immediate consequence of Theorem B, one may formulate a similar irreducibility criterion for polynomials in $r \geq 3$ variables $X_1, X_2, \ldots, X_r$ over K.

For any polynomial $f \in K[X_1, \ldots, X_r]$ we denote by $\deg_r f$ the degree of f as a polynomial in X_r with coefficients in $K[X_1, \ldots, X_{r-1}]$. The next result follows from Theorem B by writing Y for X_r, X for X_{r-1} and by replacing K with $K(X_1, \ldots, X_{r-2})$.

Theorem C (Perron) *Let K be a field, $r \geq 3$, and let $F = \sum_{i=0}^{n} a_i X_r^i \in K[X_1, \ldots, X_r]$ with $a_0, \ldots, a_{n-1} \in K[X_1, \ldots, X_{r-1}]$, $a_n \in K[X_1, \ldots, X_{r-2}]$ and $a_0 a_n \neq 0$. If*

$$\deg_{r-1} a_{n-1} > \max\{\deg_{r-1} a_0, \deg_{r-1} a_1, \ldots, \deg_{r-1} a_{n-2}\},$$

then F as a polynomial in X_r is irreducible over $K(X_1, \ldots, X_{r-1})$.

By a clever use of the triangle inequality, L. Panaitopol [4] obtained an elegant elementary proof of Theorem A, that makes no use of Rouché's Theorem. The aim of this note is to provide a new proof of Theorem B, based on ideas from [4] combined with the techniques used in [1], [2] and [3].

Before proceeding to the proof of Theorem B we note that the two conditions $a_n \in K$ and $\deg a_{n-1} > \max\{\deg a_0, \deg a_1, \ldots, \deg a_{n-2}\}$ are best possible, in the sense that there exist polynomials in $K[X, Y]$ for which either $a_n \notin K$ and $\deg a_{n-1} > \max\{\deg a_0, \deg a_1, \ldots, \deg a_{n-2}\}$, or $a_n \in K$ and $\deg a_{n-1} = \max\{\deg a_0, \deg a_1, \ldots, \deg a_{n-2}\}$, and which are reducible over $K[X]$.

To see this, one may first choose $F_1(X, Y) = XY^2 + (X^2 + 1)Y + X$. In this case $\deg a_1 > \deg a_0$, but $a_2 \notin K$, and F_1 is obviously reducible, since $F_1(X, Y) = (XY + 1)(Y + X)$. For the second case one may choose $F_2(X, Y) = Y^2 + (X + 1)Y + X$. Here $a_2 \in K$, but $\deg a_1 = \deg a_0$, and F_2 is reducible too, since $F_2(X, Y) = (Y + X)(Y + 1)$.

2 Proof of Theorem B.

We will give a proof based on the study of the location of the roots of F, regarded as a polynomial in Y with coefficients in $K[X]$. We first introduce a nonarchimedean absolute value $|\cdot|$ on $K(X)$, as follows. We fix an arbitrary real number $\rho > 1$, and for any polynomial $u(X) \in K[X]$ we define $|u(X)|$ by the equality

$$|u(X)| = \rho^{\deg u(X)}.$$

We then extend the absolute value $|\cdot|$ to $K(X)$ by multiplicativity. Thus for any $w(X) \in K(X)$, $w(X) = \frac{u(X)}{v(X)}$, with $u(X), v(X) \in K[X]$, $v(X) \neq 0$, we let $|w(X)| = \frac{|u(X)|}{|v(X)|}$. Let us note that for any non-zero element u of $K[X]$ one has $|u| \geq 1$. Let now $\overline{K(X)}$ be a fixed algebraic closure of $K(X)$, and let us fix an extension of our absolute value $|\cdot|$ to $\overline{K(X)}$, which we will also denote by $|\cdot|$.

Using our absolute value, the condition

$$\deg a_{n-1} > \max\{\deg a_0, \deg a_1, \ldots, \deg a_{n-2}\}$$

reads

$$|a_{n-1}| > \max\{|a_0|, |a_1|, \ldots, |a_{n-2}|\}. \tag{1}$$

We also have

$$|a_0| \geq |a_n| = 1. \tag{2}$$

We consider now the factorisation of the polynomial $F(X,Y)$ over $\overline{K(X)}$, say

$$F(X,Y) = a_n(X)(Y-\theta_1)\cdots(Y-\theta_n),$$

with $\theta_1, \ldots, \theta_n \in \overline{K(X)}$. Since $a_0 \neq 0$ we must have $|\theta_i| \neq 0$, $i = 1, \ldots, n$.

We will prove now that conditions (1) and (2) force F to have a single root θ with $|\theta| > 1$, and all the other roots with $|\theta| < 1$.

To see this, we will first prove that F has no roots θ with $|\theta| = 1$. Indeed, if F would have a root θ with $|\theta| = 1$, then $-a_{n-1}\theta^{n-1} = a_n\theta^n + a_{n-2}\theta^{n-2} + \cdots + a_1\theta + a_0$ and hence

$$\begin{aligned}
|a_{n-1}| &= |a_{n-1}| \cdot |\theta|^{n-1} = |a_n\theta^n + a_{n-2}\theta^{n-2} + \cdots + a_1\theta + a_0| \\
&\leq \max\{|a_n\theta^n|, |a_{n-2}\theta^{n-2}|, \ldots, |a_1\theta|, |a_0|\} \\
&= \max\{|a_n| \cdot |\theta|^n, |a_{n-2}| \cdot |\theta|^{n-2}, \ldots, |a_1| \cdot |\theta|, |a_0|\} \\
&= \max\{|a_n|, |a_{n-2}|, \ldots, |a_1|, |a_0|\},
\end{aligned}$$

which cannot hold, according to (1) and (2).

On the other hand $|\theta_1 \cdots \theta_n| = |a_0|/|a_n| \geq 1$, so F must have at least one root θ with $|\theta| > 1$, say $\theta = \theta_1$. Therefore we may write F as $F(X,Y) = (Y-\theta_1) \cdot G(X,Y)$, with

$$G(X,Y) = a_n(Y-\theta_2)\cdots(Y-\theta_n) =$$

$$= b_{n-1}Y^{n-1} + b_{n-2}Y^{n-2} + \cdots + b_1Y + b_0 \in \overline{K(X)}[Y].$$

Equating the coefficients in the equality

$$a_nY^n + \cdots + a_1Y + a_0 = (Y-\theta_1)(b_{n-1}Y^{n-1} + \cdots + b_1Y + b_0),$$

we deduce that

$$\begin{aligned}
a_0 &= -\theta_1 b_0 \\
a_i &= b_{i-1} - \theta_1 b_i \quad \text{for } i = 1, 2, \ldots, n-1 \\
a_n &= b_{n-1}.
\end{aligned} \tag{3}$$

Note that by (1) and (2) we have $|a_{n-1}| > \max\{|a_0|, |a_1|, \ldots, |a_{n-2}|, |a_n|\}$. In what follows, we will use a stronger inequality, which is the key factor for the remaining part of the proof. More precisely, since

$$\deg a_{n-1} > \max\{\deg a_0, \deg a_1, \ldots, \deg a_{n-2}\}$$

and $\deg a_0 \geq \deg a_n$, we observe that for sufficiently large ρ we actually have

$$\rho^{\deg a_{n-1}} > \rho^{\deg a_0} + \rho^{\deg a_1} + \cdots + \rho^{\deg a_{n-2}} + \rho^{\deg a_n},$$

that is

$$|a_{n-1}| > |a_0| + |a_1| + \cdots + |a_{n-2}| + |a_n|. \tag{4}$$

Using (3), (4) and the fact that our absolute value also satisfies the triangle inequality, we obtain

$$\begin{aligned}
|b_{n-2}| + |\theta_1| &\geq |b_{n-2} - \theta_1 b_{n-1}| \\
&= |a_{n-1}| > |a_0| + |a_1| + \cdots + |a_{n-2}| + |a_n| \\
&= |\theta_1 b_0| + |b_0 - \theta_1 b_1| + |b_1 - \theta_1 b_2| + \cdots + |b_{n-3} - \theta_1 b_{n-2}| + 1 \\
&\geq |\theta_1 b_0| + (|\theta_1 b_1| - |b_0|) + (|\theta_1 b_2| - |b_1|) + \cdots + \\
&\quad +(|\theta_1 b_{n-2}| - |b_{n-3}|) + 1 \\
&= |\theta_1| \cdot (|b_0| + |b_1| + \cdots + |b_{n-2}|) - (|b_0| + |b_1| + \cdots + |b_{n-3}|) + 1 \\
&= (|\theta_1| - 1) \cdot (|b_0| + |b_1| + \cdots + |b_{n-2}|) + |b_{n-2}| + 1,
\end{aligned}$$

which yields $|\theta_1| - 1 > (|\theta_1| - 1) \cdot (|b_0| + |b_1| + \cdots + |b_{n-2}|)$. After division by $|\theta_1| - 1$, we obtain

$$|b_0| + |b_1| + \cdots + |b_{n-2}| < 1. \tag{5}$$

We will prove now that (5) forces the roots $\theta_2, \ldots, \theta_n$ of G to have all absolute values strictly less than 1. Indeed, if we assume that G has a root θ with $|\theta| \geq 1$, then

$$\begin{aligned}
|\theta|^{n-1} &= |b_{n-1}\theta^{n-1}| = |b_{n-2}\theta^{n-2} + \cdots + b_1\theta + b_0| \\
&\leq |b_{n-2}\theta^{n-2}| + \cdots + |b_1\theta| + |b_0| \\
&\leq |\theta|^{n-1}(|b_{n-2}| + \cdots + |b_1| + |b_0|),
\end{aligned}$$

and hence $|b_{n-2}| + \cdots + |b_1| + |b_0| \geq 1$, which contradicts (5). Therefore $|\theta_i| < 1$ for $i = 2, \ldots, n$.

We can prove now that F is irreducible over $K(X)$. Let us assume by the contrary that F decomposes as $F(X,Y) = F_1(X,Y) \cdot F_2(X,Y)$, with $F_1,\ F_2 \in K(X)[Y]$, $\deg_Y F_1 = t \geq 1$ and $\deg_Y F_2 = s \geq 1$. By the celebrated lemma of Gauss we may in fact assume that $F_1,\ F_2 \in K[X,Y]$. Without loss of generality we may further assume that θ_1 is a root of F_2, which implies that all the roots of F_1 have absolute values strictly less than 1. On the other hand, if we write F_1 as $F_1(X,Y) = c_0 + c_1 Y + \cdots + c_t Y^t$, say, with $c_i \in K[X]$, $i = 0, \ldots, t$, then we must have $c_0 \mid a_0$ and $c_t \mid a_n$. This shows that $c_t \in K$, that is $|c_t| = 1$, so the absolute value of the product of the roots of F_1 is $|c_0|/|c_t| = |c_0| \geq 1$, which is a contradiction. This completes the proof of the theorem. $\square$

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