

Obstinate filters in residuated lattices

by

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Abstract

In this paper we introduce the notion of obstinate filter in a residuated lattice A and we state and prove some theorems which determine the relationship between this notion and other types of filters of a residuated lattice and we show whether (simple, local, quasi-local) residuated lattices have an obstinate filter. Also, we prove that, if F is an obstinate filter of a residuated lattice A , then F is a Boolean filter and A/F is the two-element chain.

Key Words: Residuated lattice, (obstinate, implicative, positive implicative, fantastic, quasi-primary) filter, (simple, local, quasi-local) residuated lattice, Boolean algebra.

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1 Introduction

In [9], Y. Zhu and Y. Xu have introduced the notion of implicative (Boolean) filter and fantastic filter of a residuated lattice. In [5], Muresan has extended the concept of quasi-primary filter and quasi-local algebra to residuated lattices and she has proven that any simple residuated lattice is a local residuated lattice and any local residuated lattice is a quasi-local residuated lattice.

In this paper we define the notion of obstinate filter of a residuated lattice and we state and prove some theorems that determine relationships between this notion and other types of filters of a residuated lattice and we show the relationship between $\text{Reg}(A)$, $\text{D}(A)$ and $\text{B}(A)$ and obstinate filters of a residuated lattice.

2 Preliminaries

We recollect some definitions and results which will be used in the following, without being cited every time they are used.

Definition 2.1. [7] *A residuated lattice is a structure $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ such that:*
(RL1) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice,

(RL2) $(A, *, 1)$ is a commutative monoid,

(RL3) $*$ and $\rightarrow$ form an adjoint pair i.e, $c \leq a \rightarrow b$ if and only if $a * c \leq b$, for all $a, b, c \in A$ (residuation),.

For any element x of a residuated lattice and positive integer n , we denote: $x^- = x \rightarrow 0$ and $x^n = \underbrace{x * \dots * x}_{n \text{ times } x}$

Lemma 2.2. [6] In each residuated lattice A , the following relations hold for all $x, y, z \in A$,

- (1) $x * (x \rightarrow y) \leq y$,
- (2) $x \leq y \rightarrow (x * y)$ and $x \leq y \rightarrow x$,
- (3) $x \leq y$ if and only if $x \rightarrow y = 1$,
- (4) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
- (5) If $x \leq y$, then $y \rightarrow z \leq x \rightarrow z$ and $z \rightarrow x \leq z \rightarrow y$,
- (6) $y \leq (y \rightarrow x) \rightarrow x$,
- (7) $y \rightarrow x \leq (z \rightarrow y) \rightarrow (z \rightarrow x)$,
- (8) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$,
- (9) $x * y \leq x, y$, hence $x * y \leq x \wedge y$ and $x * 0 = 0$,
- (10) $x \leq y$ implies $x * z \leq y * z$,
- (11) $1 \rightarrow x = x$, $x \rightarrow x = 1$, $x \leq y \rightarrow x$, $x \rightarrow 1 = 1$,
- (12) $x * x^- = 0$ and $x^- \leq x \rightarrow y$,
- (13) $x \rightarrow (y \rightarrow z) = (x * y) \rightarrow z$,
- (14) $x \leq y$ implies $y^- \leq x^-$,
- (15) $x \leq x^{--}$, $1^- = 0$, $0^- = 1$, $x^{---} = x^-$, $x^{--} \leq x^- \rightarrow x$,
- (16) $x \rightarrow (\bigwedge_{i \in I} y_i) = \bigwedge_{i \in I} (x \rightarrow y_i)$,
- (17) $(\bigvee_{i \in I} y_i) \rightarrow x = \bigwedge_{i \in I} (y_i \rightarrow x)$.

Definition 2.3. [6] An element a of a residuated lattice A is said to be regular if $a^{--} = a$. The set of all regular elements of A is denoted $\text{Reg}(A)$.

An element a of A is said to be dense iff $a^- = 0$. We denote by $Ds(A)$ the set of the dense elements of A .

We denote by $B(A)$ the Boolean center of A , that is the set of all complemented elements of the bounded lattice reduct of A .

Definition 2.4. [6] A deductive system of a residuated lattice A is a subset F of A such that $1 \in F$ and, for all $x, y \in A$ if $x, x \rightarrow y \in F$, then $y \in F$.

In [7] there has been defined a filter of a residuated lattice A to be a non-empty subset F of A such that, for all $a, b \in A$, (i) if $a, b \in F$, then $a * b \in F$, and (ii) if $a \in F$ and $a \leq b$, then $b \in F$. Note that a subset F of a residuated lattice A is a deductive system of A if and only if F is a filter of A ([7]).

Definition 2.5. [7, 9] Let F be a filter of a residuated lattice A :

★ F is said to be proper if $F \neq A$. A proper filter F is said to be prime if for all $x, y \in A$, $x \vee y \in F$ implies $x \in F$ or $y \in F$. If a proper filter F satisfies: for all $x, y \in A$, either $x \rightarrow y \in F$ or $y \rightarrow x \in F$, then F is a prime filter.

★ A proper filter F of A is said to be maximal iff it is not included in any other proper filter of A . According to [7, Corollary 1.59], a proper filter F of A is maximal iff, for all $x \in A$, the following equivalence holds: $x \notin F$ iff there exists $n \in N^* = N \setminus \{0\}$ such that $(x^n)^- \in F$.

It is a known fact that any maximal filter of a residuated lattice is a prime filter.

★ F is called a Boolean filter if $x \vee x^- \in F$, for all $x \in A$.

Definition 2.6. [9] A subset F of a residuated lattice A is called:

★ A positive implicative filter of A if $1 \in F$ and $x \rightarrow (y \rightarrow z) \in F$ and $x \rightarrow y \in F$ imply that $x \rightarrow z \in F$.

★ An implicative filter if $1 \in F$ and $x \rightarrow ((y \rightarrow z) \rightarrow y) \in F$ and $x \in F$ imply $y \in F$ or equivalently, $(x \rightarrow y) \rightarrow x \in F$ implies $x \in F$. (Implicative filters were called positive implicative filters in [8]).

★ A fantastic filter if $1 \in F$ and $z \rightarrow (y \rightarrow x) \in F$ and $z \in F$ imply $((x \rightarrow y) \rightarrow y) \rightarrow x \in F$.

★ A quasi-primary filter of A iff F is a proper filter and, for all $a, b \in A$, $(a * b)^- \in F$ implies that there exist $u \in A$ and $n \in N^*$ such that $u \vee u^- \in B(A)$, $(a^n * u)^- \in F$ and $(b^n * u^-)^- \in F$.

Definition 2.7. [8] A residuated lattice is said to be local iff it has exactly one maximal filter.

★ Simple residuated lattices are those residuated lattices which have exactly two filters (which, obviously, are the trivial filter and the improper filter). Any simple residuated lattice is local.

★ A residuated lattice A is called quasi-local iff, for any $a \in A$, there exist $e \in B(A)$ and $n \in N^*$ such that $a^n * e = 0$ and $(a^-)^n * (e^-) = 0$.

Any local residuated lattice is quasi-local.

3 Obstinate Filters

From now on, unless mentioned otherwise, $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ will be a residuated lattice, which will often be referred by its support set: A .

Definition 3.1. A filter F of A will be called an obstinate filter of A if and only if it satisfies $0 \notin F$ (i.e. F is a proper filter) and

$x, y \notin F$ imply $x \rightarrow y \in F$ and $y \rightarrow x \in F$.

Proposition 3.2. A proper filter F of A is an obstinate filter if and only if it satisfies the following condition: for all $x \in A$ if $x \notin F$ then there exists $n \geq 1$ such that $(x^-)^n \in F$.

Proof: Suppose that F is an obstinate filter and $x \notin F$. Then F is a proper filter, thus $0 \notin F$ and hence $1 = 0 \rightarrow x \in F$ and $x^- = x \rightarrow 0 \in F$. So, $(x^-)^n \in F$, for $n = 1$, which concludes the proof of direct implication.

Conversely, let $x, y \notin F$. We will show that $x \rightarrow y \in F$ and $y \rightarrow x \in F$. By the hypothesis $(x^-)^n \in F$ and $(y^-)^m \in F$, for some $n, m \geq 1$. We know that $(x^-)^n \leq x^-$ and $(y^-)^m \leq y^-$. By the definition of a filter, it follows that $x^- \in F$ and $y^- \in F$. By Lemma 2.2, we have $x^- = x \rightarrow 0 \leq x \rightarrow y$ and $y^- = y \rightarrow 0 \leq y \rightarrow x$, for all $x, y \in A$. Hence $x \rightarrow y \in F$ and $y \rightarrow x \in F$. $\square$

The following examples show that obstinate filters in residuated lattices exist and that there exist filters which are not obstinate filters.

Example 3.3. Let $A = \{0, a, b, c, d, 1\}$ with $0 < a, b < c < d < 1$ and a, b incomparable. $*$ and $\rightarrow$ defined as follows:

$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	a	a
b	0	0	b	b	b	b
c	0	a	b	c	c	c
d	0	a	b	c	c	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	b	1	b	1	1	1
b	a	a	1	1	1	1
c	0	a	b	1	1	1
d	0	a	b	d	1	1
1	0	a	b	c	d	1

Then $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ is a residuated lattice and it is clear that $F = \{b, c, d, 1\}$ is a filter. Using Proposition 3.2, it is easy to check that F is an obstinate filter of A .

Example 3.4. Let $A = \{0, a, b, c, d, 1\}$, with $0 < a, b, c, d < 1$ and a, b, c, d are pairwise incomparable. $*$ and $\rightarrow$ defined as follows:

$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	b	d	d	a
b	c	b	b	0	0	b
c	b	d	0	d	d	c
d	b	d	0	d	d	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	b	c	c	1
b	c	a	1	c	c	1
c	b	a	b	1	a	1
d	b	a	b	a	1	1
1	0	a	b	c	d	1

Then $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ is a residuated lattice and it is clear that $F = \{c, d, 1\}$ is a filter of A and $a \notin F$ and $(a^-)^n = (0)^n = 0 \notin F$, for all $n \geq 1$. Hence by Proposition 3.2, F is not an obstinate filter of A .

Now, we apply the definition of obstinate filters and we obtain some characterizations of obstinate filters and describe the relation between obstinate filters and other filters in a residuated lattice.

Remark 3.5. $\text{Reg}(A)$ is not an obstinate filter in any residuated lattice A . Actually, $\text{Reg}(A)$ is not a proper filter in any residuated lattice A . Indeed, $\text{Reg}(A)$ always contains 0, thus it is a filter iff it is equal to A , thus, when it is a filter, $\text{Reg}(A)$ is the improper filter A .

Theorem 3.6. Let F be an obstinate filter of A . Then F is a maximal filter of A .

Proof: Let us suppose that F is obstinate filter which is not maximal. So there exists a proper filter G strictly greater than F (with respect to set-inclusion). Let $a \in G \setminus F$. Then $(a^-)^n \in F$ for some $n \geq 1$. We know that $(a^-)^n \leq a^-$. By the definition of a filter, it follows that $a^- \in F$, thus also $a^- \in G$. So $a \in G$, $a^- \in G$, hence, by definition of a filter, $a * a^- = 0 \in G$, which is a contradiction to the fact that G is proper. $\square$

Remark 3.7. Theorem 3.6 shows that, for instance, if $\{1\}$ is an obstinate filter of A , then $\{1\}$ is a maximal filter of A , hence $\{1\}$ is the only maximal filter of A and thus $\{1\}$ is the only obstinate filter of A , and, moreover, $\{1\}$ is the only proper filter of A , hence A is simple.

The following example shows that $Ds(A)$ need not be an obstinate filter

Example 3.8. In Example 3.3, we check that $F = \{b, c, d, 1\}$ is an obstinate filter. It easy to check that $Ds(A) = \{c, d, 1\}$, and $Ds(A) \subsetneq F \subsetneq A$, hence $Ds(A)$ is not a maximal filter and thus $Ds(A)$ is not an obstinate filter.

The next example shows that the converse of Theorem 3.6 is not true.

Example 3.9. Let $A = \{0, a, b, c, d, e, f, 1\}$, with $0 < d < e < f < a < 1$ and $0 < d < c < b < a < 1$ and both b and c incomparable with e and f . $*$ and $\rightarrow$ defined as follows:

$*$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	c	c	c	0	d	d	a
b	0	c	c	c	0	0	d	b
c	0	c	c	c	0	0	0	c
d	0	0	0	0	0	0	0	d
e	0	d	0	0	0	d	d	e
f	0	d	d	0	0	d	d	f
1	0	a	b	c	d	e	f	1

$\rightarrow$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	d	1	a	a	f	f	f	1
b	e	1	1	a	f	f	f	1
c	f	1	1	1	f	f	f	1
d	a	1	1	1	1	1	1	1
e	b	1	a	a	a	1	1	1
f	c	1	a	a	a	a	1	1
1	1	a	b	c	d	e	f	1

Then $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ is a residuated lattice and it is clear that $F = \{e, f, 1\}$ is a maximal filter of A but $a, c \notin F$ and $a \rightarrow c \notin F$ thus F is not an obstinate filter of A .

Theorem 3.10. Let F be a proper filter of A . Then F is an obstinate filter if and only if $x \in F$ or $x^- \in F$, for all $x \in A$.

Proof: Assume that F is an obstinate filter and $x \notin F$. Since F is an obstinate filter, we get that $(x^-)^n \in F$, for some $n \geq 1$. We know that $(x^-)^n \leq x^-$, thus $x^- \in F$ and the direct implication is now proved.

Conversely, let $x \notin F$. To prove that F is an obstinate filter, we only need to show that $(x^-)^n \in F$, for some $n \geq 1$. By the hypothesis, we get that $x^- \in F$. Hence F is an obstinate filter of A . $\square$

Remark 3.11. In Boolean algebras, obstinate filters coincide with maximal filters. Indeed, any Boolean algebra can be organized as a residuated lattice, with the multiplication equal to its meet and its residuum to its Boolean implication.

Theorem 3.12. Let F be a filter of A . Then the following conditions are equivalent:

- (i) F is a maximal and Boolean filter,
- (ii) F is a prime and Boolean filter,
- (iii) F is an obstinate filter.

Proof: (i) $\Rightarrow$ (ii) By the fact that any maximal filter is a prime filter.

(ii) $\Rightarrow$ (iii) We have for all $x \in A$, $x \vee x^- \in F$, since F is a Boolean filter. Thus $x \in F$ or $x^- \in F$, because F is a prime filter. Hence F is an obstinate filter.

(iii) $\Rightarrow$ (i) Suppose that F is an obstinate filter. It follows from Theorem 3.6 that F is a maximal filter. By Theorem 3.10, $x \in F$ or $x^- \in F$, for all $x \in A$. We know $x \leq x \vee x^-$ and $x^- \leq x \vee x^-$, thus $x \vee x^- \in F$. $\square$

Theorem 3.13. Let F be a filter of A . For any $x, y \in A$, the following conditions are equivalent:

- (i) F is a maximal and implicative filter,
- (ii) F is a maximal and positive implicative filter,
- (iii) F is an obstinate filter.

Proof: (i) $\Rightarrow$ (ii) By Theorem 7.13[9].

(ii) $\Rightarrow$ (iii) Let $y \notin F$. At first, we show that, $A_y = \{u \in A \mid y \rightarrow u \in F\}$ is a filter. $1 \in A_y$, since $y \rightarrow 1 = 1 \in F$. If $x, x \rightarrow t \in A_y$, then $y \rightarrow x, y \rightarrow (x \rightarrow t) \in F$, hence, since F is a positive implicative filter, it follows that $y \rightarrow t \in F$, thus $t \in A_y$. If $x \in F$, since $x \leq y \rightarrow x$, we get $y \rightarrow x \in F$, thus $x \in A_y$, hence $F \subseteq A_y \subseteq A$. Now, since $y \notin F$, but $y \rightarrow y = 1 \in F$ and hence $y \in A_y$, it follows that $F \subsetneq A_y$, so, since F is a maximal filter and $y \notin F$, we get that $A_y = A$. Then every $x \in A$ satisfies $x \in A_y$, and so $y \rightarrow x \in F$. Similarly, if $x, y \in A$ such that $x \notin F$, then $x \rightarrow y \in F$.

(iii) $\Rightarrow$ (i) Assume on the contrary that F is not an implicative filter, thus there exist $x, y \in A$ such that $(x \rightarrow y) \rightarrow x \in F$ and $x \notin F$. We consider two cases: $y \in F$ or $y \notin F$. If $y \in F$, then $x \rightarrow y \in F$, hence $x \in F$. If $y \notin F$ by the hypothesis we have $x \rightarrow y \in F$ and again $x \in F$. In every cases we get a contradiction. Hence F is an implicative filter. According to Theorem 3.6, F is a maximal filter. $\square$

Proposition 3.14. If F is an obstinate filter, then F is a fantastic filter but the converse is not true.

Proof: If F is an obstinate filter of A , then F is an implicative filter of A , hence is a fantastic filter of A [3, Theorem 4.5]. The following example disproves the converse implication. $\square$

Example 3.15. In Example 3.4, we can check that $F = \{1\}$ is not an obstinate filter. It is easy to check that F is a fantastic filter.

The following example shows that a quasi-primary filter need not be an obstinate filter.

Example 3.16. Let $A = \{0, a, b, 1\}$, with $0 < a < b < 1$. $*$ and $\rightarrow$ defined as follow:

$*$	0	a	b	1	$\rightarrow$	0	a	b	1
0	0	0	0	0	0	1	1	1	1
a	0	a	a	a	a	0	1	1	1
b	0	a	b	b	b	0	a	1	1
1	0	a	b	1	1	0	a	b	1

Then $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ is a residuated lattice and it is clear that $F = \{b, 1\}$ is not an obstinate filter and is a quasi-primary filter of A .

Theorem 3.17. Any obstinate filter is a quasi-primary filter.

Proof: Let F be an obstinate filter. We consider three cases:

case (1) Let $a, b \notin F$. For $u = 0$, $n = 1$ we get that $0 \vee 0^- = 0 \vee 1 = 1 \in B(A)$, $(a^n * 0)^- = 1 \in F$ and $(b^n * 1)^- = b^- \in F$.

case (2) Let $a \in F$, $b \notin F$. The proof from case (1) is valid here, as well.

case (3) Let $a, b \in F$. Then $a * b \in F$, hence $(a * b)^- \notin F$. □

Theorem 3.18. The two-element chain is the only simple residuated lattice which has an obstinate filter.

Proof: It is easy to check that, in the two-element chain, the trivial filter is obstinate filter. Now let A be a simple residuated lattice which is not the two-element chain. Then A has at least one element x which is different from 0 and 1. Assume by absurdum that A has an obstinate filter, F . Then F is a proper filter of A , hence F is the trivial filter: $F = \{1\}$. So $x \neq 1$, thus $x \notin F$, and F is an obstinate filter, hence $x^- \in F = \{1\}$ by Theorem 3.10. Thus $x \rightarrow 0 = x^- = 1$, that is $x \leq 0$, which means that $x = 0$. This is a contradiction to the choice of x . Therefore A has no obstinate filter. □

Remark 3.19. The residuated lattice of Example 3.4 is local (it is not simple) and has no obstinate filter.

Example 3.20. The residuated lattice of Example 3.3 is quasi-local (it isn't local) residuated lattice. It is clear that $F = \{b, c, d, 1\}$ is an obstinate filter of A .

Example 3.21. The residuated lattice of Example 3.9 is quasi-local (it isn't local) residuated lattice. It is clear that A has no obstinate filter of A .

Corollary 3.22. Let F be an obstinate filter of A . Then A/F is the two-element chain (the standard Boolean-algebra).

Proof: By Theorem 3.10, we obtain that A/F is two-element chain. □

Every algebra in the following results will be an arbitrary residuated lattice.

Corollary 3.23. *Let $\{F_\alpha\}_{\alpha \in I}$ be a family of filters of $\{A_\alpha\}_{\alpha \in I}$. Then $\prod_{\alpha \in I} F_\alpha$ is a filter of $\{A_\alpha\}_{\alpha \in I}$, see [2]. If $\{F_\alpha\}_{\alpha \in I}$ are obstinate filters of $\{A_\alpha\}_{\alpha \in I}$, then it is clear that $\prod_{\alpha \in I} F_\alpha$ is an obstinate filter of $\{A_\alpha\}_{\alpha \in I}$.*

Theorem 3.24. *Let f be a RL-homomorphism from A into B and G be an obstinate filter of B . Then $f^{-1}(G)$ is an obstinate filter of A .*

Proof: Let $x \in A$ such that $x \notin f^{-1}(G)$. Then $f(x) \notin G$, thus, by Theorem 3.10, $[f(x)]^- \in G$, hence $f(x^-) \in G$. Then we get that $x^- \in f^{-1}(G)$. Hence $f^{-1}(G)$ is an obstinate filter of A . $\square$

Remark 3.25. *Let f be a RL-homomorphism from A into B ($f \in \text{Hom}(A, B)$). Then $\text{Ker}(f) = \{x \in A : f(x) = 1\}$ is a filter of A .*

Lemma 3.26. *Let $f \in \text{Hom}(A, B)$ be an homomorphism and F be a filter of A . If $\text{Ker}(f) \subseteq F$, then $f^{-1}(f(F)) = F$.*

Proof: Obviously, $F \subseteq f^{-1}(f(F))$. For the inverse inclusion assume $x \in f^{-1}(f(F))$, that is $f(x) \in f(F)$. Thus there is $y \in F$ such that $f(x) = f(y)$, so $f(y \rightarrow x) = f(y) \rightarrow f(x) = 1$, hence $y \rightarrow x \in \text{Ker}(f) \subseteq F$. Since $y, y \rightarrow x \in F$, by the definition of a deductive system we get that $x \in F$. Therefore $f^{-1}(f(F)) \subseteq F$. $\square$

Theorem 3.27. *If F is an obstinate filter of A , then there exists $f \in \text{Hom}(A, A)$ such that $\text{ker}(f) = F$.*

Proof: Suppose F is an obstinate filter of A , since A has an obstinate filter, and thus proper filter, then $0 \neq 1$. We define f as follows

$$f(x) = \begin{cases} 1 & \text{if } x \in F, \\ 0 & \text{if } x \in A \setminus F \end{cases}$$

In order to verify that $f \in \text{Hom}(A, A)$, we shall consider two arbitrary elements $x, y \in A$ and we will divide the proof into four cases.

Case (1): $x, y \in F$:

(1_a) If $x, y \in F$, then by the definition of a filter we get that $x * y \in F$. Hence $f(x) = 1 = f(y)$ and $f(x * y) = 1$. On the other hand, $f(x) * f(y) = 1 * 1 = 1$. Therefore $f(x * y) = f(x) * f(y)$.

(1_{a'}) we have $x, y \in F$ and, by Lemma 2.2, we have $y \leq x \rightarrow y$. We get that $x \rightarrow y \in F$. Hence $f(x \rightarrow y) = 1$. On the other hand, $f(x) \rightarrow f(y) = 1 \rightarrow 1 = 1$. Therefore $f(x \rightarrow y) = f(x) \rightarrow f(y)$.

(1_{a''}) in this case, $x \wedge y \in F$, thus $f(x \wedge y) = 1$. On the other hand, $f(x) \wedge f(y) = 1 \wedge 1 = 1$. Therefore $f(x \wedge y) = f(x) \wedge f(y)$.

(1_{a'''}) since $x, y \in F$, we have $x \vee y \in F$, thus $f(x \vee y) = 1$. On the other hand, $f(x) \vee f(y) = 1 \vee 1 = 1$. Therefore $f(x \vee y) = f(x) \vee f(y)$.

Case (2): $x, y \notin F$:

(2_a) If $x, y \notin F$, then $x * y \notin F$. So $f(x * y) = 0$. On the other hand $f(x) * f(y) = 0 * 0 = 0$. It follows that $f(x * y) = f(x) * f(y)$.

(2_{a'}) we have $x, y \notin F$, then $x \rightarrow y \in F$ because F is an obstinate filter, and so $f(x \rightarrow y) = 1$. On the other hand, $f(x) \rightarrow f(y) = 0 \rightarrow 0 = 1$. It follows that $f(x \rightarrow y) = f(x) \rightarrow f(y)$.

(2_{a''}) since $x, y \notin F$, then $x \wedge y \notin F$, and so $f(x \wedge y) = 0$. On the other hand, $f(x) \wedge f(y) = 0 \wedge 0 = 0$. It follows that $f(x \wedge y) = f(x) \wedge f(y)$.

(2_{a'''}) we have $x, y \notin F$, then $y \rightarrow x \in F$ because F is an obstinate filter, consider $(x \vee y) \rightarrow x = (x \rightarrow x) \wedge (y \rightarrow x) = y \rightarrow x$. Then $x \vee y \notin F$, and so $f(x \vee y) = 0$. On the other hand, $f(x) \vee f(y) = 0 \vee 0 = 0$. It follows that $f(x \vee y) = f(x) \vee f(y)$.

Case (3): $x \notin F, y \in F$:

(3_a) If $x \notin F$ and $y \in F$, then $x * y \notin F$. So $f(x * y) = 0$. On the other hand $f(x) * f(y) = 0 * 1 = 0$. It follows that $f(x * y) = f(x) * f(y)$.

(3_{a'}) we have $x \notin F$ and $y \in F$. We have $y \leq x \rightarrow y$ hence $x \rightarrow y \in F$. Then $f(x \rightarrow y) = 1$. On the other hand $f(x) \rightarrow f(y) = 0 \rightarrow 1 = 1$. It follows that $f(x \rightarrow y) = f(x) \rightarrow f(y)$.

(3_{a''}) in this case $x \wedge y \notin F$, and so $f(x \wedge y) = 0$. On the other hand $f(x) \wedge f(y) = 1 \wedge 0 = 0$. It follow that $f(x \wedge y) = f(x) \wedge f(y)$.

(3_{a'''}) we have $x \vee y \in F$, and so $f(x \vee y) = 1$. On the other hand, $f(x) \vee f(y) = 0 \vee 1 = 1$. It follows that $f(x \vee y) = f(x) \vee f(y)$.

Case (4): $x \in F, y \notin F$:

(4_a) If $x \in F$ and $y \notin F$, then $x * y \notin F$. So $f(x * y) = 0$. On the other hand $f(x) * f(y) = 1 * 0 = 0$. It follows that $f(x * y) = f(x) * f(y)$.

(4_{a'}) we have $x \in F$ and $y \notin F$, then $x \rightarrow y \notin F$. So $f(x \rightarrow y) = 0$. On the other hand $f(x) \rightarrow f(y) = 1 \rightarrow 0 = 0$. It follow that $f(x \rightarrow y) = f(x) \rightarrow f(y)$.

(4_{a''}) in this case $x \wedge y \notin F$, and so $f(x \wedge y) = 0$. On the other hand $f(x) \wedge f(y) = 0 \wedge 1 = 0$. It follow that $f(x \wedge y) = f(x) \wedge f(y)$.

(4_{a'''}) we have $x \vee y \in F$, and so $f(x \vee y) = 1$. On the other hand, $f(x) \vee f(y) = 1 \vee 0 = 1$. It follows that $f(x \vee y) = f(x) \vee f(y)$.

Summarizing all the above we have proven that $f \in \text{Hom}(A, A)$. It is clear that $\text{Ker}(f) = f^{-1}(1) = F$. $\square$

4 Conclusion and future research

In this paper, we have introduced the notion of obstinate filters in residuated lattices. We have established properties of obstinate filters in residuated lattices. We proved the relationships between obstinate filters and other types of filters in residuated lattices for example, we proved that obstinate filters are (implicative) Boolean filters and F is an obstinate filter if and only if F is a maximal and (implicative) Boolean filter. We have shown that, if F is an obstinate filter of a residuated lattice A , then A/F is the two-element chain.

In our future work, we will consider the notion of n -fold obstinate (filter) residuated lattices as schematic extensions of residuated lattice and try to define other types of filters in residuated lattices.

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