Multiplicity results for a class of two-point boundary value systems investigated via variational methods
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Abstract
An existence result of multiple solutions for a class of two-point boundary value equations depending upon a positive parameter is established. Our main tool is a recent three critical points theorem due to Bonanno and Marano [G. Bonanno, S.A. Marano, On the structure of the critical set of non-differentiable functions with a weak compactness condition, Appl. Anal. 89 (2010), 1-10].

Key Words: Three solutions; Critical point; Variational methods; Two-point boundary value system.

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1 Introduction
In this paper, we are interested in ensuring the existence of at least three weak solutions for the following two-point boundary value system

\[
\begin{aligned}
&\left(\phi_{p_i}(u'_i)\right)' + \lambda F_{u_i}(x, u_1, \ldots, u_n) = a_i(x)|u_i|^{p_i-2}u_i \quad \text{in } (a, b), \\
&u'_i(a) = u'_i(b) = 0
\end{aligned}
\]

for \(1 \leq i \leq n\), where \(p_i > 1\), \(\phi_{p_i}(t) := |t|^{p_i-2}t\) and \(a_i \in L^\infty([a, b])\) with \(\text{ess inf}_{(a, b)} a_i \geq 0\) for \(1 \leq i \leq n\), \(\lambda\) is a positive parameter, \(F : [a, b] \times \mathbb{R}^n \to \mathbb{R}\) is a measurable function with respect to \(x \in [a, b]\) for every \((t_1, \ldots, t_n) \in \mathbb{R}^n\) and is \(C^1\) with respect to \((t_1, \ldots, t_n) \in \mathbb{R}^n\) for a.e. \(x \in [a, b]\), \(F_{u_i}\) denotes the partial derivative of \(F\) with respect to \(u_i\), and \(F(x, 0, \ldots, 0) = 0\) for a.e. \(x \in [a, b]\).

Boundary value problems for ordinary differential equations play a fundamental role both in theory and applications. To establish the existence and multiplicity of solutions to nonlinear differential problems is very important as well as the application of such results in the physical reality. In fact, it is well known that the mathematical modelling of important questions in different fields of research, such as mechanical engineering, control systems, economics, computer
science and many others, leads naturally to the consideration of nonlinear differential equations. Conditions that guarantee the existence of multiple solutions to differential equations are of interest because physical processes described by differential equations can exhibit more than one solution. For example, certain chemical reactions in tubular reactors can be mathematically described by a nonlinear two-point boundary value problem with the interest in seeing if multiple steady-states to the problem exist. For a treatment of chemical reactor theory and multiple solutions see [2, Section 7] and references therein, and for additional approaches to the existence of multiple solutions to boundary value problems, see [13] and [14] and references therein. The existence and multiplicity of solutions for two-point boundary value problems have been widely investigated (see, for instance, [1, 4, 9, 10, 11, 12, 15] and references therein).

Throughout this paper, we let \( X \) be the Cartesian product of the \( n \) Sobolev spaces \( W^{1,p_i}([a,b]) \) for \( 1 \leq i \leq n \), i.e., \( X = W^{1,p_1}([a,b]) \times W^{1,p_2}([a,b]) \times \cdots \times W^{1,p_n}([a,b]) \) equipped with the norm

\[
\|(u_1, \ldots, u_n)\| := \sum_{i=1}^{n} \|u_i\|_{p_i},
\]

where

\[
\|u_i\|_{p_i} := \left( \int_a^b \left( |u_i'(x)|^{p_i} + a_i(x)|u_i(x)|^{p_i} \right) dx \right)^{\frac{1}{p_i}}
\]

for \( 1 \leq i \leq n \), which is equivalent to the usual one.

Let

\[
k := \max \left\{ \sup_{u_i \in W^{1,p_i}([a,b]) \setminus \{0\}} \max_{x \in [a,b]} \frac{|u_i(x)|^{p_i}}{\|u_i\|_{p_i}} : \text{for} \ 1 \leq i \leq n \right\} \tag{2}
\]

Since \( p_i > 1 \) for \( 1 \leq i \leq n \), the embedding \( X \hookrightarrow (C^0([a,b]))^n \) is compact, and so \( k < +\infty \). It follows from Proposition 4.1 of [3] that

\[
\sup_{u_i \in W^{1,p_i}([a,b]) \setminus \{0\}} \max_{x \in [a,b]} \frac{|u_i(x)|^{p_i}}{\|u_i\|_{p_i}} > \frac{1}{\|a_i\|_1} \quad \text{for} \ 1 \leq i \leq n,
\]

where \( \|a_i\|_1 := \int_a^b |a_i(x)| dx \) for \( 1 \leq i \leq n \), and so \( \frac{1}{\|a_i\|_1} \leq k \) for \( 1 \leq i \leq n \). In addition, it is known [3] that for \( 1 \leq i \leq n \) we have

\[
\sup_{u_i \in W^{1,p_i}([a,b]) \setminus \{0\}} \max_{x \in [a,b]} \frac{|u_i(x)|}{\|u_i\|_{p_i}} = 2^{\frac{n-1}{p_i}} \max \left\{ \left( \frac{1}{\|a_i\|_1} \right)^{\frac{1}{p_i}}, (a+b)(b-a) \frac{n-1}{p_i} \frac{\|a_i\|_\infty}{\|a_i\|_1} \right\}. \tag{3}
\]

We mean by a weak solution of system (1), any \( u = (u_1, \ldots, u_n) \in X \) such that

\[
\int_a^b \sum_{i=1}^{n} \phi_{p_i} (u'_i(x)) v_i(x) dx - \lambda \int_a^b \sum_{i=1}^{n} F_{u_i}(x, u_1(x), \ldots, u_n(x)) v_i(x) dx
\]

\[
+ \int_a^b \sum_{i=1}^{n} a_i(x)|u_i(x)|^{p_i-2} u_i(x)v_i(x) dx = 0
\]

for all \( v = (v_1, \ldots, v_n) \in X \).
2 Main results

In the present section we prove an existence result for system (1). The main tool is a critical
point theorem that we recall here in a convenient form (Theorem 1). This result has been
obtained in [6] and it is a more precise version of Theorem 3.2 of [5]. We also refer the reader
to the recent papers [7] and [8] where an analogous variational approach has been developed
on studying elliptic problems with subcritical growth.

Theorem 1 (Theorem 3.6 of [6]) Let \( X \) be a reflexive real Banach space; \( \Phi : X \to \mathbb{R} \) be
a coercive, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous
functional whose Gâteaux derivative admits a continuous inverse on \( X^* \); \( \Psi : X \to \mathbb{R} \) be a
continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that
\[
\Phi(0) = \Psi(0) = 0.
\]
Assume that there exist \( r > 0 \) and \( x \in X \), with \( r < \Phi(x) \), such that
\[(a_1) \sup_{\|x\| \leq r} \frac{\Phi(x)}{\Psi(x)} < \frac{\Psi(x)}{\Psi(\bar{x})};\]
\[(a_2) \text{ for each } \lambda \in \Lambda_r := \frac{\Phi(x)}{\Psi(x)} \text{, the functional } \Phi - \lambda \Psi \text{ is coercive.}\]
Then, for each \( \lambda \in \Lambda_r \) the functional \( \Phi - \lambda \Psi \) has at least three distinct critical points in \( X \).

For any \( \gamma > 0 \), we denote by \( K(\gamma) \) the set
\[
\left\{ (t_1, \ldots, t_n) \in \mathbb{R}^n : \sum_{i=1}^n \frac{|t_i|^{p_i}}{p_i} \leq \gamma \right\}.
\]
This set will be used in some of our hypotheses with appropriate choices of \( \gamma \).

We state our main result as follows.

Theorem 2 Assume that there exist a positive constant \( r \) and a function \( w = (w_1, \ldots, w_n) \in X \)
such that
\[(i) \sum_{i=1}^n \|w_i\|_p > r;\]
\[(ii) \left( r \Pi_{i=1}^n p_i \right) \frac{\int_a^b F(x,w_1(x),\ldots,w_n(x))dx}{\sum_{i=1}^n (\Pi_{j=1,j\neq i}^n p_j) \|w_i\|_p} - \int_a^b \sup_{(t_1,\ldots,t_n) \in K(kr)} F(x,t_1,\ldots,t_n)dx > 0;\]
\[(iii) \limsup_{(t_1,\ldots,t_n) \to (+\infty,\ldots,\infty)} \frac{F(x,t_1,\ldots,t_n)}{\sum_{i=1}^n \frac{|t_i|^{p_i}}{p_i}} \leq \frac{\int_a^b \sup_{(t_1,\ldots,t_n) \in K(kr)} F(x,t_1,\ldots,t_n)dx}{kr(b-a)} \text{ uniformly with respect to } x \in [a,b].\]
Then, for each \( \lambda \in \left[ \frac{\sum_{i=1}^n (\Pi_{j=1,j\neq i}^n p_j) \|w_i\|_p}{(\Pi_{i=1}^n p_i) \int_a^b F(x,w_1(x),\ldots,w_n(x))dx} , \int_a^b \sup_{(t_1,\ldots,t_n) \in K(kr)} F(x,t_1,\ldots,t_n)dx \right] \), system
\[(1) \text{ admits at least three weak solutions in } X.\]
Proof: For each \( u = (u_1, \ldots, u_n) \in X \), define \( \Phi, \Psi : X \to \mathbb{R} \) as

\[
\Phi(u) := \sum_{i=1}^{n} \frac{\|u_i\|_{p_i}^{p_i}}{p_i}, \quad \text{and} \quad \Psi(u) := \int_{a}^{b} F(x, u_1(x), \ldots, u_n(x))dx.
\]

It is well known that \( \Phi \) and \( \Psi \) are well defined and continuously Gâteaux differentiable functionals with

\[
\Phi'(u)(v) = \int_{a}^{b} \sum_{i=1}^{n} |u_i'(x)|^{p_i-2} u_i'(x) v_i'(x) dx + \int_{a}^{b} \sum_{i=1}^{n} a_i(x) |u_i(x)|^{p_i-2} u_i(x) v_i(x) dx
\]

and

\[
\Psi'(u)(v) = \int_{a}^{b} \sum_{i=1}^{n} F_{u_i}(x, u_1(x), \ldots, u_n(x)) v_i(x) dx
\]

for every \( u = (u_1, \ldots, u_n), v = (v_1, \ldots, v_n) \in X \), as well as \( \Psi' : X \to X^* \) is continuous and compact operator (see Proposition 26.2 of [17]). Furthermore, by Proposition 25.20 of [17], \( \Phi \) is sequentially weakly lower semicontinuous. Also, \( \Phi \) is coercive and \( \Phi(0) = \Psi(0) = 0 \) in \( X \).

Then, for each \( u \in \Phi^{-1}((-\infty, r]) \), we have that

\[
\sup_{x \in [a, b]} |u_i(x)|^{p_i} \leq k \|u_i\|_{p_i}^{p_i}
\]

for \( 1 \leq i \leq n \) (see (2)), we have that

\[
\sup_{x \in [a, b]} \sum_{i=1}^{n} \frac{|u_i(x)|^{p_i}}{p_i} \leq k \sum_{i=1}^{n} \frac{\|u_i\|_{p_i}^{p_i}}{p_i}
\]

for each \( u = (u_1, \ldots, u_n) \in X \). From (4), for each \( r > 0 \) we obtain

\[
\Phi^{-1}((-\infty, r]) = \left\{ u = (u_1, \ldots, u_n) \in X : \Phi(u) \leq r \right\}
\]

\[
= \left\{ u = (u_1, \ldots, u_n) \in X : \sum_{i=1}^{n} \frac{|u_i(x)|^{p_i}}{p_i} \leq r \right\}
\]

\[
\subseteq \left\{ u = (u_1, \ldots, u_n) \in X : \sum_{i=1}^{n} \frac{|u_i(x)|^{p_i}}{p_i} \leq kr \quad \text{for all} \quad x \in [a, b] \right\}.
\]

Then,

\[
\sup_{u \in \Phi^{-1}((-\infty, r])} \Psi(u) = \sup_{u \in \Phi^{-1}((-\infty, r])} \int_{a}^{b} F(x, u_1(x), \ldots, u_n(x))dx
\]

\[
\leq \int_{a}^{b} \sup_{(t_1, \ldots, t_n) \in K(kr)} F(x, t_1, \ldots, t_n)dx.
\]
Therefore, from the condition (ii), we have
\[
\sup_{u \in \Phi^{-1}((-\infty,r])} \Psi(u) \leq \int_a^b \sup_{(t_1,\ldots,t_n) \in K(kr)} F(x,t_1,\ldots,t_n)dx
\]
\[
< \left(r \prod_{i=1}^n p_i \right)^{\frac{b-a}{p_i}} \sum_{i=1}^n \left( \prod_{j=1,j\neq i}^n p_j \right) \|w_i\|_{p_i}^{p_i}
\]
\[
= r \int_a^b F(x,u_1(x),\ldots,u_n(x))dx
\]
\[
= r \frac{\Psi(u)}{\Phi(u)}
\]
from which (a1) of Theorem 1 follows.

Now, fixed \( \lambda \in \) \( \left[ \sum_{i=1}^n \left( \prod_{j=1,j\neq i}^n p_j \right) \|w_i\|_{p_i}^{p_i} \right] \) \( \frac{r}{r} \sum_{i=1}^n \sup_{(t_1,\ldots,t_n) \in K(kr)} F(x,t_1,\ldots,t_n)dx \), due to (iii), we can find \( \gamma, \vartheta \in \mathbb{R} \) with
\[
0 < \gamma < \frac{\int_a^b \sup_{(t_1,\ldots,t_n) \in K(kr)} F(x,t_1,\ldots,t_n)dx}{r}
\]
such that
\[
k(b-a)F(x,t_1,\ldots,t_n) \leq \gamma \sum_{i=1}^n \frac{|t_i|^{p_i}}{p_i} + \vartheta
\]
for all \( x \in [a,b] \) and for all \( (t_1,\ldots,t_n) \in \mathbb{R}^n \). Fixed \( (u_1,\ldots,u_n) \in X \), bearing in mind (4), we have
\[
\Phi(u) - \lambda \Psi(u) = \sum_{i=1}^n \frac{|u_i|^{p_i}}{p_i} - \lambda \int_a^b F(x,u_1(x),\ldots,u_n(x))dx
\]
\[
\geq \sum_{i=1}^n \frac{|u_i|^{p_i}}{p_i} - \lambda \gamma \frac{1}{k(b-a)} \left( \sum_{i=1}^n \frac{1}{p_i} \int_a^b |u_i(x)|^{p_i}dx \right) - \lambda \frac{\vartheta}{k(b-a)}
\]
\[
\geq \sum_{i=1}^n \frac{|u_i|^{p_i}}{p_i} - \lambda \gamma \frac{1}{k(b-a)} \left( \sum_{i=1}^n \frac{1}{p_i} \int_a^b \sup_{(t_1,\ldots,t_n) \in K(kr)} F(x,t_1,\ldots,t_n)dx \right) - \lambda \frac{\vartheta}{k(b-a)}
\]
\[
\geq \left(1 - \frac{\gamma r}{\int_a^b \sup_{(t_1,\ldots,t_n) \in K(kr)} F(x,t_1,\ldots,t_n)dx} \right) \sum_{i=1}^n \frac{|u_i|^{p_i}}{p_i} - \lambda \frac{\vartheta}{k(b-a)}
\]
Thus,
\[
\lim_{\|u\| \to +\infty} (\Phi(u) - \lambda \Psi(u)) = +\infty,
\]
i.e., \( \Phi - \lambda \Psi \) is coercive. Now, all the hypotheses of Theorem 1 are satisfied. Also note that the solutions of the equation \( \Phi'(u) - \lambda \Psi'(u) = 0 \) are exactly the weak solutions of (1). Thus for
Assume that there exist \( \lambda \) and \( \tau \) with \( \theta_1 < \tau \) for \( 1 \leq i \leq n \) such that

\[
\begin{align*}
\sum_{i=1}^n \left( \prod_{j=1, j \neq i}^n (p_j) \right) \sum_{i=1}^n \left( \prod_{j=1, j \neq i}^n (p_j) \right)^{\theta_1 p_i} \int_a^b F(x, w_1(x), ..., w_n(x)) dx \\
\int_a^b \sup_{(t_1, ..., t_n) \in K(kr)} F(x, t_1, ..., t_n) dx \\
\end{align*}
\]

system (1) admits at least three weak solutions in \( X \).

Remark 1 The classical and the weak solutions of system (1) coincide, provided that the functions \( F_i(1 \leq i \leq n) \) be continuous in \([a, b] \times \mathbb{R}^n\).

Next, we want to give a verifiable consequence of Theorem 2 for a fixed text function \( w \).

Corollary 1 Assume that there exist \( n+1 \) positive constants \( \theta \) and \( \tau \) with \( \theta_1 < \tau \) for \( 1 \leq i \leq n \) such that

\[
\begin{align*}
(j) & \quad \frac{1}{n} \sum_{i=1}^n \left( \prod_{j=1, j \neq i}^n (p_j) \right)^{\theta_1 p_i} \int_a^b F(x, \tau, ..., \tau) dx - \int_a^b \sup_{(t_1, ..., t_n) \in K(kr)} F(x, t_1, ..., t_n) dx > 0; \\
(j) & \quad \limsup_{(t_1, ..., t_n) \to (+\infty, ..., +\infty)} \frac{\int_a^b \sup_{(t_1, ..., t_n) \in K(kr)} F(x, t_1, ..., t_n) dx}{(b-a) \sum_{i=1}^n \left( \prod_{j=1, j \neq i}^n (p_j) \right)^{\theta_1 p_i}} < \sup_{(t_1, ..., t_n) \in K(kr)} F(x, t_1, ..., t_n) dx \text{ uniformly with respect to } x \in [a, b].
\end{align*}
\]

Then, for each \( \lambda \in \left[ \frac{1}{n} \sum_{i=1}^n \left( \prod_{j=1, j \neq i}^n (p_j) \right)^{\theta_1 p_i} \int_a^b F(x, \tau, ..., \tau) dx, \left( k \prod_{i=1}^n \left( \prod_{j=1, j \neq i}^n (p_j) \right)^{\theta_1 p_i} \right) \int_a^b \sup_{(t_1, ..., t_n) \in K(kr)} F(x, t_1, ..., t_n) dx \right] \), system (1) admits at least three weak solutions in \( X \).

Proof: Choose \( w(x) = (w_1(x), ..., w_n(x)) = (\tau, ..., \tau) \) for every \( x \in [a, b] \). Then we have

\[
\begin{align*}
\sum_{i=1}^n \frac{\|w_i\|^{p_i}_{p_i}}{p_i} & = \sum_{i=1}^n \frac{\theta_1 p_i}{p_i} \|a_i\|_1. \\
\text{Put } r & \quad := \frac{1}{n} \sum_{i=1}^n \frac{\theta_1 p_i}{p_i}. \quad \text{Now since } \theta_1 < \tau, \text{ bearing in mind that}
\end{align*}
\]

\[
\begin{align*}
\sum_{i=1}^n \frac{\|w_i\|^{p_i}_{p_i}}{p_i} & \leq k \text{ for } 1 \leq i \leq n, \text{ one has } \sum_{i=1}^n \frac{\|w_i\|^{p_i}_{p_i}}{p_i} > r \text{ which is (i) of Theorem 2. Also, since}
\end{align*}
\]

\[
\begin{align*}
\left( \prod_{i=1}^n (p_i) \right)^{\theta_1 p_i} \int_a^b F(x, w_1(x), ..., w_n(x)) dx & = \frac{1}{k} \sum_{i=1}^n \left( \prod_{j=1, j \neq i}^n (p_j) \right)^{\theta_1 p_i} \int_a^b F(x, \tau, ..., \tau) dx \\
\int_a^b \sup_{(t_1, ..., t_n) \in K(kr)} F(x, t_1, ..., t_n) dx & = \frac{1}{k} \sum_{i=1}^n \left( \prod_{j=1, j \neq i}^n (p_j) \right)^{\theta_1 p_i} \int_a^b F(x, \tau, ..., \tau) dx,
\end{align*}
\]

(j) guarantees (ii). Thus, all the assumptions of Theorem 2 are satisfied and the proof is complete.

We now point out a special situation of Corollary 1, in which the function \( F \) has separable variables.

Corollary 2 Let \( f \) be a continuous function in \([a, b] \) and \( \tilde{f}_i \) for \( 1 \leq i \leq n \) be \( C^1 \) for a.e. \( x \in [a, b] \). Assume that there exist \( n+1 \) positive constants \( \theta_i \) and \( \tau \) with \( \theta_i < \tau \) for \( 1 \leq i \leq n \) such that

\[
\begin{align*}
\sum_{i=1}^n \left( \prod_{j=1, j \neq i}^n (p_j) \right)^{\theta_1 p_i} \int_a^b F(x, w_1(x), ..., w_n(x)) dx & = \frac{1}{k} \sum_{i=1}^n \left( \prod_{j=1, j \neq i}^n (p_j) \right)^{\theta_1 p_i} \int_a^b F(x, \tau, ..., \tau) dx \\
\int_a^b \sup_{(t_1, ..., t_n) \in K(kr)} F(x, t_1, ..., t_n) dx & = \frac{1}{k} \sum_{i=1}^n \left( \prod_{j=1, j \neq i}^n (p_j) \right)^{\theta_1 p_i} \int_a^b F(x, \tau, ..., \tau) dx,
\end{align*}
\]
constants

Let

\[ \int \]

Then, for each

\[ \text{Multiplicity results} \]

\[ \text{Hence, the proof is complete.} \]

and from (k)-(kkk), it is easy to verify that all the assumptions of Corollary 1 are satisfied.

\[ \text{Note that} \]

\[ \text{for} \]

\[ 1 \]

\[ \leq \lambda \]

\[ \k \]

\[ \lambda \in \left[ \sum_{i=1}^{n} \left( \prod_{j=1,j \neq i}^{n} p_j \right)^{p_i} \| a_i \|_1 \right] \left( \prod_{i=1}^{n} p_i^{p_i} \right) \left( \prod_{(1, \ldots, t_n) \in K(\sum_{i=1}^{n} p_i^{p_i})} \prod_{i=1}^{n} f_i(t_i) \right) \int_{a}^{b} f(x) dx \]

\[ \text{for each} \]

\[ \text{uniformly with respect to} \]

\[ \text{for} \]

\[ 1 \leq i \leq n. \]

\[ \text{Proof:} \]

\[ F(x, u_1, \ldots, u_n) = f(x) \left( \prod_{i=1}^{n} f_i(u_i) \right) \text{ for each} \]

\[ (x, u_1, \ldots, u_n) \in [a, b] \times \mathbb{R}^n, \]

and note that

\[ \int_{a}^{b} \sup_{(t_1, \ldots, t_n) \in K(\sum_{i=1}^{n} p_i^{p_i})} F(x, t_1, \ldots, t_n) dx = \int_{a}^{b} f(x) dx \]

\[ \max_{(t_1, \ldots, t_n) \in K(\sum_{i=1}^{n} p_i^{p_i})} \prod_{i=1}^{n} f_i(t_i) \]

and

\[ \int_{a}^{b} F(x, \tau, \ldots, \tau) dx = \prod_{i=1}^{n} f_i(\tau) \int_{a}^{b} f(x) dx, \]

and from (k)-(kkk), it is easy to verify that all the assumptions of Corollary 1 are satisfied. Hence, the proof is complete.

**Corollary 3** Let \( F : \mathbb{R}^n \to \mathbb{R} \) be a \( C^1 \) function and assume that there exist \( n + 1 \) positive constants \( \theta_i \) and \( \tau \) with \( \theta_i < \tau \) for \( 1 \leq i \leq n \) such that

\[ \frac{1}{k} \sum_{i=1}^{n} \left( \prod_{j=1,j \neq i}^{n} p_j \right)^{p_i} F(\tau, \ldots, \tau) - \max_{(t_1, \ldots, t_n) \in K(\sum_{i=1}^{n} p_i^{p_i})} \prod_{i=1}^{n} f_i(t_i) > 0; \]

\[ \limsup_{(t_1, \ldots, t_n) \to (+\infty, \ldots, +\infty)} \frac{F(t_1, \ldots, t_n)}{\sum_{i=1}^{n} t_i^{p_i}} \leq \max_{(t_1, \ldots, t_n) \in K(\sum_{i=1}^{n} p_i^{p_i})} \frac{F(t_1, \ldots, t_n)}{\sum_{i=1}^{n} t_i^{p_i}}. \]

Then, for each

\[ \lambda \in \left[ \frac{1}{k} \sum_{i=1}^{n} \left( \prod_{j=1,j \neq i}^{n} p_j \right)^{p_i} \tau^{p_i} \| a_i \|_1 \right] \left( \prod_{i=1}^{n} p_i^{p_i} \right) \left( \prod_{(t_1, \ldots, t_n) \in K(\sum_{i=1}^{n} p_i^{p_i})} \prod_{i=1}^{n} f_i(t_i) \right) \int_{a}^{b} f(x) dx \]

\[ \text{the system} \]

\[ \left\{ \begin{array}{l}
(\phi_{p_i}(u_i)')' + \lambda F_i (u_1, \ldots, u_n) = a_i(x)|u_i|^{p_i-2}u_i \quad \text{in} \ (a, b), \\
u_i(a) = u_i(b) = 0
\end{array} \right. \]
for $1 \leq i \leq n$, admits at least three weak solutions in $X$.

We now look at a consequence of Corollary 3 in the ordinary case $p_i = 2$ for $1 \leq i \leq n$. For simplicity, we put $(a, b) = (0, 1)$. Note that in this situation we have (see (3))

$$k = 2 \max\{\|a_i\|_1^{-1}, \|a_i\|_2^2\|a_i\|_1^{-2} : 1 \leq i \leq n\}.$$

**Corollary 4** Let $F : \mathbb{R}^n \to \mathbb{R}$ be a $C^1$ function and assume that there exist $n + 1$ positive constants $\theta_i$ and $\tau$ with $\theta_i < \tau$ for $1 \leq i \leq n$ such that

(m) $\left(\sum_{i=1}^n \theta_i^2 \right) \frac{F(t_1, \ldots, t_n)}{K_1} - \max_{(t_1, \ldots, t_n) \in K_1} F(t_1, \ldots, t_n) > 0,$

where $K_1 := \left\{ (t_1, \ldots, t_n) \in \mathbb{R}^n : \sum_{i=1}^n t_i^2 \leq \sum_{i=1}^n \theta_i^2 \right\}$;

(mm) $\limsup_{(t_1, \ldots, t_n) \to (+\infty, \ldots, +\infty)} \frac{F(t_1, \ldots, t_n)}{\sum_{i=1}^n t_i^2} \leq 0.$

Then, for each $\lambda \in \left[ \frac{\tau^2 \sum_{i=1}^n \|a_i\|_1}{2F(\tau, \ldots, \tau)}, \frac{2\max_{(t_1, \ldots, t_n) \in K_1} F(t_1, \ldots, t_n)}{\sum_{i=1}^n \theta_i^2} \right]$, the system

$$\begin{cases} u_i'' + \lambda F(u_1, \ldots, u_n) = a_i(x)u_i & \text{in } (0, 1), \\ u_i(0) = u_i'(1) = 0 \end{cases}$$

for $1 \leq i \leq n$, admits at least three weak solutions in $(W^{1,2}([0, 1]))^n$.

Finally, we conclude this paper by giving an immediate consequence of Corollary 4 and an example of it when $n = 1$. Now we have

$$k = 2 \max\{\|a\|_1^{-1}, \|a\|_2^2\|a\|_1^{-2}\}.$$

**Corollary 5** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Put $F(t) = \int_0^t f(\xi)\,d\xi$ for each $t \in \mathbb{R}$. Assume that there exist two positive constants $\theta$ and $\tau$ with $\theta < \tau$ such that

(n) $\frac{\theta^2 F(\tau)}{2c\tau^2} - \max_{t \in [-\theta, \theta]} F(t) > 0$ where $c := \max\{1, \|a\|_2^2\|a\|_1^{-1}\};$

(nn) $\limsup_{|t| \to +\infty} \frac{F(t)}{t^2} \leq 0.$

Then, for each $\lambda \in \left[ \frac{\tau^2 \|a\|_1}{2F(\tau), \frac{2\max_{t \in [-\theta, \theta]} F(t)}{\sum_{i=1}^n \theta_i^2}} \right]$, the problem

$$\begin{cases} u'' + \lambda f(u) = a(x)u & \text{in } (0, 1), \\ u(0) = u'(1) = 0 \end{cases}$$

admits at least three classical solutions in $C^2([0, 1])$.

**Example 1** Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(t) = t^{12}e^{-t^2}(13 - 2t^2)$ for all $t \in \mathbb{R}$. By the expression of $f$ we have $F(t) = \int_0^t f(\xi)\,d\xi = t^{13}e^{-t^2}$ for all $t \in \mathbb{R}$. In fact, by choosing $\tau = 2$, $\theta = 1$ and $a(x) = x$, we see that $k = 8$. A simple computation shows

$$\frac{\theta^2 F(\tau)}{2c\tau^2} - \max_{t \in [-\theta, \theta]} F(t) = \frac{29}{e^4} - \frac{1}{e} > 0.$$
Moreover,
\[
\limsup_{|t| \to +\infty} \frac{F(t)}{t^2} = \limsup_{|t| \to +\infty} te^{-t^2} = 0.
\]
So, by Corollary 5, for each \( \lambda \in \left[ \frac{1}{e^{12}e^{-u^2}}, \frac{e}{10} \right] \), the problem
\[
\begin{align*}
u'' + \lambda u^{12} e^{-u^2} (13 - 2u^2) &= xu \quad \text{in } (0,1), \\
u'(0) &= u'(1) = 0
\end{align*}
\]
admits at least three classical solutions in \( C^2([0,1]) \).

References


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