Bull. Math. Soc. Sci. Math. Roumanie
Tome 57(105) No. 1, 2014, 101–108

Some existence results for a system of operatorial equations

by

Ioan A. Rus and Marcel-Adrian Şerban

Abstract

In this paper we study a system of operatorial equations in terms of vector-valued distances. The basic tools are some classical fixed point principles and data dependence of the fixed points in the case of Perov fixed point theorem. Applications to some systems of functional integral equations are also given.

Key Words: Vector-valued distance, matrix convergent to zero, fixed point principle, system of operatorial equations, system of integral equations.
2010 Mathematics Subject Classification: Primary 47H10
Secondary 54H25, 47J05, 45G15.

1 Introduction

Let X_i, $i = \overline{1,p}$, be some nonempty sets and $f : \prod_{i=1}^{p} X_i \to \prod_{i=1}^{p} X_i$ be an operator. In this case the fixed point equation

$$x = f(x), \tag{1.1}$$

where $x = (x_1, \ldots, x_p)$ and $f = (f_1, \ldots, f_p)$, takes the following form

$$\begin{cases} x_1 = f_1(x_1, \ldots, x_p) \\ \vdots \\ x_p = f_p(x_1, \ldots, x_p) \end{cases} \tag{1.2}$$

In this paper we study the system (1.2) using the following technique: we split the system in two parts $f = \left(\tilde{f}_1, \tilde{f}_2\right)$, $f : X \times Y \to X \times Y$, where $X = \prod_{i=1}^{m} X_i$ and $Y = \prod_{i=m+1}^{p} X_i$, $m < p$. We consider the case of (X, d) a metric space with a vector-valued distance, $(d : X \times X \to \mathbb{R}^m_+)$, and (Y, τ) a Hausdorff topological space with the *fixed point property*. A topological space Y has the fixed point property if any continuous operator $g : Y \to Y$ has a fixed point. A general principle for the existence of the fixed point of operator $f : X \times Y \to X \times Y$, can be formulated combining the Perov fixed point principle, the continuous dependence of the fixed points and the the fixed point property of the space Y. For other fixed point results using the vectorial distance and matrix convergent to zero technique, see [1], [11], [6], [8], [16], [3], [4], [5], [7], [9], [13].

We begin our considerations with some notations, notions and Perov fixed point principle which shall be useful in presentation of our results.

Let (X, d) be a metric space. We will use the following symbols:
If $f : X \to X$ is an operator then $F_f := \{x \in X |\ x = f(x)\}$ denotes the fixed point set of the operator f.

Definition 1.1. *A matrix $S \in \mathbb{R}_+^{m\times m}$ is called a matrix convergent to zero iff $S^k \to 0$ as $k \to +\infty$.*

Theorem 1.1. *(see [1], [6], [8], [11], [14]) Let $S \in \mathbb{R}_+^{m\times m}$. The following statements are equivalent:*

(i) S is a matrix convergent to zero;

(ii) $S^k x \to 0$ as $k \to +\infty$, $\forall x \in \mathbb{R}^m$;

(iii) $I_m - S$ is non-singular and

$$(I_m - S)^{-1} = I_m + S + S^2 + \ldots$$

(iv) $I_m - S$ is non-singular and $(I_m - S)^{-1}$ has nonnegative elements;

(v) $\lambda \in \mathbb{C}$, $\det(S - \lambda I_m) = 0$ imply $|\lambda| < 1$;

(vi) there exists at least one subordinate matrix norm such that $\|S\| < 1$.

The matrices convergent to zero were used by A. I. Perov [5] to generalize the contraction principle in the case of metric spaces with a vector-valued distance.

Definition 1.2. *(see [5], [11], [13]) Let (X, d) be a complete metric space with $d : X \times X \to \mathbb{R}_+^m$ a vector-valued distance and $f : X \to X$. The operator f is called an S-contraction if there exists a matrix $S \in \mathbb{R}_+^{m\times m}$ such that:*

(i) S is a matrix convergent to zero;

(ii) $d(f(x), f(y)) \le S d(x, y)$, for all $x, y \in X$.

Theorem 1.2 (Perov)**.** *(see [13]) Let (X, d) be a complete metric space with $d : X \times X \to \mathbb{R}_+^m$ a vector-valued distance and $f : X \to X$ be an S-contraction. Then:*

(i) $F_f = F_{f^n} = \{x^*\}$, for all $n \in \mathbb{N}^*$;

(ii) $f^n(x) \overset{d}{\to} x^*$ as $n \to +\infty$, for all $x \in X$;

(iii) $d(f^n(x), x^*) \le (I_m - S)^{-1} S^n d(x, f(x))$, for all $x \in X$ and $n \in \mathbb{N}^*$;

(iv) $d(x, x^*) \le (I_m - S)^{-1} d(x, f(x))$, for all $x \in X$.

2 Existence results via Perov's fixed point theorem

Let (X_i, d_i), $i = \overline{1,m}$, be some complete metric spaces and let $X := \prod_{i=1}^{m} X_i$ their cartesian product. Let $d : X \to \mathbb{R}_+^m$

$$d(x,y) := \begin{pmatrix} d_1(x_1,y_1) \\ \vdots \\ d_m(x_m,y_m) \end{pmatrix}$$

a vector-valued metric on X.

Let (Y,τ) be a Hausdorff topological space and $f : X \times Y \to X \times Y$, $f = (f_1, f_2)$, an operator.

Theorem 2.1. *We suppose that:*

(i) f *is continuous;*

(ii) there exists a matrix $S \in \mathbb{R}_+^{m\times m}$ *convergent to zero such that*

$$d(f_1(u,y), f_1(v,y)) \le Sd(u,v),$$

for all $u, v \in X$ *and* $y \in Y$*;*

(iii) (Y,τ) *is a topological space with the fixed point property.*

Then the operator f *has at least a fixed point.*

Proof: Let us consider the operator $f_1(\cdot, y) : X \to X$. By the Perov's theorem (see [11], [13], [6], ...) this operator has a unique fixed point $x^*(y)$ for all $y \in Y$. From the continuity of $f_1 : X \times Y \to X$ it follows that the operator

$$x^* : Y \to X, \ y \mapsto x^*(y),$$

is continuous. Indeed, for $y_1, y_2 \in Y$ we have

$$\begin{aligned} & d(x^*(y_1), x^*(y_2)) = d(f_1(x^*(y_1), y_1), f_1(x^*(y_2), y_2)) \le \\ \le\ & d(f_1(x^*(y_1), y_1), f_1(x^*(y_2), y_1)) + d(f_1(x^*(y_2), y_1), f_1(x^*(y_2), y_2)) \le \\ \le\ & Sd(x^*(y_1), x^*(y_2)) + d(f_1(x^*(y_2), y_1), f_1(x^*(y_2), y_2)) \end{aligned}$$

So, we have

$$d(x^*(y_1), x^*(y_2)) \le (I_m - S)^{-1} d(f_1(x^*(y_2), y_1), f_1(x^*(y_2), y_2)) \to 0 \text{ as } y_1 \to y_2.$$

We consider the operator $h : Y \to Y$, defined by

$$h : Y \to Y, \ y \mapsto f_2(x^*(y), y).$$

Since $x^*(\cdot)$ and f_2 are continuous then h is continuous and from (iii) we have that $F_h \ne \emptyset$. Let $y^* \in F_h$ then it is easy to see that $(x^*(y^*), y^*) \in F_f$, therefore $F_f \ne \emptyset$. □

Remark 2.1. *For examples of topological spaces with fixed point property see [10], [11], [13], ...*

From Theorem 2.1 we have for Y a compact convex subset of a Banach space:

Theorem 2.2. *We suppose that:*

(i) f *is continuous;*

(ii) there exists a matrix $S \in \mathbb{R}_{+}^{m\times m}$ *convergent to zero such that*

$$d\left(f_1\left(u,y\right),f_1\left(v,y\right)\right) \leq Sd\left(u,v\right),$$

for all $u,v\in X$ *and* $y\in Y$.

Then the operator f *has at least a fixed point.*

Proof: Since Y a compact convex subset of a Banach space, hence Y has the fixed point property, so the conditions from Theorem 2.1 are satisfied. □

Remark 2.2. *In the case of* $m=1$*, from Theorem 2.2 we have a result given by C. Avramescu in [2] and by I.A. Rus in [12].*

Remark 2.3. *For the fixed point theory on cartesian product see, for example, [14], [15] and the references therein.*

3 Applications

Example 3.1. *Let us consider the following system of functional-integral equations*

$$\begin{cases} x\left(t\right) &= \int\limits_0^1 K\left(t,s,x\left(s\right),y\left(s\right)\right)ds + g\left(t\right), \qquad t\in[0;1] \\ y\left(t\right) &= \int\limits_0^1 H\left(t,s,x\left(s\right),y\left(s\right),y\left(y\left(s\right)\right)\right)ds, \quad t\in[0;1] \end{cases} \tag{3.1}$$

where $K\in C\left([0;1]\times[0;1]\times\mathbb{R}^m\times[0;1],\mathbb{R}^m\right)$, $g\in C\left([0;1],\mathbb{R}^m\right)$,
$H\in C\left([0;1]\times[0;1]\times\mathbb{R}^m\times[0;1]\times[0;1],\mathbb{R}\right)$.

Let $X_1=X_2=\ldots=X_m:=C\left[0;1\right]$ *with the Cebyshev norm*

$$|x|_\infty = \max_{t\in[0;1]}|x\left(t\right)|.$$

and $X:=\prod\limits_{i=1}^m X_i = C\left([0;1],\mathbb{R}^m\right)$ *with the vector-valued norm*

$$\|x\| = \begin{pmatrix} |x_1|_\infty \\ \vdots \\ |x_m|_\infty \end{pmatrix}.$$

Let

$$Y := C_L\left([0;1],[0;1]\right) = \begin{array}{l} \{y \in C[0;1] \mid y(t) \in [0;1], \ \forall t \in [0;1] \ \text{and} \\ |y(t_1) - y(t_2)| \leq L|t_1 - t_2|, \ \forall t_1, t_2 \in [0;1]\}. \end{array}$$

We remark that Y is a convex compact subset of the Banach space $(C[0;1], |\cdot|_\infty)$.

From the Theorem 2.2 we have:

Theorem 3.1. *We suppose that:*

(i) there exists a matrix $S \in \mathbb{R}_+^{m\times m}$ convergent to zero such that

$$\begin{pmatrix} |K_1(t,s,u,w) - K_1(t,s,v,w)| \\ \vdots \\ |K_m(t,s,u,w) - K_m(t,s,v,w)| \end{pmatrix} \leq S \begin{pmatrix} |u_1 - v_1| \\ \vdots \\ |u_m - v_m| \end{pmatrix},$$

for all $t, s, w \in [0;1]$, $u, v \in \mathbb{R}^m$;

(ii) $0 \leq H(t,s,u,v,w) \leq 1$, for all $t, s, v, w \in [0;1]$, $u \in \mathbb{R}^m$;

(iii) $|H(t_1,s,u,v,w) - H(t_2,s,u,v,w)| \leq L|t_1 - t_2|$, for all $t_1, t_2, s, v, w \in [0;1]$, $u \in \mathbb{R}^m$.

Then the system (3.1) has at least a solution in $X \times Y$.

Proof: For $(x,y) \in X \times Y$ we consider the operators

$$f_1(x,y)(t) = \int_0^1 K(t,s,x(s),y(s))\,ds + g(t), \ t \in [0;1],$$

$$f_2(x,y)(t) = \int_0^1 H(t,s,x(s),y(s),y(y(s)))\,ds, \quad t \in [0;1]$$

and $f = (f_1, f_2)$. From the continuity of data, from conditions (ii) and (iii) it follows that $f : X \times Y \to X \times Y$ and f is continuous.

From condition (i) and the remark that $Y = C_L([0;1],[0;1])$ is a convex compact subset of the Banach space $(C[0;1], |\cdot|_\infty)$ we are in the conditions of the Theorem 2.2. □

As an application of the Theorem 2.1 we present the following example:

Example 3.2. *Let us consider the following system of functional-integral equations*

$$\begin{cases} x(t) &= \int_\Omega K(t,s,x(s),y(s))\,ds + g(t), & t \in \overline{\Omega} \\ y(t) &= \int_\Omega H(t,s,x(s),y(s),y(\gamma(s)))\,ds + h(t), & t \in \overline{\Omega} \end{cases} \tag{3.2}$$

where $\Omega \subset \mathbb{R}^q$ *is an open bounded domain,* $K \in C\left(\overline{\Omega} \times \overline{\Omega} \times \mathbb{R}^m \times \mathbb{R}^p, \mathbb{R}^m\right)$, $g \in C\left(\overline{\Omega}, \mathbb{R}^m\right)$, $H \in C(\overline{\Omega} \times \overline{\Omega} \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^p, \mathbb{R}^p)$, $h \in C\left(\overline{\Omega}, \mathbb{R}^p\right)$, $\gamma \in C\left(\overline{\Omega}, \overline{\Omega}\right)$.

Let $X_1 = X_2 = \ldots = X_m := C\left(\overline{\Omega}\right)$ *with the Cebyshev norm* $|\cdot|_\infty$ *and* $X := \prod_{i=1}^{m} X_i = C\left(\overline{\Omega}, \mathbb{R}^m\right)$ *with the vector-valued norm*

$$\|x\|_X = \begin{pmatrix} |x_1|_\infty \\ \vdots \\ |x_m|_\infty \end{pmatrix}.$$

Let $Y_1 = Y_2 = \ldots = Y_p := C\left(\overline{\Omega}\right)$ *with the Cebyshev norm* $|\cdot|_\infty$ *and* $Y := \prod_{i=1}^{m} Y_i = C\left(\overline{\Omega}, \mathbb{R}^p\right)$ *with the vector-valued norm*

$$\|y\|_Y = \begin{pmatrix} |y_1|_\infty \\ \vdots \\ |y_p|_\infty \end{pmatrix}.$$

From the Theorem 2.1 we have:

Theorem 3.2. *We suppose that:*

(i) there exists a matrix $S \in \mathbb{R}_+^{m \times m}$ *with mes* $\left(\overline{\Omega}\right) \cdot S$ *convergent to zero such that*

$$\begin{pmatrix} |K_1(t,s,u,w) - K_1(t,s,v,w)| \\ \vdots \\ |K_m(t,s,u,w) - K_m(t,s,v,w)| \end{pmatrix} \leq S \begin{pmatrix} |u_1 - v_1| \\ \vdots \\ |u_m - v_m| \end{pmatrix},$$

for all $t, s \in \overline{\Omega}$, $u, v \in \mathbb{R}^m$, $w \in \mathbb{R}^p$;

(ii) there exists a matrix $T \in \mathbb{R}_+^{p \times p}$ *with mes* $\left(\overline{\Omega}\right) \cdot T$ *convergent to zero and* $M \in \mathbb{R}_+^p$ *such that*

$$\begin{pmatrix} |H_1(t,s,u,v,w)| \\ \vdots \\ |H_p(t,s,u,v,w)| \end{pmatrix} \leq T \begin{pmatrix} \max\{|v_1|, |w_1|\} \\ \vdots \\ \max\{|v_p|, |w_p|\} \end{pmatrix} + M,$$

for all $t, s \in \overline{\Omega}$, $u \in \mathbb{R}^m$, $v, w \in \mathbb{R}^p$.

Then the system (3.2) has at least a solution in $X \times Y$.

Proof: For $(x, y) \in X \times Y$ we consider the operators

$$f_1(x,y)(t) = \int_\Omega K(t,s,x(s),y(s))\,ds + g(t),\ t \in \overline{\Omega},$$

$$f_2(x,y)(t) = \int_\Omega H(t,s,x(s),y(s),y(\gamma(s)))\,ds + h(t),\ t \in \overline{\Omega}$$

and $f = (f_1, f_2)$. From the continuity of data we have that $f : X \times Y \to X \times Y$ and f is continuous.

From condition (i) we get

$$\|f_1(x,y) - f_1(\bar{x},y)\|_X \le mes\left(\overline{\Omega}\right) \cdot S \cdot \|x - \bar{x}\|_X ,$$

for all $x, \bar{x} \in X$, $y \in Y$, so $f_1(\cdot, y)$ satisfies the Perov theorem for any $y \in Y$. We denote by $x^*(y)$ the unique fixed point of $f_1(\cdot, y)$ and the application $x^* : Y \to X$, $y \mapsto x^*(y)$ is continuous (see the proof of Theorem 2.1). We define the operator $h : Y \to Y$ by

$$h(y) = f_2(x^*(y), y) .$$

Let $R \in \mathbb{R}^p_+$, we denote by

$$D_R := \{y \in Y \mid \|y\|_Y \le R\} .$$

Condition (ii) implies that there exists an $R^0 \in \mathbb{R}^p_+$ such that $f_2(X, D_R) \subset D_R$ for all $R \ge R^0$. Indeed, for $x \in X$, $y \in D_R$ we have

$$\|f_2(x,y)\|_Y \le mes\left(\overline{\Omega}\right) \cdot T \cdot R + mes\left(\overline{\Omega}\right) \cdot M + \|h\|_Y ,$$

so, to have an R such that

$$mes\left(\overline{\Omega}\right) TR + mes\left(\overline{\Omega}\right) M + \|h\|_Y \le R \Leftrightarrow \left(I_p - mes\left(\overline{\Omega}\right) T\right)^{-1} \cdot \left(mes\left(\overline{\Omega}\right) M + \|h\|_Y\right) \le R.$$

thus, we can take $R^0 := \left(I_p - mes\left(\overline{\Omega}\right) \cdot T\right)^{-1} \cdot \left(mes\left(\overline{\Omega}\right) \cdot M + \|h\|_Y\right)$. Since $f_2(X, D_R) \subset D_R$ for all $R \ge R^0$ then $h(D_R) \subset D_R$ for all $R \ge R^0$. We remark that $\overline{co\, h(D_R)}$ is a compact convex subset and the subset $X \times \overline{co\, h(D_R)}$ is invariant for the operator f. Since the subset $\overline{co\, h(D_R)}$ is a topological space with the fixed point property then all the condition of Theorem 2.1 are satisfied and we get the conclusion. □

Acknowledgement. This work was supported by a grant of the Romanian National Authority for Scientific Research, CNCS – UEFISCDI, project number PN-II-ID-PCE-2011-3-0094.

References

[1] G. Allaire, S.M. Kaber, *Numerical Linear Algebra*, Springer, 2008.

[2] C. Avramescu, Asupra unei teoreme de punct fix, *Studii şi Cercetări Mat.*, **22**(1970), 215-221.

[3] M. Dobriţoiu, System of integral equations with modified argument, *Carpathian J. Math.*, **24**(2008), No. 2, 26-36.

[4] M. Dobriţoiu, M.A. Şerban, A system of integral equations arising from infectious diseases, via Picard operators, *Carpathian J. Math.*, **26**(2010), No. 2, 170-183.

[5] A.I. Perov, On the Cauchy problem for a system of ordinary differential equations, *Priblijen. Metod Res. Dif. Urav.*, Kiev, 1964, (in Russian).

[6] R. Precup, The role of convergent to zero matrices in the study of semiliniar operator systems, *Math. Comput. Modelling*, **49**(2009), No. 3-4, 703–708.

[7] R. Precup, Componentwise compression-expansion conditions for systems of nonlinear operator equations and applications. Mathematical models in engineering, biology and medicine, 284–293, AIP Conf. Proc., 1124, Amer. Inst. Phys., Melville, NY, 2009.

[8] F. Robert, *Matrices nonnégatives et normes vectorielles*, Univ. Sc. et Médicale de Grenoble, 1973/1974.

[9] I.A. Rus, *Generalized Contractions and Applications*, Cluj University Press, Cluj-Napoca, 2001.

[10] I.A. Rus, *Fixed Point Structure Theory*, Cluj University Press, Cluj-Napoca, 2006.

[11] I.A. Rus, *Principles and Applications of the Fixed Point Theory*, (Romanian), Dacia, Cluj-Napoca, 1979.

[12] I.A. Rus, On the fixed points of the mappings defined on a cartesian product (III), (in Romanian), *Studia Univ. Babeş-Bolyai Math.*, **24**(1979), 55-56.

[13] I.A. Rus, A. Petruşel, G. Petruşel, *Fixed Point Theory*, Cluj University Press, Cluj-Napoca, 2008.

[14] M.A. Şerban, Fixed point theorems on cartesian product, *Fixed Point Theory*, **9**(2008), No. 1, 331-350.

[15] M.A. Şerban, *Fixed Point Theory for Operators on Cartesian Product*, (in Romanian), Cluj University Press, Cluj-Napoca, 2002.

[16] P.P. Zabrejko, K-metric and K-normed linear spaces: survey, *Collect. Math.*, **48**(1997), 825-859.

Received: 28.01.2013,

Accepted: 05.08.2013.

Department of Mathematics,
Babeş-Bolyai University of Cluj-Napoca
E-mail: {iarus, mserban}@math.ubbcluj.ro