On the intervals of a third between Farey fractions
by
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Dedicated to the memory of Laurențiu Panaitopol (1940-2008)
on the occasion of his 70th anniversary

Abstract

The spacing distribution between Farey points has drawn attention in recent years. It was found that the gaps \(\gamma_{j+1} - \gamma_j\) between consecutive elements of the Farey sequence produce, as \(Q \to \infty\), a limiting measure. Numerical computations suggest that for any \(d \geq 2\), the gaps \(\gamma_{j+d} - \gamma_j\) also produce a limiting measure whose support is distinguished by remarkable topological features. Here we prove the existence of the spacing distribution for \(d = 2\) and characterize completely the corresponding support of the measure.

Key Words: Spacing distribution, Farey fractions.

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1 Introduction

Let \(\mathcal{F}_Q = \{\gamma_1, \ldots, \gamma_N\}\) be the Farey sequence of order \(Q\), which is defined to be the set of all subunitary irreducible fractions with denominators \(\leq Q\), arranged in ascending order. For any interval \(I \subset [0, 1]\), we write \(\mathcal{F}_Q(I) = \mathcal{F}_Q \cap I\). The cardinality of \(\mathcal{F}_Q(I)\) is well known to be \(N_I(Q) = 3|I|Q^2/\pi^2 + O(Q \log Q)\). When \(I = [0, 1]\) we write shortly \(N(Q)\) instead of \(N_{[0,1]}(Q)\). Since \(\mathcal{F}_Q\) contains a large number of fractions obtained by a combined process of division, sieving and sorting of integers from \([1, Q]\), one would apriori expect little or even no special structure in the set of all differences between consecutive fractions (which we also call intervals of a second). Though, this expectation is not fulfilled. This is sustained from many points of view by a series of authors, such as Franel [4],

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Kanemitsu, Sita Rama Chandra Rao and Siva Rama Sarma [9], Hall and Tenenbaum [7], [8], Hall [5], Augustin, Boca and the authors [1], who have studied the set of gaps between consecutive Farey fractions. A regularity is expected also in the set of larger gaps $\gamma(d+1) - \gamma'$, where $\gamma'$ runs over $\{\gamma_1, \ldots, \gamma_{N-d}\}$ and $d \geq 2$. (We use up-scripts, such as $\gamma', \gamma'', \gamma'''$, to write consecutive elements of $\mathcal{F}_Q$.) It is our object to treat here the case $d = 2$, that is, the case of intervals of a third.

Geometrical representations of the set of pairs of neighbor intervals of fractions from $\mathcal{F}_Q$ created for different values of $Q$ reveals sets of points whose density concentrates on different parts of the plane. The aesthetical qualities of the pictures catches attention immediately. For any $d \geq 1$ they look like a swallow and the main topological distinctions are in the number of folds of the tail. Thus, when $d = 1$ (neighbor pairs of intervals of a second) the swallow has a one-fold tail (see [1]). When $d = 2$, the case treated in the present paper, the swallow has a two-fold tail (see Figure 1) and in Section 3 we have calculated explicitly the equations of the frontier. In the cases $d \geq 3$ the tail appears always to have a three-folded tail, but this is more complex and its characterization will appear in a separate paper.

Given $N$ real numbers $x_1 \leq x_2 \leq \cdots \leq x_N$ with mean spacing 1, we consider the $h$-th level of intervals of a third probability $\mu_{2,h}$ on $R_+^n$, defined, for $f \in C_c([0, \infty))$, by

$$\int_{[0, \infty)^h} f d\mu_{2,h} = \frac{1}{N - h - 1} \sum_{j=1}^{N-h-1} f(x_{j+2} - x_j, x_{j+3} - x_{j+1}, \ldots, x_{j+h+1} - x_{j+h-1}).$$

In our case, we normalize $\mathcal{F}_Q$ to get the sequence $x_j = N(Q, I)\gamma_j/|I|$, $1 \leq j \leq N(Q, I)$ with mean spacing equal to one. Accordingly, we get the sequence $(\mu_{2,h})_{Q \geq 1}$ of the $h$-th level of intervals of a third probabilities on $[0, \infty)^h$. We show that this sequence converges, as $Q \to \infty$, to a probability measure $\mu_{2,h}$, which is independent of $I$, and can be expressed explicitly.

For any $\gamma_i = a_i/q_i$ and $\gamma_j = a_j/q_j$ in $\mathcal{F}_Q$, we set $\Delta(\gamma_i, \gamma_j) = \Delta(i, j) = -\left[\frac{a_i}{q_i}, \frac{a_j}{q_j}\right]$. This is the numerator of the difference $\gamma_j - \gamma_i$. It is well known that $\Delta(\gamma', \gamma'') = -1$ for any consecutive elements of $\mathcal{F}_Q$, and it turns out that this equality is responsible for the existence of the $h$-spacing distribution of the Farey sequence. Though, this relation is no longer true for larger intervals, but there is a convenient replacement. To see this, let us note that a Farey fraction can be uniquely determined by its two predecessors. Indeed, if $\frac{a}{q} < \frac{a'}{q'} < \frac{a''}{q''}$ are consecutive fractions of $\mathcal{F}_Q$, we have $d'' = ka'' - d$ and $d''' = kq'' - d'$, where $k = \Delta(\gamma', \gamma'') = \left[\frac{a'+a''}{q'}\right]$.

The basic idea of our procedure is to parametrize the set of $h$-tuples of intervals of a third in terms of just two variables that run over a completely described
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Figure 1: The support of $\mu_{2,2}$.

domain. The set of pairs of consecutive denominators of fractions in $\mathfrak{F}_Q$ are exactly the elements of

$$\{(q', q''): 1 \leq q', q'' \leq Q, q' + q'' > Q \text{ and } (q', q'') = 1\}.$$  

Since we are mainly interested in what happens when $Q \to \infty$, we reduce the scale $Q$ times, and consider the background triangle $T = \{(x, y): \ 0 < x \leq 1, \ x + y > 1\}$, called the Farey triangle. We split it into a series of polygons as follows. Firstly, for each $(x, y) \in \mathbb{R}^2$, we set $L_0(x, y) = x, L_1(x, y) = y$, and then, for $i \geq 2$, we define recursively:

$$L_i(x, y) = \left[\frac{1 + L_{i-2}(x, y)}{L_{i-1}(x, y)}\right] L_{i-1}(x, y) - L_{i-2}(x, y).$$  

Then, as in [3], we consider the map

$$k : T \to (\mathbb{N}^*)^h, \quad k(x, y) = (k_1(x, y), \ldots, k_h(x, y)).$$
where \( k_i(x, y) = \left[ \frac{1 + L_{i-1}(x, y)}{L_i(x, y)} \right] \). The functions \( k_i(x, y) \) are locally constant, and the subsets of \( T \) on which they are constant plays a special role. Thus, for any \( k \in (\mathbb{N}^*)^h \), we get the convex polygon

\[
T_k = \{(x, y) \in T: \; k(x, y) = k\}.
\]

Notice that \( T = \bigcup_{k \in (\mathbb{N}^*)^h} T_k \) and \( T_k \cap T_{k'} = \emptyset \) whenever \( k \neq k' \).

Next we consider the application \( \Phi_{2,h} : T \to (0, \infty)^h \) defined by

\[
\Phi_{2,h}(x, y) = \frac{3}{\pi^2} \left( \frac{k_1(x, y)}{L_0(x, y) L_2(x, y)} + \frac{k_2(x, y)}{L_1(x, y) L_3(x, y)} + \cdots + \frac{k_h(x, y)}{L_{h-1}(x, y) L_{h+1}(x, y)} \right).
\]

Our main result shows that, indeed, for \( Q \to \infty \), the sequence \( (\mu_{2,h}^Q)_{Q \geq 1} \) converges to a measure and \( \Phi_{2,h}(x, y) \) is the needed tool to describe its support.

**Theorem 1.** The sequence \( (\mu_{2,h}^Q)_{Q \geq 1} \) converges weakly to a probability measure \( \mu_{2,h} \), which is independent of \( I \). The support \( D_{2,h} \) of \( \mu_{2,h} \) is the closure of the range of \( \Phi_{2,h} \), and

\[
\mu_{2,h}(C) = 2 \text{Area}(\Phi_{2,h}^{-1}(C)),
\]

for any parallelepiped \( C = \prod_{j=1}^h (\alpha_j, \beta_j) \subset (0, \infty)^h \).

In Table 1 from Section 3 we provide explicit formulae for all the pieces that form \( D_{2,2} \).

## 2 The Existence of the Limiting Measure

It is plain that in order to prove Theorem 1, it suffices to see the effect of \( \mu_{2,h} \) on bounded parallelepipeds. For any \( C = \prod_{j=1}^h (\alpha_j, \beta_j) \subset (0, \infty)^h \), we define

\[
\mu_{2,h}^{Q, I}(C) := \frac{1}{N_I(Q)} \cdot \# \left\{ \gamma_j \in \mathcal{F}_I(Q): \; \frac{\alpha_i}{N_I(Q)} < \gamma_j < \frac{\beta_i}{N_I(Q)} \quad \text{for} \; i = 1, \ldots, h \right\}.
\]

We have to show that the sequence \( \{\mu_{2,h}^{Q, I}\}_{Q} \) is convergent when \( Q \to \infty \) and the limit is independent of \( I \). In the beginning we treat the case of the complete interval \( I = [0, 1] \).

### 2.1 The case \( I = [0, 1] \)

In the following we write shortly \( \mu_{2,h}^Q \) instead of \( \mu_{2,h}^{Q, [0,1]} \).

With the notations from the Introduction, we see that \( \gamma_j = k_{j+1}/q_{j+1}q_{j+1} \). Then \( \mu_{2,h}^Q(C) \) can be written as

\[
\mu_{2,h}^Q(C) = \frac{1}{N(Q)} \cdot \# \left\{ \gamma_j \in \mathcal{F}(Q): \; \frac{N(Q)}{\alpha_i} < \frac{q_{j+i+1}q_{j+i+1}}{k_{j+i}} < \frac{N(Q)}{\beta_i} \quad \text{for} \; i = 1, \ldots, h \right\}.
\]
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Knowing that \( q_{j+i} = Q L_i(q_j/Q, q_{j+1}/Q) \), we consider the set

\[
\Omega^Q(C) = \left\{ (x, y) \in QT : \frac{N(Q)}{Q^2} \frac{L_{i-1}(\tilde{y}, \tilde{y})}{k_i(\tilde{y}, \tilde{y})} < \frac{N(Q)}{Q^{2\alpha_i}} \right\}.
\] (2)

Since neighbor denominators in \( \mathcal{F}_Q \) are always coprime, relation (1) turns into

\[
\mu^Q_{2,h}(C) = \frac{1}{N(Q)} \cdot \# \left\{ (x, y) \in \Omega^Q(C) \cap N^2 : \gcd(x, y) = 1 \right\}.
\]

Next, we select the points with coprime coordinates using Möbius summation (cf. [1, Lemma 2]), and we find that

\[
\mu^Q_{2,h}(C) = \frac{6Q^2}{\pi^2 N(Q)} \text{Area}(\Omega(C)) + O\left( \text{length}(\partial\Omega^Q(C)) \log Q \right).
\] (3)

Splitting \( T \) into the series of polygons \( T_k \), we see that the error term in (3) is \( O(Q \log Q) \). In the main term, we replace \( \Omega^Q(C) \) by the bounded set \( \Omega(C) = \Omega^Q(C)/Q \). These yield

\[
\mu^Q_{2,h}(C) = \frac{6Q^2}{\pi^2 N(Q)} \text{Area}(\Omega(C)) + O\left( \frac{\log Q}{Q} \right).
\] (4)

It remains to replace in (4) the set \( \Omega(C) \) by a set as in (2), but with bounds independent of \( Q \) in the corresponding inequalities. This set is

\[
\mathcal{D}(C) := \left\{ (x, y) \in QT : \frac{3}{\pi^2 \beta_i} \frac{L_{i-1}(x, y)}{k_i(x, y)} < \frac{3}{\pi^2 \alpha_i} \right\}.
\] (5)

Notice that \( \mathcal{D}(C) \) is exactly \( \Phi_{2,h}^{-1}(C) \). The replacement does not change the error term because, via \( N(Q) = 3Q^2/\pi^2 + O(Q \log Q) \), we have:

\[
\max_{1 \leq i \leq h} \left\{ \frac{|N(Q)/\alpha_i Q^2 - 3/\pi^2 \alpha_i|}{|N(Q)/\beta_i Q^2 - 3/\pi^2 \beta_i|} \right\} = O(C\left( \frac{\log Q}{Q} \right)),
\] (6)

which implies

\[
\text{Area}(\Omega(C) \triangle \mathcal{D}(C)) = O(C\left( \frac{\log Q}{Q} \right)).
\] (7)

Therefore, by (5) and (7), we get

\[
\mu^Q_{2,h}(C) = 2 \text{Area}(\mathcal{D}(C)) + O\left( \frac{\log Q}{Q} \right).
\] (8)

In particular, this gives \( \mu_{2,h}(C) = \lim_{Q \to \infty} \mu^Q_{2,h}(C) = 2 \text{Area}(\mathcal{D}(C)) \), concluding the proof of the theorem when \( I = [0, 1] \).
2.2 The short interval case

Suppose now that \( I \subset [0, 1] \) is fixed. In order to impose the condition that only the fractions from \( I \) are involved in the calculations, we employ the fundamental property of neighbor fractions in \( \mathcal{G}_Q \). This says that if \( \gamma' = a'/q' \) and \( \gamma'' = a''/q'' \) are consecutive then \( a''q' - a'q'' = 1 \). Consequently, \( a'' \equiv (q'')^{-1} \pmod{q''} \), and this allows us to write the fraction \( a''/q'' \) in terms of \( q' \) and \( q'' \). Thus

\[
a''/q'' \in \mathcal{I} \iff (q')^{-1} \pmod{q''} \in q''\mathcal{I}.
\]

This time we have to estimate

\[
\mu_{2, h}^{Q, I}(C) = \frac{1}{N_2(Q)} \cdot \# \Omega_{I}^{Q},
\]

where

\[
\Omega_{I}^{Q} = \left\{ (q', q'') \in Q\mathcal{I} : \frac{N_2(Q)}{|(q')^2 R_i|} \leq \frac{L_{i-1}(\frac{q'}{r}, \frac{q''}{r})}{k_i(\frac{q'}{r}, \frac{q''}{r})} < \frac{N_2(Q)}{|(q'')^2 R_i|}, \quad \text{for } i = 1, \ldots, h; \quad (q')^{-1} \pmod{q''} \in q''\mathcal{I} \right\}.
\]

We may write (9) as

\[
\mu_{2, h}^{Q, I}(C) = \frac{1}{N_2(Q)} \sum_{q=1}^{Q} N_q(I^Q(q), q\mathcal{I}),
\]

where

\[
N_q(I_1, I_2) = \# \{(m, n) \in I_1 \times I_2 : mn \equiv 1 \pmod{q} \},
\]

for any \( I_1, I_2 \subset [0, Q - 1] \) and

\[
I^Q(q) = \left\{ x \in (Q - q, Q) : \frac{N_2(Q)}{|(q')^2 R_i|} \leq \frac{L_{i-1}(\frac{q'}{r}, \frac{q''}{r})}{k_i(\frac{q'}{r}, \frac{q''}{r})} < \frac{N_2(Q)}{|(q'')^2 R_i|}, \quad \text{for } i = 1, \ldots, h \right\}.
\]

For the best available technique to estimate the size of \( N_q(I_1, I_2) \) one requires bounds for Kloosterman sums (cf. [2]). This is done when \( I_1 \) and \( I_2 \) are intervals, but it may be easily extended for finite unions of subintervals of \( [0, Q - 1] \) (as the set \( I^Q(q) \) is), even with the same formula. For our needs here, it suffices a version with a slightly weaker term:

\[
N_q(I^Q(q), q\mathcal{I}) = \frac{\varphi(q)|I^Q(q)| \cdot |\mathcal{I}|}{q} + O_{C, \varepsilon}(q^{1/2 + \varepsilon}).
\]

Inserting (11) into (10), we get

\[
\mu_{2, h}^{Q, I}(C) = \frac{|\mathcal{I}|}{N_2(Q)} \sum_{q=1}^{Q} \frac{\varphi(q)|I^Q(q)|}{q} + O_{C, \varepsilon}(Q^{-1/2 + \varepsilon}).
\]

To calculate the sum in (12), we employ the Euler-MacLaurin formula, noticing the fact that \( |J^Q_C(q)| \), as a function of \( q \), is piecewise continuous differentiable on \([0, 1]\). We obtain

\[
\sum_{q=1}^{Q} \frac{\varphi(q)|J^Q_C(q)|}{q} = \frac{1}{\zeta(2)} \int_{1}^{Q} |J^Q_C(q)| \, dq + O\left( \sum_{1 \leq q \leq Q} |J^Q_C(q)| + \int_{1}^{Q} \frac{\partial}{\partial q} |J^Q_C(q)| \, dq \right) \log Q.
\]

The size of the error term is estimated observing, firstly, that \( |J^Q_C(q)| \leq Q \). Secondly, by the definition of \( J^Q_C(q) \) it follows that there exists a partition of \([1, Q]\) in finitely many intervals with the property that the cardinality of \( J^Q_C(q) \) is monotonic on each of them. Therefore

\[
\int_{1}^{Q} \frac{\partial}{\partial q} |J^Q_C(q)| \, dq = O_C(Q).
\]

Then, forgathering (13), (14), (6) in (12) and using again the estimate \( N_I(Q) = 3|I|Q^{2}/\pi^{2} + O(Q \log Q) \), we obtain

\[
\mu_{2,h}^{Q,I}(C) = \frac{6|I|}{\pi^2 N_I(Q)} \int_{1}^{Q} |J^Q_C(q)| \, dq + O_{C,\varepsilon}(Q^{-1/2+\varepsilon})
\]

\[
= 2 \text{Area} (D(Q)) + O_{C,\varepsilon}(Q^{-1/2+\varepsilon}).
\]

This concludes the proof of the theorem.

### 3 The Support of the Limiting Measure

For \( h = 1 \), we have \( D_{2,1} = [6/\pi^2, \infty) \). For \( h \geq 2 \), by Theorem 1, it follows that \( D_{2,h} \) is a countable union of hyper-surfaces in \([6/\pi^2, \infty)^h \).

The support \( D_{2,h} \) has some striking features. Let us see them in the case \( h = 2 \). We write \( k = (k, l) \) and observe that

\[
T_{k,l} = \left\{ (x, y) \in T_k : \frac{1 + (l+1)x}{k(l+1) - 1} < y \leq \frac{1 + lx}{kl - 1} \right\}.
\]

Roughly speaking, by definition we find that \( T_{k} \) corresponds to the set of 3-tuples \((\gamma', \gamma'', \gamma''')\) of consecutive elements of \( \mathcal{F}_Q \) with the property that \( \Delta(\gamma', \gamma'''') = k \). Similarly, \( T_{k,l} \) corresponds to the set of 4-tuples \((\gamma', \gamma'', \gamma''', \gamma''''\) of consecutive elements of \( \mathcal{F}_Q \) with the property that \( \Delta(\gamma', \gamma''') = k \) and \( \Delta(\gamma'', \gamma''') = l \). We
remark that $T_{1,1} = \emptyset$, and also $T_{k,l} = \emptyset$ whenever both $k$ and $l$ are $\geq 2$, except when $(k, l) \in \{(2,2); (2,3); (2,4); (3,2); (4,2)\}$. Notice that the symmetry of the Farey sequence of order $Q$ with respect to $1/2$ produces a sort of balance between the polygons $T_{k,l}$ and $T_{l,k}$.

Then the support $D_{2,h}$ is the closure of the image of the function $\Phi_{2,2}$, which can be written as

$$\Phi_{2,2}(x, y) = \frac{3}{\pi^2} \left( \frac{k}{xz}, \frac{l}{yt} \right),$$

in which $z = x - ky$, $t = y - lt$, for $(x, y) \in T_{k,l}$. A tedious, but elementary, computation allows us to find precisely the boundaries of $\Phi_{2,2}(T_{k,l})$. The image obtained is shown in Figure 1 and the equations are listed in Table 1. All the functions that produce the equations of the boundaries of $\Phi_{2,2}(T_{k,l})$ are either of the form

$$\frac{at}{a + bt + c \sqrt{t(t - d)}},$$

with $t$ in a certain interval that might be unbounded, or the symmetric with respect to $x = y$ of such a curve. Here $a, b, c, d, e$ are integers.

We conclude by making a few remarks. Firstly, we mention that $\Phi_{2,2}$ has a symmetrization effect, namely, it makes $\Phi_{2,2}(T_{m,n})$ and $\Phi_{2,2}(T_{n,m})$ to be symmetric with respect to the first diagonal $y = x$, for any $m, n \geq 1$. The diamond $\Phi_{2,2}(T_{2,2})$ is the single nonempty domain $\Phi_{2,2}(T_{k,l})$ that has $y = x$ as axis of symmetry. The top of the beak of the swallow $D_{2,h}$ has coordinates $(6/\pi^2, 6/\pi^2)$. The asymptotes of the wings are $y = 6/\pi^2$ and $x = 6/\pi^2$. The highest density is on a region situated in the neck, where many components of the swallow overlap partially or completely.

Table 1 below lists all the equations of the boundaries of $\Phi_{2,2}(T_{k,l})$. In the head of the table $MN$ represents an edge of $T_{k,l}$ (listed in counterclockwise order, starting either from the East or from the North side) and $g_{MN}(t)$ is a parametrization of $\Phi_{2,2}(MN)$.

Table 1: The edges of $D_{2,2}$.  

<table>
<thead>
<tr>
<th>$k, l$</th>
<th>$MN$</th>
<th>$\frac{\pi^2}{4} g_{MN}(t)$</th>
<th>the domain of $t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 2</td>
<td>(1, 4); (0, 1)</td>
<td>$\frac{6t}{\sqrt{t(t - 4)}}$</td>
<td>$\frac{7}{2} \leq t \leq \infty$</td>
</tr>
<tr>
<td>1, 2</td>
<td>(0, 1); (1/5, 4/5)</td>
<td>$\frac{16t}{12 + 3t + 5\sqrt{t(t - 8)}}$</td>
<td>$\frac{25}{3} \leq t \leq \infty$</td>
</tr>
<tr>
<td>1, 2</td>
<td>(1/5, 4/5); (1/4, 1)</td>
<td>$\frac{16t}{12 - 3t + 5\sqrt{t(t + 8)}}$</td>
<td>$\frac{7}{2} \leq t \leq \frac{25}{3}$</td>
</tr>
<tr>
<td>1, 3</td>
<td>(1/2, 1); (1/4, 1)</td>
<td>$\frac{6t}{t + 3\sqrt{t(t - 4)}}$</td>
<td>$4 \leq t \leq \frac{7}{2}$</td>
</tr>
</tbody>
</table>

$^1$Remark that the edges of $\Phi_{2,2}(T_{2,2})$ are close to being, but are not exactly straight lines. The same applies for the edges of the diamonds in the tail.
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<table>
<thead>
<tr>
<th>$k, \ell$</th>
<th>$\mathcal{M} \mathcal{N}$</th>
<th>$\frac{\pi^2}{3} g_{\mathcal{M} \mathcal{N}}(t)$</th>
<th>the domain of $t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 1</td>
<td>(1/2, 1); (1/3, 1/2)</td>
<td>$-t+3\sqrt{t(t+8)}$</td>
<td>$-\frac{t}{2} \leq t \leq \frac{22}{3}$</td>
</tr>
<tr>
<td>1, 2</td>
<td>(1/3, 1/2); (1/4, 1/3)</td>
<td>$-20+7\sqrt{t(t+8)}$</td>
<td>$8 \leq t \leq \frac{22}{3}$</td>
</tr>
<tr>
<td>1, 2</td>
<td>(1/4, 1/3); (1/5, 1/4)</td>
<td>$-20+7\sqrt{t(t+8)}$</td>
<td>$8 \leq t \leq \frac{22}{3}$</td>
</tr>
<tr>
<td>1, 2</td>
<td>(1/5, 1/4); (1/6, 1/5)</td>
<td>$-24-7t+11\sqrt{t(t+12)}$</td>
<td>$4 \leq t \leq \frac{22}{3}$</td>
</tr>
<tr>
<td>1, 3</td>
<td>(1/6, 1/5); (1/7, 1/6)</td>
<td>$-2t+\sqrt{t(t+4)}$</td>
<td>$\frac{22}{5} \leq t \leq 4$</td>
</tr>
<tr>
<td>1, 3</td>
<td>(1/7, 1/6); (1/8, 1/7)</td>
<td>$-t+3\sqrt{t(t+12)}$</td>
<td>$4 \leq t \leq \frac{22}{3}$</td>
</tr>
<tr>
<td>1, 3</td>
<td>(1/8, 1/7); (1/9, 1/8)</td>
<td>$-28+11t-13\sqrt{t(t-8)}$</td>
<td>$\frac{19}{6} \leq t \leq 9$</td>
</tr>
<tr>
<td>1, 3</td>
<td>(1/9, 1/8); (1/10, 1/9)</td>
<td>$-40-13t+10\sqrt{t(t+16)}$</td>
<td>$\frac{22}{6} \leq t \leq 9$</td>
</tr>
<tr>
<td>1, 4</td>
<td>(1/10, 1/9); (1/11, 1/10)</td>
<td>$\frac{2t}{(t-2)t-\sqrt{t(t-4)}}$</td>
<td>$\frac{t^2}{2(t-2)} \leq t \leq \frac{(t+1)^2}{2(t-1)}$</td>
</tr>
<tr>
<td>1, 4</td>
<td>(1/11, 1/10); (1/12, 1/11)</td>
<td>$\frac{2(t-1)t}{(t-2)t-\sqrt{t(t+4)}}$</td>
<td>$\frac{t^2}{2(t-2)} \leq t \leq \frac{(t+1)^2}{2(t-3)}$</td>
</tr>
<tr>
<td>1, 4</td>
<td>(1/12, 1/11); (1/13, 1/12)</td>
<td>$\frac{2t-\sqrt{t(t+12)}}{4+4t-3t-3t\sqrt{t(t-8)}}$</td>
<td>$\frac{(t+1)^2}{2(t-3)} \leq t \leq \frac{(t+1)^2}{2(t-2)}$</td>
</tr>
<tr>
<td>1, 5</td>
<td>(1/13, 1/12); (1/14, 1/13)</td>
<td>$\frac{2t-\sqrt{t(t+12)}}{4+4t-3t-3t\sqrt{t(t-8)}}$</td>
<td>$\frac{(t+1)^2}{2(t-3)} \leq t \leq \frac{(t+1)^2}{2(t-2)}$</td>
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<tr>
<td>1, 5</td>
<td>(1/14, 1/13): (1/15, 1/14)</td>
<td>$\frac{4t^2t^2}{(1-2)t+\sqrt{t(t+6)}}$</td>
<td>$\frac{(t+1)^2}{2(t-1)} \leq t \leq \frac{(t+2)^2}{2(t-2)}$</td>
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<td>$\frac{4t}{(t+2)(t-2)}$</td>
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<td>$-12+4t-3\sqrt{t(t-6)}$</td>
<td>$6 \leq t \leq \frac{19}{4}$</td>
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<td>$\frac{2t}{24-2t+7\sqrt{t(t-6)}}$</td>
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<table>
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<th>$k, l$</th>
<th>$M/N$</th>
<th>$\frac{n^2}{3} g_{M/N} (t)$</th>
<th>the domain of $t$</th>
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<td>2, 3</td>
<td>$\left( \frac{1}{5}, \frac{2}{5} \right); \left( \frac{1}{7}, \frac{2}{7} \right)$</td>
<td>$\frac{144}{5} t \geq 56 - 225 + 24 \sqrt{t(t+1)}$</td>
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<td>$\frac{94}{5} t \geq (t+3)(2t-3)$</td>
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<td>$\frac{-72+31t-11 \sqrt{3t(3t-16)}}{4} t \geq -60 - 23t + 9 \sqrt{3t(3t+20)}$</td>
<td>$\frac{14}{5} \leq t \leq \frac{142}{20}$</td>
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<td>$\frac{-168+79t-27 \sqrt{3t(3t-16)}}{20} t \geq -72 - 11t + 7 \sqrt{3t(3t-16)}$</td>
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<td>$\frac{164}{5} t \geq (t+4)(3t-4)$</td>
<td>$4 \leq t \leq \frac{20}{3}$</td>
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<td>$\frac{-24}{5} t \geq -80 + 37t - 38 \sqrt{t(t-5)}$</td>
<td>$9 \leq t \leq \frac{29}{3}$</td>
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<td>$\frac{144}{5} t \geq -56 - 23t + 26 \sqrt{t(t+7)}$</td>
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<td>$k \leq \frac{k(k+1)}{k-1}$</td>
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<td>$\frac{4(k+1)^2 t^2}{(t-k+1)(t+k)} \geq \frac{k(k+1) t}{(k+2)(k-4k-4)}$</td>
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On the Intervals of a Third between Farey Fractions

References


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