Extending and contracting maximal ideals in the function rings of pointfree topology

by

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Abstract

By first describing the fixed maximal ideals of $\mathcal{R}L$ and those of its bounded part, $\mathcal{R}^*L$, we show that every fixed maximal ideal of the bigger ring contracts to a fixed maximal ideal of the smaller ring, and every fixed maximal ideal of the smaller ring extends to a fixed maximal ideal of the bigger ring. However, the only instance where every maximal ideal of $\mathcal{R}^*L$ extends to a maximal ideal of $\mathcal{R}L$ is when the two rings coincide. This allows the deduction that $\mathcal{R}L$ is integral over $\mathcal{R}^*L$ if and only if $L$ is pseudocompact.

Key Words: Frame, ring of real-valued continuous functions on a frame, maximal ideal, fixed maximal ideal, contraction of an ideal, extension of an ideal.

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1 Introduction

By a “function ring” we mean a ring isomorphic to $\mathcal{R}L$ for some completely regular frame $L$. As is well-known, the rings $C(X)$, for Tychonoff spaces $X$, are function rings in this sense because of the isomorphism $C(X) \cong \mathcal{R}(\mathcal{O}X)$. Following Banaschewski [3], we refer to any ring isomorphic to some $C(X)$ as a “classical function ring”. As observed in [9], not every function ring is classical.

If one wants to investigate fixed maximal ideals of $\mathcal{R}^*L$, one cannot merely look at the classical case since the property of being fixed is not “algebraic”. In describing fixed maximal ideals of $\mathcal{R}^*L$, we first index them with points of $\beta L$ (Proposition 3.8), and then, using the isomorphism $\mathcal{R}^*L \cong \mathcal{R}(\beta L)$, index them with points of $L$ (Proposition 3.12). This then immediately shows that every fixed maximal ideal of $\mathcal{R}L$ contracts to a fixed maximal ideal of $\mathcal{R}^*L$. The subtler dual of this, namely, that every fixed maximal ideal of $\mathcal{R}^*L$ extends to a fixed maximal ideal of $\mathcal{R}L$ requires more work.

Even in the classical case (see [9, p. 57]), not every maximal ideal of $C^*(X)$ extends to a maximal ideal of $C(X)$, nor is every maximal ideal of $C^*(X)$ a contraction of a maximal ideal of $C(X)$. It turns out that these phenomena, in the general case, of course, are equivalent,
and occur precisely when the frame is pseudocompact (Proposition 4.4). In other words, this happens only in the trivial case where \( \mathcal{R}^*L = RL \).

2 Preliminaries

Our references for the general theory of frames is [10]. All our frames are assumed to be completely regular, spaces are Tychonoff, and rings are commutative with unity. We denote the set of all points of \( L \) by \( Pt(L) \). The right adjoint of a frame homomorphism \( h \) is denoted by \( h^* \). If \( h: L \to M \) is a frame homomorphism, then \( h^*(p) \in Pt(L) \) for each \( p \in Pt(M) \). On the other hand, if \( h \) is onto and \( q \in Pt(L) \) with \( h(q) \neq 1 \), then \( h(q) \in Pt(M) \). By a character of a frame \( L \) we mean a frame homomorphism \( L \to 2 \), where \( 2 \) denotes the two-element frame.

Regarding the frame of reals \( L(\mathbb{R}) \) and the \( f \)-ring \( RL \) of continuous real-valued functions on \( L \), we use the notation of [2]. The bounded part, in the \( f \)-ring sense, of \( RL \) is denoted by \( R^*L \) and is characterized by:

\[
\varphi \in R^*L \iff \varphi(p, q) = 1 \text{ for some } p, q \in Q.
\]

In the event that \( R^*L = RL \), the frame \( L \) is said to be pseudocompact. We shall freely use properties of the cozero map without comment. The cozero part of \( L \) will be denoted by \( CozL \).

Every frame homomorphism \( h: L \to M \) induces a ring homomorphism \( \mathcal{R}h: RL \to RM \) which sends an element \( \alpha \) of \( RL \) to the composite \( ha \). Furthermore,

\[
coz(ha) = (ha)((-), (0)) = h(((-), (0))) = h(coz \alpha).
\]

3 Describing fixed maximal ideals

We start by first describing all maximal ideals of \( R^*L \) in a manner similar to the description of maximal ideals of \( RL \). In [4] we associated with each \( 1_{\beta L} \neq I \in \beta L \) the ideal \( M^I \) of \( RL \) defined by

\[
M^I = \{ \alpha \in RL \mid rL(coz \alpha) \subseteq I \}.
\]

It is shown in [4] that:

(1) Maximal ideals of \( RL \) are precisely the ideals \( M^I \), for \( I \in Pt(\beta L) \).

(2) Every prime ideal of \( RL \) is contained in a unique maximal ideal.

(3) If \( M^I = M^J \), then \( I = J \).

An ideal \( Q \) of \( RL \) or \( R^*L \) is fixed if \( \bigvee\{coz \alpha \mid \alpha \in Q\} < 1 \). This is the exact counterpart of the familiar classical notion concerning ideals of \( C(X) \) and \( C^*(X) \).
As shown in [9, Theorem 4.6], fixed maximal ideals of \( C(X) \) are in one-one correspondence with the points of \( X \), and they are precisely the sets
\[
M_p = \{ f \in C(X) \mid f(p) = 0 \}.
\]
We show below that the situation in \( RL \) is similar. Let us introduce the following types of fixed ideals of \( RL \).

**Definition 3.1** For each \( a \in L \) with \( a < 1 \), define the subset \( M_a \) of \( RL \) by
\[
M_a = \{ \alpha \in RL \mid \text{coz} \alpha \leq a \}.
\]
Clearly, \( M_a \) is an ideal, and, in fact, \( M_a = M_{rL(a)} \). Since \( rL(a) = rL(b) \) implies \( a = b \), it follows that \( M_a = M_b \) implies \( a = b \). Furthermore, \( M_a \) is fixed. The following lemma from [6] helps identify all fixed maximal ideals of \( RL \).

**Lemma 3.2** For any \( I \in \beta L \), \( \bigvee \{ \text{coz} \alpha \mid \alpha \in M_I \} = \bigvee I \). Hence, the ideal \( M_I \) is fixed if and only if \( \bigvee I < 1 \).

**Proposition 3.3** The fixed maximal ideals of \( RL \) are precisely the ideals \( M_p \) for \( p \in Pt(L) \). They are distinct for distinct points.

**Proof:** Since, for any \( p \in Pt(L) \), \( M_p = M_{rL(p)}^{*} \), and \( rL(p) \) is a point of \( \beta L \) with \( \bigvee I < 1 \), it follows that \( M_p \) is a fixed maximal ideal. Conversely, let \( Q \) be a fixed maximal ideal of \( \beta L \). Then, from the discussion above, \( Q = M_I \) for some point \( I \) of \( \beta L \) with \( \bigvee I < 1 \). Write \( p = \bigvee I \), and note that \( p \in Pt(L) \). We show that \( M_I = M_p \). If \( \alpha \in M_I \), then \( rL(\text{coz} \alpha) \subseteq I \). Taking joins yields \( \text{coz} \alpha \leq \bigvee I = p \), so that \( M_I \subseteq M_p \). Since \( M_I \) is a maximal ideal and \( M_p \) is a proper ideal, we have \( M_I = M_p \). □

**Corollary 3.4** The fixed maximal ideals of \( RL \) are precisely the sets \( \text{Ker} (RL) \) for the characters \( \xi \) of \( L \). They are different for distinct characters.

**Proof:** Let \( M_p \) be a fixed maximal ideal of \( RL \). Let \( \xi : L \rightarrow 2 \) be the character of \( L \) determined by \( p \). For any \( \alpha \in RL \), one checks easily that \( \alpha \in M_p \) if and only if \( \alpha \in \text{Ker} (RL) \). It follows therefore that \( M_p = \text{Ker} (RL) \). On the other hand, let \( \zeta : L \rightarrow 2 \) be a character of \( L \). Then \( \zeta(0) \) is a point of \( L \). A calculation similar to the one above shows that any \( \gamma \in RL \) is in \( \text{Ker} (RL) \) if and only if \( \text{coz} \gamma \leq \zeta(0) \); so that \( \text{Ker} (RL) = M_{\zeta(0)} \). □

These results yield a spatiality criterion in terms of fixed maximal ideals. It is shown in [7, Lemma 3.1] that a regular frame is spatial if and only if every element different from the top is below some point of the frame. Using this characterization, we deduce from the results above the following corollary.

**Corollary 3.5** The following are equivalent for a completely regular frame \( L \).
1. \(L\) is spatial.
2. Every fixed ideal of \(\mathcal{R}L\) is contained in a fixed maximal ideal.
3. Every fixed ideal of \(\mathcal{R}L\) is contained in \(\text{Ker}(\mathcal{R}\xi)\) for some character \(\xi: L \to 2\).

In fact, for spatial frames \(L\), maximal ideals of \(\mathcal{R}L\) are characterizable in terms of kernels of ring homomorphisms induced by characters of \(L\). Namely:

**Corollary 3.6** If \(L\) is spatial, then an ideal of \(\mathcal{R}L\) is fixed if and only if it is contained in \(\text{Ker}(\mathcal{R}\xi)\) for some character \(\xi: L \to 2\) of \(L\).

We now turn our attention to describing maximal ideals of the ring \(\mathcal{R}^*L\). We denote by \(t_L\) the ring isomorphism

\[
t_L: \mathcal{R}(\beta L) \to \mathcal{R}^*L \quad \text{given by} \quad t_L(\alpha) = j_L\alpha,
\]

the inverse of which we will denote by \(\varphi \mapsto \varphi^\beta\).

**Definition 3.7** For each \(I \in \beta L\) with \(I < 1_{\beta L}\), the ideal \(M^I\) of \(\mathcal{R}^*L\) is defined by

\[
M^I = \{\alpha \in \mathcal{R}^*L \mid \cos(\alpha^\beta) \subseteq I\}.
\]

Maximal ideals of \(\mathcal{R}^*L\) are among these ideals. In establishing this we exploit the isomorphism just described, and the fact (proved in [5]) that a completely regular frame \(L\) is compact if and only if every maximal ideal of \(\mathcal{R}L\) is fixed. Note that, in accordance with Definition 3.1, for any \(I \in \text{Pt}(\beta L)\), \(M_I\) is the fixed maximal ideal of \(\mathcal{R}(\beta L)\) associated with the point \(I\).

**Proposition 3.8** Maximal ideals of \(\mathcal{R}^*L\) are precisely the ideals \(M^I\), for \(I \in \text{Pt}(\beta L)\). They are distinct for distinct \(I\).

**Proof:** Consider the isomorphism \(t_L: \mathcal{R}(\beta L) \to \mathcal{R}^*L\) described above. Maximal ideals of \(\mathcal{R}^*L\) are precisely the images under \(t_L\) of the maximal ideals of \(\mathcal{R}(\beta L)\). Since \(\beta L\) is compact, its maximal ideals are all fixed, and are therefore precisely the ideals \(M_I\), for \(I \in \text{Pt}(\beta L)\), by Proposition 3.3. Therefore it suffices to show that, for each \(I \in \text{Pt}(\beta L)\), \(M^I = t_L[M_I]\). Since \(M^I\) is a proper ideal, it suffices, by the maximality of \(t_L[M_I]\), to show that \(t_L[M_I] \subseteq M^I\). Let \(\alpha \in M_I\). Then \(\cos \alpha \subseteq I\). Now \(\alpha = t_L^{-1}(t_L(\alpha)) = (j_L\alpha)^\beta\), and therefore \(\cos((j_L\alpha)^\beta) \subseteq I\). Consequently, \(j_L\alpha \in M^I\), by definition. Thus, \(t_L(\alpha) \in M^I\), as required. The latter part of the proposition follows because \(I \neq J\) in \(\text{Pt}(\beta L)\) implies \(M_I \neq M_J\), as observed earlier. \(\square\)

Now that we know what maximal ideals of \(\mathcal{R}^*L\) look like, we identify the fixed ones among them.

**Proposition 3.9** For any \(I \in \text{Pt}(\beta L)\), the maximal ideal \(M^I\) is fixed if and only if \(\bigvee I < 1\).
Proof: Suppose $M^*I$ is fixed and put $s = \bigvee \{\text{coz } \alpha \mid \alpha \in M^*I\}$. Then $s < 1$. Let $x \in I$. Since $I$ is a completely regular ideal, there is a cozero element $c$ such that $x \leq c \in I$. Take a $\gamma \in R^*L$ such that $c = \text{coz } \gamma$. Now,

$$\bigvee \text{coz } (\gamma^\beta) = \text{coz } \gamma = c \in I.$$  

Thus, for each $d \in \text{coz } (\gamma^\beta)$, $d \leq c \in I$, and therefore $d \in I$. This shows that $\text{coz } (\gamma^\beta) \subseteq I$, and therefore $\gamma \in M^*I$, whence $x \leq c \leq s$. As $x$ is arbitrary, it follows that $\bigvee I \leq s < 1$.

Conversely, suppose $\bigvee I < 1$ and let $\varphi \in M^*I$. Then $\text{coz } (\varphi^\beta) \subseteq I$, and therefore

$$\text{coz } \varphi = \bigvee \text{coz } (\varphi^\beta) \leq \bigvee I < 1.$$  

Consequently,

$$\bigvee \{\text{coz } \alpha \mid \alpha \in M^*I\} \leq \bigvee I < 1,$$

so that $M^*I$ is fixed.

As before, we have now described maximal ideals of $R^*L$, but by a condition (namely, “$\text{coz } (\varphi^\beta) \subseteq I$”) which involves the Stone-Čech compactification of $L$. To replace this condition with one at the level of $L$ itself, we introduce the following definition.

Definition 3.10 For each $a \in L$ with $a < 1$, let $M^*_a$ be the subset of $R^*L$ defined by

$$M^*_a = \{\alpha \in R^*L \mid \text{coz } \alpha \leq a\}.$$  

Clearly, $M^*_a$ is a fixed ideal of $R^*L$, and $M^*_a = M_a \cap R^*L$. We will show that, in analogy with Proposition 3.3, the fixed maximal ideals of $R^*L$ are precisely the ideals $M^*_p$, for $p \in Pt(L)$. We need a lemma.

Lemma 3.11 Let $\varphi \in R^*L$ and $I$ be a point of $\beta L$ with $\bigvee I < 1$. Then

$$\text{coz } (\varphi^\beta) \subseteq I \iff \text{coz } \varphi \leq \bigvee I.$$  

Proof: ($\Rightarrow$): Since $\text{coz } \varphi = \bigvee \text{coz } (\varphi^\beta)$, if $\text{coz } (\varphi^\beta) \subseteq I$, then

$$\text{coz } \varphi = \bigvee \text{coz } (\varphi^\beta) \leq \bigvee I.$$  

($\Leftarrow$): If not, then $\text{coz } (\varphi^\beta) \lor I = 1_{\beta L}$ since $I$ is a point of $\beta L$. Therefore

$$\bigvee \text{coz } (\varphi^\beta) \lor \bigvee I = 1,$$

that is, $\text{coz } \varphi \lor \bigvee I = 1$, and hence $\bigvee I = 1$ since the present hypothesis is that $\text{coz } \varphi \leq \bigvee I$, contradicting the initial assumption about $I$.

Proposition 3.12 The fixed maximal ideals of $R^*L$ are precisely the ideals $M^*_p$, for $p \in Pt(L)$. They are distinct for distinct points.
Proof: In view of Propositions 3.8 and 3.9, it suffices to prove that:

1. each of the ideals \( M_p^* \), \( p \) a point of \( L \), is of the form \( M^*I \) for some \( I \in Pt(\beta L) \) with \( \bigvee I < 1 \), and

2. for each \( I \in Pt(\beta L) \) with \( \bigvee I < 1 \), \( M^*I = M_p^* \), for some point \( p \) of \( L \).

Regarding (1); if \( p \) is a point of \( L \), then \( r_L(p) \) is a point of \( \beta L \) with \( \bigvee r_L(p) = p < 1 \). Therefore, in light of Lemma 3.11, for any \( \varphi \in R^*L \) we have that \( coz \varphi \leq p \) if and only if \( coz (\varphi^\beta) \leq r_L(p) \); the consequence of which is that

\[
M_p^* = \{ \varphi \in R^*L \mid coz \varphi \leq p \} = \{ \varphi \in R^*L \mid coz (\varphi^\beta) \leq r_L(p) \} = M^*r_L(p).
\]

Regarding (2); let \( I \) be a point of \( \beta L \) with \( \bigvee I < 1 \). Then \( \bigvee I \) is a point of \( L \), and

\[
M^*I = \{ \varphi \in R^*L \mid coz (\varphi^\beta) \subseteq I \} = \{ \varphi \in R^*L \mid coz \varphi \leq \bigvee I \} = M^*\bigvee I.
\]

That these ideals are distinct for distinct points follows as in the case of \( RL \) since, for any \( a \in L, a = \bigvee \{ coz \varphi \mid \varphi \in R^*L \text{ and } coz \varphi \leq a \} \).

4 Extending and contracting

We now come to extending and contracting maximal ideals. Recall that if \( B \) is a subring of a ring \( A \) and \( I \) is an ideal of \( A \), then \( I \cap B \) is an ideal of \( B \) called the contraction of \( I \) and denoted by \( I^c \). On the other hand, if \( J \) is an ideal of \( B \), then the (possibly improper) ideal of \( A \) generated by \( J \) is called the extension of \( J \) and is denoted by \( J^e \). Using properties of \( f \)-rings with bounded inversion, it is not hard to show that, for any \( \alpha \in RL, \frac{\alpha}{1 + \alpha} \in R^*L \).

**Lemma 4.1** The following results hold for any \( I \in Pt(\beta L) \).

1. \( (M^I)^e \subseteq M^*I \).
2. \( M^I \subseteq (M^*I)^c \).

**Proof:** (1) Let \( \alpha \in (M^I)^c = M^I \cap R^*L \). Then \( \alpha \in R^*L \) and \( r_L(coz \alpha) \subseteq I \). Since \( \alpha \in R^*L \), the element \( \alpha^\beta \) of \( R(\beta L) \) is defined, and we have \( \alpha = j_L \alpha^\beta \), so that

\[
coz \alpha = coz (j_L \alpha^\beta) = j_L (coz \alpha^\beta).
\]

Since \( r_L \) is the right adjoint of \( j_L \), this implies

\[
coz \alpha^\beta \leq r_L (coz \alpha) \subseteq I.
\]
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showing that \( \alpha \in M^*. \) Therefore \((M^*)^c \subseteq M^*.\)

(2) Let \( \alpha \in M^I. \) Since \( \alpha = \frac{\alpha}{1 + \alpha^2} \cdot (1 + \alpha^2) \), if we can show that \( \frac{\alpha}{1 + \alpha^2} \in M^* \), it will follow that \( \alpha \in (M^*)^c \), and we shall be done. Since \( \frac{\alpha}{1 + \alpha^2} \) is bounded, the element \((\frac{\alpha}{1 + \alpha^2})^2\) of \( R(\beta L) \) exists, and

\[ j_L(coz\left(\frac{\alpha}{1 + \alpha^2}\right)^2) = coz\left(j_L \cdot \left(\frac{\alpha}{1 + \alpha^2}\right)^2\right) = coz\left(\frac{\alpha}{1 + \alpha^2}\right) = coz \alpha, \]

which implies

\[ coz\left(\frac{\alpha}{1 + \alpha^2}\right)^2 \subseteq r_L(coz \alpha) \subseteq I, \]

the last step since \( \alpha \in M^I. \) Therefore \( \alpha \in (M^*)^c \), and hence \( M^I \subseteq (M^*)^c. \) \( \square \)

An ideal \( I \) of \( L \) is said to be \( \sigma \)-proper if, for any countable \( S \subseteq I, \forall S < 1. \) We recall from [1, Proposition 1.17] that if \( A \) is a subring of \( B, \) an ideal of \( A \) and \( b \) an ideal of \( B, \) then

\[ a \subseteq a^{cc}, \ b^{cc} \subseteq b, \ a^e = a^{ccc} \] and \( b^e = b^{ccc}. \)

**Proposition 4.2** For any \( I \in Pt(\beta L), \) the following conditions are equivalent.

(1) \((M^I)^c = M^I.\)

(2) \( M^*I \) contains no unit of \( RL. \)

(3) \( I \) is \( \sigma \)-proper.

(4) \((M^*)^c = M^I.\)

**Proof:** (1) \( \Leftrightarrow \) (2): Clearly, if \( M^{*I} = M^I \cap R^*L, \) then \( M^{*I} \) cannot contain a unit of \( RL, \) otherwise \( M^I \) would be improper. Conversely, assume \( M^{*I} \) contains no unit of \( RL. \) Let \( \alpha \in M^{*I}. \) We show that \( r_L(coz \alpha) \subseteq I. \) If not, then \( r_L(coz \alpha) \lor I = 1, \) since \( I \in Pt(\beta L). \) By normality, pick \( J \in Coz(\beta L) \) such that \( J \subseteq I \) and \( r_L(coz \alpha) \lor J = 1, \) so that

\[ coz \alpha \lor J = 1. \]

Now take \( \varphi \in R(\beta L) \) such that \( J = coz \varphi. \) Then \( coz \varphi \subseteq I. \) But \( \varphi = (j_L \varphi)^2, \) so we have \( coz \left((j_L \varphi)^2\right) \subseteq I, \) which implies \( j_L \varphi \in M^{*I}. \) Thus, \( \alpha^2 + (j_L \varphi)^2 \) is an element of \( M^{*I} \) which is a unit of \( RL \) since, by what we observed above,

\[ coz \left(\alpha^2 + (j_L \varphi)^2\right) = coz \alpha \lor coz (j_L \varphi) = coz \alpha \lor V coz \varphi = coz \alpha \lor V J = 1. \]

Therefore \( r_L(coz \alpha) \subseteq I, \) which implies \( \alpha \in M^I. \) Therefore \( M^{*I} \subseteq (M^I)^c, \) and hence equality, in view of the lemma above.

(2) \( \Rightarrow \) (3): If \( I \) is not \( \sigma \)-proper, then, as shown in the \((\Rightarrow) \) part of the proof of [5, Proposition 4.4], there is a \( \gamma \in R^*L \) such that \( coz (\gamma^2) \subseteq I \) and \( V \gamma^2(0, -) = 1. \) But this implies \( \gamma \in M^{*I} \) and

\[ coz \gamma = coz (j_L \gamma^2) = V coz (\gamma^2) \geq V \gamma^2(0, -) = 1, \]
whence $M^*I$ contains a unit of $RL$.

$(3) \Rightarrow (2)$: Suppose $M^*I$ contains a unit $\alpha$ of $RL$. Consider the element $\alpha^\beta$ of $R(\beta L)$. Since $\alpha \in M^I$, $coz (\alpha^\beta) \subseteq I$. For each $n \in \mathbb{N}$, pick $J_n \in \beta L$ such that $J_n \ll coz(\alpha^\beta)$ and $coz(\alpha^\beta) = \bigvee_{\beta L} \{J_n \mid n \in \mathbb{N}\}$.

Put $c_n = \bigvee J_n$, and note that $c_n \in I$ since $J_n \ll I$. Now, since $\alpha$ is invertible, we have

$$1 = coz \alpha = j_{\beta L}(coz(\alpha^\beta)) = j_{\beta L} \left( \bigvee_{n} J_n \right) = \bigvee_{n} j_{\beta L}(J_n) = \bigvee c_n,$$

so that $I$ is not $\sigma$-proper.

$(1) \Rightarrow (4)$: If $(M^I)^e = M^I$, then $(M^*I)^e = M^I$ since $(M^*I)^e = (M^*I)^{ece} = (M^I)^{ece} = (M^I)^{ce} \subseteq M^I \subseteq (M^*I)^e$,

the last containment being the second part of the lemma above.

$(4) \Rightarrow (1)$: If $(M^*I)^e = M^I$, then the chain of containments

$$M^*I \subseteq (M^*I)^{ece} \subseteq (M^I)^e \subseteq M^*I,$$

the last in view of Lemma 4.1, shows that $M^*I = (M^I)^e$. \qed

We can now summarize the results above, concerning fixed maximal ideals, as follows.

**Corollary 4.3** The fixed maximal ideals of $R^*L$ are precisely the contractions of the fixed maximal ideals of $RL$, and, conversely, the fixed maximal ideals of $RL$ are precisely the extensions of the fixed maximal ideals of $R^*L$. Furthermore, the contraction and extension maps are bijections, and inverses of each other when restricted to the sets of fixed maximal ideals.

In closing, we validate the claim made in the last sentence in the abstract. In one of the implications we will use the following fact. Putting together the results in [8, Proposition 4.4 and Corollary 3.7], we have that a completely regular frame $L$ is pseudocompact if and only if every point of $\beta L$ is $\sigma$-proper.

**Proposition 4.4** The following are equivalent for a frame $L$:

$(1)$ Every maximal ideal of $RL$ contracts to a maximal ideal of $R^*L$.

$(2)$ Every maximal ideal of $R^*L$ extends to a maximal ideal of $RL$.

$(3)$ $L$ is pseudocompact.
Proof: (1) ⇒ (2): Consider any maximal ideal $M^I$ of $RL$. Since $(M^I)^c \subseteq M^I$, by Lemma 4.1, and since $(M^I)^c$ is a maximal ideal if we assume (1), it follows that $(M^I)^c = M^I$. Therefore $(M^I)^c = (M^I)^{ce} \subseteq M^I,$ which implies $(M^I)^c = M^I$ by Lemma 4.1. Thus, (2) is implied by (1).

(2) ⇒ (3): Let $I \in Pt(\beta L)$. By (2), $(M^I)^c$ is a proper ideal. Since $M^I \subseteq (M^I)^c$, by Lemma 4.1, we have that $M^I = (M^I)^{ce}$, and hence $I$ is $\sigma$-proper, by Proposition 4.2. Therefore $L$ is pseudocompact by the remarks above.

(3) ⇒ (1): This is trivial since, under the hypothesis of (3), $RL = R^L$.

Finally, it is well known that if a ring $A$ is integral over a subring $B$, then every maximal ideal of $A$ contracts to a maximal ideal of $B$. We deduce, therefore, from this last result that $RL$ is integral over $R^L$ if and only if $L$ is pseudocompact. At the classical level, the deduction is that $C(X)$ is integral over $C^*(X)$ if and only if $X$ is pseudocompact, because $X$ is pseudocompact if and only if $\mathcal{D}X$ is pseudocompact, $C(X) \cong R(\mathcal{D}X)$ and $C^*(X) \cong R^*(\mathcal{D}X)$ since $C^*(X) \cong C(\beta X) \cong R(\mathcal{D}(\beta X)) \cong R(\beta(\mathcal{D}X)) \cong R^*(\mathcal{D}X)$.

In conclusion, we state explicitly that $RL$ is integral over $R^L$ iff $RL = R^L$.

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References


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