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Numerical solution of nonlinear stochastic integral equation by stochastic operational matrix based on Bernstein polynomials

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Abstract

In this paper, a practical and computational numerical method based on Bernstein polynomials for solving nonlinear stochastic integral equations is presented. Stochastic operational matrix of Bernstein polynomials is determined. The main idea is that it reduces the stochastic integral equation to a system of algebraic equations. Thus we can solve the problem by iteration methods. Numerical example illustrates the efficiency and accuracy of the method.

Key Words: Bernstein polynomial; Brownian motion; Itô integral; Stochastic operational matrix; Nonlinear stochastic integral equation.
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1 Introduction

Mathematical modeling of real life problems usually causes functional equations, like ordinary or partial differential equations, stochastic differential equations, stochastic integral and stochastic integro-differential equations. They play a prominent role in range of application areas including biology, chemistry, epidemiology, mechanics, economics and finance([1]-[4]).
Stochastic differential equations arise naturally in various engineering problems([5]), where the effects of random 'noise' perturbations to a system are being considered. For example in the problem of tracking satelite, we know that it's motion will obey Newton's law to a very high degree of accuracy, so in theory we can integrate the trajectories from the initial point. However in practice there are other random effects which perturb the motion.
Consider the ordinary differential equation as

$$x'(t) = \lambda b(t, x(t)), \quad x(t_0) = x_0. \tag{1}$$

The stochastic version of (1) can be written in differential form as([4])

$$dx(t) = \lambda_1 b(t, x(t))dt + \lambda_2 \sigma(t, x(t))dB(t), \quad x(t_0) = x_0, \tag{2}$$

where, λ_1, λ_2 are parameters and $x(t), b(t, x(t)), \sigma(t, x(t))$ for $t \in [0, T)$ are stochastic processes defined on the some probability space (Ω, F, P). Also $x(t)$ is unknown function and $B(t)$ is Brownian motion. Due to the irregularity of the Brownian motion one can only interpert the stochastic differential equation in terms of the stochastic integral equations as

$$x(t) = x_0 + \lambda_1 \int_0^t b(s, x(s))ds + \lambda_2 \int_0^t \sigma(s, x(s))dB(s). \tag{3}$$

The main problem is to calculate the third term of right hand side in equation (3) that is called Itô integral. Numerous papers have been focusing on the existence solution of equation (3) ([6]-[8]). Also some papers have been presented numerical methods to solve stochastic differential equations and stochastic integral equations ([9]-[16]), but the nonlinear stochastic differential equations are still difficult to solve either numerically or theoretically.

Bernstein polynomials and their operational matrix have been frequently used in the solution of integral equations, differential equations and approximation theory ([17]-[23]). In this work, we derived stochastic operational matrix based on Bernstein polynomials for solving Itô integral in Eq.(3). Furthermore, we find numerical solution of nonlinear stochastic integral equation(NSIE) (3).

This paper is organized as follows.

In Section 2 we review some of the basic theory of the stochastic calculus and Bernstein polynomials. In Section 3 we introduce stochastic operational matrix. In Section 4 we apply Bernstein polynomial approximation, their operational matrix and stochastic operational matrix with collocation method to reduce the NSIE to a system of algebraic equations that can be solved by Newton's method. Section 5 shows convergence of the method. In Section 6 the presented method is tested with an example. Finally, Section 7 gives some brief conclusions.

2 Preliminaries and notations

2.1 Stochastic calculus

Consider random variable X with distribution f_x, so

$$E[X^p] = \int_{-\infty}^{\infty} x^p f_x dx < \infty.$$

Suppose $p \geq 2$ and denote $L^p(\Omega, H)$ the collection of all strongly measurable, p-th integrable H-valued random variables. It is routine to check that $L^p(\Omega, H)$ is a Banach space with

$$\|V\|_{L^p(\Omega,H)} := [E\|V\|^p]^{\frac{1}{p}},$$

for each $V \in L^p(\Omega, H)$. Here we consider $L^2(\Omega, H)$.

Definition 2.1. The sequence $\{X_n\}$ converge to X in L^2 if for each n, $E(|X_n|^2) < \infty$ and $E(\|X_n - X\|)^2 \to 0$ as $n \to \infty$ [3].

Suppose $0 \leq s \leq T$, let $\upsilon = \upsilon(s, T)$ be the class of functions that $f(t, \omega) : [0, \infty] \times \Omega \to R^n$, satisfy

(i) the function $(t,\omega) \to f(t,\omega)$ is $\beta \times \mathcal{F}$ measurable, where β is the Borel algebra.
(ii) f is adapted to $\mathcal{F}_t$.
(iii) $E\big[\int_s^T f(t,\omega)^2 dt\big] < \infty$.

Definition 2.2. (The Itô integral[4]). Let $f \in \upsilon(s,T)$, then the Itô integral of f is defined by

$$\int_s^T f(t,\omega)dB(t)(w) = \lim_{n\to\infty} \int_s^T \varphi_n(t,\omega)dB(t)(\omega),$$

where $\{\varphi_n\}$ is the sequence of elementary functions such that,

$$E\big[\int_s^T (f-\varphi_n)^2 dt\big] \to 0 \qquad a.s, \quad n \to \infty.$$

Theorem 2.3. (The Itô isometry[4]) Let $f \in \nu(S,T)$, then

$$E\big[(\int_S^T f(t,\omega)dB(t)(\omega))^2\big] = E\big[\int_S^T f^2(t,\omega)dt\big].$$

2.2 Bernstein polynomials

The Bernstein polynomials of nth-degree are defined as

$$\beta_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i} = \sum_{k=0}^{n-i} (-1)^k \binom{n}{i}\binom{n-i}{k} t^{i+k}, \quad t \in [0,1],$$

for $i = 0,1,...,n$. Now consider

$$\Phi(t) = [\beta_{0,n}(t), \beta_{1,n}(t), ..., \beta_{n,n}(t)]^T,$$

we can write

$$\Phi(t) = AT_n(t), \tag{4}$$

where, $T_n(t) = \begin{bmatrix} 1 & t & \dots & t^n \end{bmatrix}^T$ and A is an $(n+1)\times(n+1)$ upper triangular matrix with

$$A_{i+1} = \Big[\overbrace{0,0,...,0}^{i \text{ times}}, (-1)^0\tbinom{n}{i}\tbinom{n-i}{0}, (-1)^1\tbinom{n}{i}\tbinom{n-i}{1}, ..., (-1)^{n-i}\tbinom{n}{i}\tbinom{n-i}{n-i}\Big].$$

Since, $L^2[0,1]$ is a Hilbert space with the inner product $(f,g) = \int_0^1 f(x)g(x)dx$, any function $f(x) \in L^2[0,1]$ can be expanded in Bernstein basis ([19]) as

$$f(t) \simeq B_n(f(t)) = C^T\Phi(t), \tag{5}$$

where, $C^T = (f(t),\Phi(t))\mathbf{D}^{-1}$, $(f(t),\Phi(t)) = \int_0^1 f(t)\Phi(t)dt$. $\mathbf{D} = (\Phi(t),\Phi(t))$ is an $(n+1)\times(n+1)$ matrix and is called dual matrix of $\Phi(t)$. The elements of $\mathbf{D}$ are specified in([17]). The integration of $\Phi(t)$ is approximated as

$$\int_0^t \Phi(s)ds \simeq \mathbf{P}\Phi(t), \tag{6}$$

where, $\mathbf{P}$ is an $(n+1)\times(n+1)$ operational matrix([17]).

3 Stochastic operational matrix based on Bernstein polynomials

Let

$$\int_0^t \Phi(s)dB(s) = \int_0^t AT_n(s)dB(s) = A\big[\int_0^t dB(s), \int_0^t sdB(s), ..., \int_0^t s^n dB(s)\big]^T. \tag{7}$$

We can write

$$\begin{bmatrix} \int_0^t dB(s) \\ \int_0^t sdB(s) \\ \vdots \\ \int_0^t s^n dB(s) \end{bmatrix} = B(t)T_n(t) - \begin{bmatrix} 0 \\ \int_0^t B(s)ds \\ \vdots \\ n\int_0^t s^{n-1}B(s)ds \end{bmatrix} = M_n(t) = \big(m_i\big)_{i=0,1,...,n},$$

where

$$m_i = t^i B(t) - i\int_0^t s^{i-1}B(s)ds, \qquad i = 0, ..., n.$$

By using composite trapezium rule we get

$$m_i \simeq t^i B(t) - \frac{it}{4}\big(2(\frac{t}{2})^{i-1}B(\frac{t}{2}) + t^{i-1}B(t)\big) = \big[(1-\frac{i}{4})B(t) - \frac{i}{2^i}B(\frac{t}{2})\big]t^i, \quad i = 0, ..., n.$$

Also we approximate $B(t)$ and $B(\frac{t}{2})$, for $0 \le t \le 1$, by $B(.5)$ and $B(.25)$. After replacing these approximations in (7) , we obtain

$$AM_n(t) = A\begin{bmatrix} B(.5) & 0 & \dots & 0 \\ 0 & \frac{3}{4}B(.5) - \frac{1}{2}B(.25) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (1-\frac{n}{4})B(.5) - \frac{n}{2^n}B(.25) \end{bmatrix}\begin{bmatrix} 1 \\ t \\ \vdots \\ t^n \end{bmatrix}.$$

Put

$$D_s = \begin{bmatrix} B(.5) & 0 & \dots & 0 \\ 0 & \frac{3}{4}B(.5) - \frac{1}{2}B(.25) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (1-\frac{n}{4})B(.5) - \frac{n}{2^n}B(.25) \end{bmatrix},$$

then

$$AM_n(t) = AD_sT_n(t) = AD_sA^{-1}\Phi(t) = P_s\Phi(t),$$

where $P_s = AD_sA^{-1}$ is $(n+1)\times(n+1)$ stochastic operational matrix. Therefore

$$\int_0^t \Phi(s)dB(s) \simeq P_s\Phi(t). \tag{8}$$

4 Implementation of Bernstein operational matrices for solving NSIE

Consider the nonlinear stochastic integral equation (3) and let

$$z_1(t) = b(t, x(t)), \quad z_2(t) = \sigma(t, x(t)). \tag{9}$$

First, we find the collocation approximation for $z_1(t)$ and $z_2(t)$. By substituting Eqs.(9) in Eq.(3) we get

$$\begin{cases} z_1(t) = b(t, \lambda_1 \int_0^t z_1(s)ds + \lambda_2 \int_0^t z_2(s)dB(s) + x_0), \\ z_2(t) = \sigma(t, \lambda_1 \int_0^t z_1(s)ds + \lambda_2 \int_0^t z_2(s)dB(s) + x_0). \end{cases} \tag{10}$$

The Bernstein polynomials approximation of $z_1(t)$ and $z_2(t)$ can be written as

$$z_1(t) \simeq B_n(z_1(t)) = Z_1^T \Phi(t), \quad z_2(t) \simeq B_n(z_2(t)) = Z_2^T \Phi(t), \tag{11}$$

which Z_1 and Z_2 are defined by (5). By using (11),(6)and(8) we have

$$\int_0^t z_1(s)ds \simeq Z_1^T \int_0^t \Phi(s)ds = Z_1^T P\Phi(t), \tag{12}$$

and

$$\int_0^t z_2(s)dB(s) \simeq Z_2^T \int_0^t \Phi(s)dB(s) = Z_2^T P_s\Phi(t). \tag{13}$$

After substituting the approximate equations (11),(12) and (13) in (10), we get

$$\begin{cases} Z_1^T\Phi(t) = b(t, \lambda_1 Z_1^T P\Phi(t) + \lambda_2 Z_2^T P_s\Phi(t) + x_0), \\ Z_2^T\Phi(t) = \sigma(t, \lambda_1 Z_1^T P\Phi(t) + \lambda_2 Z_2^T P_s\Phi(t) + x_0). \end{cases} \tag{14}$$

Now, we collocate Eqs.(14) in $n+1$ Newton-cotes nodes , $t_i = \frac{(2i-1)}{2(n+1)}$, $i = 1, 2, ..., n+1$, then we rewrite Eqs.(14) as

$$\begin{cases} Z_1^T\Phi(t_i) = b(t_i, \lambda_1 Z_1^T P\Phi(t_i) + \lambda_2 Z_2^T P_s\Phi(t_i) + x_0), \\ Z_2^T\Phi(t_i) = \sigma(t_i, \lambda_1 Z_1^T P\Phi(t_i) + \lambda_2 Z_2^T P_s\Phi(t_i) + x_0), \end{cases} \quad i = 1, 2, ..., n+1. \tag{15}$$

After solving nonlinear system (15) with Newton's method, we obtain Z_1^T and Z_2^T. Finally, we can approximate Eq. (3) as follows

$$x_n(t) = \lambda_1 Z_1^T P\Phi(t) + \lambda_2 Z_2^T P_s\Phi(t) + x_0.$$

5 Convergence analysis

Theorem 5.1. For all function f in $C[0,1]$, the sequence $\{B_n(f); n = 1, 2, ...\}$ converges uniformly to f.

Proof. see[24].

Theorem 5.1 shows that for any $f \in [0,1]$ and for any ε, there exists n such that inequality

$$\|B_n(f) - f\| < \varepsilon,$$

holds.
We suppose $\|.\|$ be the L^2 norm on $[0,1]$. Let us $e_n(t) = x(t) - x_n(t)$ be an error function of approximate solution $x_n(t)$ to the exact solution $x(t)$,

$$x(t) - x_n(t) = \lambda_1 \int_0^t \big(z_1(s) - \hat{z}_1(s)\big)ds + \lambda_2 \int_0^t \big(z_2(s) - \hat{z}_2(s)\big)dB(s), \tag{16}$$

where $z_i(t), \quad i = 1,2$ are defined in (9), also $\hat{z}_i(t), i = 1,2$ is approximated form of $z_i(t)$ by Bernstein approximation

$$\hat{z}_1(s) = B_n(b(s, x_n(s))), \qquad \hat{z}_2(s) = B_n(\sigma(s, x_n(s))),$$

and

$$z_1^n(s) = b(s, x_n(s)), \qquad z_2^n(s) = \sigma(s, x_n(s)),$$

Theorem 5.2. Let $x(t)$ be exact solution and $x_n(t)$ be the Bernstein approximate solution of (3). Also assume that
(i) For every T and N, there is a constant D depending only on T and N such that for all $|x|, |y| \leq N$ and all $0 \leq t \leq T$,

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq D|x - y|.$$

(ii) Coefficients satisfy the linear growth condition

$$|b(t,x)| + |\sigma(t,x)| \leq D(1 + |x|).$$

(iii) $E(|x|^2) < \infty$.
Then $x_n(t)$ converges to $x(t)$ in L^2.

Proof.

$$e_n(t) = \lambda_1 \int_0^t (z_1(s) - \hat{z}_1(s))ds + \lambda_2 \int_0^t (z_2(s) - \hat{z}_2(s))dB(s),$$

$$E\|e_n(t)\|^2 \leq 2\big(|\lambda_1|^2 E\| \int_0^t (z_1(s) - \hat{z}_1(s))ds\|^2 + |\lambda_2|^2 E\| \int_0^t (z_2(s) - \hat{z}_2(s))dB(s)\|^2\big),$$

by the Itô isometry, we get

$$\begin{aligned} E\|e_n(t)\|^2 &\leq 2\big[|\lambda_1|^2 \int_0^t E\|(z_1(s) - \hat{z}_1(s))\|^2 ds + |\lambda_2|^2 \int_0^t E\|(z_2(s) - \hat{z}_2(s))\|^2 ds\big] \\ &\leq 8\big[|\lambda_1|^2 \int_0^t E\|(z_1(s) - z_1^n(s))\|^2 ds + |\lambda_1|^2 \int_0^t E\|(z_1^n(s) - \hat{z}_1(s))\|^2 ds\big] \\ &\quad + |\lambda_2|^2 \int_0^t E\|(z_2(s) - z_2^n(s))\|^2 ds + |\lambda_2|^2 \int_0^t E\|(z_2^n(s) - \hat{z}_2(s))\|^2 ds\big]. \end{aligned}$$

By Theorem 5.1, there exists $n > 0$ such that for any ε,

$$E\|z_j^n(s) - \hat{z}_j(s)\|^2 \leq \varepsilon = \frac{\varepsilon_1}{16|\lambda_j|^2}, \; j = 1,2,$$

so,

$$E\|e_n(t)\|^2 \leq \varepsilon_1 + 8\big[|\lambda_1|^2 \int_0^t E\|z_1(s) - z_1^n(s)\|^2 ds + |\lambda_2|^2 \int_0^t E\|z_2(s)) - z_2^n(s)\|^2 ds\big],$$

by using Lipschitz condition

$$E\|e_n(t)\|^2 \leq \varepsilon_1 + 8(|\lambda_1|^2 + |\lambda_2|^2)D^2 \int_0^t E\|e_n(s)\|^2 ds). \quad (17)$$

Hence from (17) and Gronwall inequality we get

$$E\|e_n(t)\|^2 \longrightarrow 0,$$

so, $x_n(t) \longrightarrow x(t)$ in L^2.

6 Numerical example

Let $x(t)$ be the exact solution and $y(t)$ be the Bernstein approximation solution, then we define the error for some points in the interval $[0, 1)$ as

$$\|E(t_i)\|_\infty = Max|x(t_i) - y(t_i)|, \quad 0 \leq t_i < 1.$$

Example. Consider the nonlinear stochastic Volterra integral equation as follows (population growth problem)([16])

$$x(t) = 0.5 + \int_0^t x(s)(1 - x(s))ds + \int_0^t x(s)dB(s), \quad t \in [0, 1], \quad (18)$$

with the exact solution

$$x(t) = \frac{e^{0.5t+B(t)}}{2 + \int_0^t e^{0.5s+B(s)}ds},$$

where $x(t)$ is an unknown stochastic process defined on the probability space $(\Omega, \mathcal{F}, P)$ and $B(t)$ is a Brownian motion process. The numerical results are shown in Tables 1 and 2. $\overline{x}_E$ is the errors mean and s_E is the standard deviation of errors in k iteration.

Table 1: Mean, standard deviation and Confidence Interval for error mean. $n = 8, \ k = 500.$

t_i	$\overline{x}_E$	s_E	0.95 Confidence Interval	
			Lowerbound	*Upperbound*
0	0.000144	0.000264	0.000120	0.000167
0.1	0.021256	0.004160	0.020891	0.021620
0.2	0.044276	0.006677	0.043690	0.044861
0.3	0.070860	0.009800	0.070001	0.071719
0.4	0.096359	0.007710	0.095683	0.097034
0.5	0.117920	0.012309	0.116841	0.118999
0.6	0.141850	0.012924	0.140717	0.142983
0.7	0.162259	0.015772	0.160877	0.163641
0.8	0.188730	0.020347	0.186947	0.190513
0.9	0.209053	0.022928	0.207043	0.211063

Table 2: Mean, standard deviation and Confidence Interval for error mean. $n = 13,\ k = 500$.

t_i	$\overline{x}_E$	s_E	0.95 Confidence Interval	
			Lowerbound	*Upperbound*
0	0.000415	0.000772	0.000347	0.000482
0.1	0.006726	0.025251	0.004512	0.008939
0.2	0.014065	0.041921	0.010390	0.017739
0.3	0.001199	0.069176	0.004864	0.007262
0.4	0.028622	0.094439	0.020344	0.036899
0.5	0.055273	0.117777	0.044949	0.065596
0.6	0.087075	0.131612	0.075538	0.098611
0.7	0.123044	0.144593	0.110370	0.135718
0.8	0.152556	0.150813	0.139337	0.165775
0.9	0.177911	0.153433	0.164462	0.191360

7 Conclusion

This paper suggested a numerical method to solve NSIE by using Bernstein polynomials and their operational matrices that introduced in ([20]), also we derived and used the stochastic operational matrix of Bernstein polynomials to transform our NSIE to a nonlinear system of algebraic equations that can be solved by Newton's method. The main advantage of this method is its efficiency and simple applicability. The accuracy is comparatively good in comparison with methods that are applied directly to solve nonlinear stochastic differential equation.

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