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Nonlinear Neumann problems driven by a nonhomogeneous differential operator

by

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Abstract

We study a nonlinear parametric Neumann problem driven by a nonhomogeneous quasilinear elliptic differential operator $\operatorname{div}(a(x, \nabla u))$, a special case of which is the p-Laplacian. The reaction term is a nonlinearity function f which exhibits $(p-1)$-subcritical growth. By using variational methods, we prove a multiplicity result on the existence of weak solutions for such problems. An explicit example of an application is also presented.

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1 Introduction

In this paper we study the existence of multiple solutions for the following Neumann problem,

$$\begin{cases} -\operatorname{div}(a(x, \nabla u)) + |u|^{p-2}u = (\lambda k(x) + \mu) f(u) & \text{in } \Omega \\ \dfrac{\partial u}{\partial n_a} = 0 & \text{on } \partial\Omega. \end{cases} \tag{$N_{\lambda,\mu}$}$$

Here and in the sequel, Ω is a bounded, connected domain in $(\mathbb{R}^N, |\cdot|)$ with smooth boundary $\partial\Omega$, $p > 1$, $a : \bar{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ is a suitable Carathéodory map which is strictly monotone in the $\xi \in \mathbb{R}^N$ variable and $\partial u/\partial n_a := a(x, \nabla u) \cdot n$, where n is the outward unit normal vector on $\partial\Omega$. Further, λ and μ are positive real parameters, $k \in L^\infty(\Omega)_+$ and finally, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function which is $(p-1)$-sublinear at infinity. We cite a recent monograph by Kristály, Rădulescu and Varga [12] as a general reference on variational methods.

Recently, problems involving p-Laplacian-like operators have been studied by several authors under different boundary conditions and by using different technical approaches.

For instance, Dirichlet problems involving a general operator in divergence form were studied by De Nápoli and Mariani in [5] by imposing symmetry condition on the map $\xi \mapsto a(a, \xi)$. In the cited paper the existence of one weak solution was proved by exploiting the standard mountain pass geometry and requiring, among other assumptions, that the nonlinearity f has a $(p-1)$-superlinear behaviour at infinity. The non-uniform case was successively considered by Duc and Vu in [6] who extended the result of [5] under the key hypothesis that the map a fulfills a suitable growth condition.

In [11], by using variational methods, Kristály, Lisei and Varga studied the analogue of the above case for a uniform Dirichlet problem with parameter, obtaining the existence of three weak solutions requiring that the nonlinearity f has a $(p-1)$-sublinear growth at infinity.

Successively, Yang, Geng and Yan [25] proved the existence of three weak solutions for singular p-Laplacian type equations. Finally, Papageorgiou, Rocha and Staicu in [20] considered a nonsmooth p-Laplacian problem in divergence form, obtaining the existence of at least two nontrivial weak solutions. See also the contributions obtained by Servadei in [22] for related multiplicity results.

The study of the corresponding Neumann problem is in some sense lagging behind. Superlinear Neumann problems were studied by Aizicovici, Papageorgiou and Staicu [1] and Gasiński-Papageorgiou [9]. In [1] the differential operator is the p-Laplacian and the superlinear reaction term satisfies the celebrated (AR)-condition. In [9] the differential operator is nonhomogeneous incorporating the p-Laplacian, but for the superlinear case the authors prove only an existence theorem and do not have multiplicity results. Related to this paper are also the nice works [10, 19] and references therein.

Our goal in this paper is to prove a multiplicity result for Neumann problem $(N_{\lambda,\mu})$ by using a critical point result due to Ricceri (see Theorem 2.1). More precisely, for a suitable $\mu = \mu_0$ and λ sufficiently small, the existence of multiple solutions for problem (N_{λ,μ_0}) will be obtained requiring that the nonlinearity f has a $(p-1)$-linear growth in addition to a suitable oscillating behaviour of the associated potential (see condition $(\mathrm{h}_m^{\mu_0})$). We also emphasize that our hypotheses on a, following the approach given in [9], are considerably weaker than the corresponding ones in [5, 11], where $a(x,\xi) =: \nabla_\xi A(x,\xi)$, with $A \in C(\bar{\Omega} \times \mathbb{R}^N)$ and for every $x \in \bar{\Omega}$, $A(x,\cdot) \in C^1(\mathbb{R}^N)$. Moreover, they assume that for every $x \in \bar{\Omega}$, the function $\xi \mapsto A(x,\xi)$ is a strongly convex function.

This requirement, in the special case of the p-Laplacian operator $\mathrm{div}(|\nabla u|^{p-2}\nabla u)$ implies that $p \geq 2$. In contrast, in our approach we only have that for every $x \in \bar{\Omega}$, the map $\xi \mapsto A(x,\xi)$ is strictly convex. So, for p-Laplacian equations we allow any $p > 1$.

The plan of the paper is as follows. Section 2 is devoted to our abstract framework, while Section 3 is dedicated to the main results. A concrete example of an application is then presented (see Example 3.8).

2 Abstract framework

Let $W^{1,p}(\Omega)$ $(p > 1)$ be the usual Sobolev space, equipped with the norm

$$\|u\| := \Big(\int_\Omega (|\nabla u(x)|^p + |u(x)|^p)dx \Big)^{1/p}.$$

Further, let (for semplicity of notation) $W^{-1,p}(\Omega)$ be its topological dual and denote the duality brackets for the pair $(W^{-1,p}(\Omega), W^{1,p}(\Omega))$ by $\langle \cdot, \cdot \rangle$. Indicate by p^* the critical exponent of the

Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$.

Recall that if $p < N$ then $p^* = Np/(N-p)$ and for every $q \in [1, p^*]$ there exists a positive constant c_q such that

$$\|u\|_{L^q(\Omega)} \leq c_q \|u\|\,, \tag{1}$$

for every $u \in W^{1,p}(\Omega)$. Moreover, when $p \geq N$, this inequality holds for any $q \in [1, +\infty[$, since $p^* = +\infty$.

Our main tool will be the following abstract critical point theorem due to Ricceri [21].

Theorem 2.1. *Let H be a separable and reflexive real Banach space and let $\mathcal{N}, \mathcal{G} : H \to \mathbb{R}$ be sequentially weakly lower semicontinuous and continuously Gâteaux differentiable functionals, with $\mathcal{N}$ coercive.*

Assume that the functional $J_\lambda := \mathcal{N} + \lambda \mathcal{G}$ satisfies the Palais-Smale condition for every $\lambda > 0$ small enough and that the set of all global minima of $\mathcal{N}$ has at least m connected components in the weak topology, with $m \geq 2$.

Then for every $\eta > \inf\limits_{u \in H} \mathcal{N}(u)$, there exists $\bar{\lambda} > 0$ such that for every $\lambda \in (0, \bar{\lambda})$, the functional J_λ has at least $m+1$ critical points, m of which are lying in the set $\mathcal{N}^{-1}((-\infty, \eta))$.

For the sake of completeness, we also recall that a C^1-functional $J : X \to \mathbb{R}$, where X is a real Banach space with topological dual X^*, satisfies the Palais-Smale condition at level $\alpha \in \mathbb{R}$, (briefly $(\mathrm{PS})_\alpha$) when

$(\mathrm{PS})_\alpha$ *Every sequence $\{u_n\}$ in X such that*

$$J(u_n) \to \alpha, \qquad \text{and} \qquad \|J'(u_n)\|_{X^*} \to 0,$$

possesses a convergent subsequence.

Finally, we say that J satisfies the Palais-Smale condition (in short (PS)) if $(\mathrm{PS})_\alpha$ holds for every $\alpha \in \mathbb{R}$.

3 The main result

In the sequel, let $\Omega \subset \mathbb{R}^N$ be a bounded and connected Euclidean domain. Assume that there exists a function $A : \bar{\Omega} \times \mathbb{R}^N \to \mathbb{R}$, with gradient $a(x,\xi) := \nabla_\xi A(x,\xi) : \bar{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$, such that the following conditions hold:

(α_1) *For all $\xi \in \mathbb{R}^N$, the function $x \mapsto A(x,\xi)$ is measurable*;

(α_2) *For almost all $x \in \bar{\Omega}$, the function $\xi \mapsto A(x,\xi)$* is C^1, *strictly convex, and $A(x,0) = 0$*;

(α_3) *For almost all $x \in \bar{\Omega}$* and all $\xi \in \mathbb{R}^N$, *we assume*

$$|a(x,\xi)| \leq a_0(x) + c_0 |\xi|^{p-1},$$

with $a_0 \in L^\infty(\Omega)_+$, $c_0 > 0$ and $p > 1$;

(α_4) *For almost all* $x \in \bar{\Omega}$ and all $\xi \in \mathbb{R}^N$, *we suppose*

$$a(x,\xi) \cdot \xi \leq pA(x,\xi);$$

(α_5) *There exists* $\kappa > 0$ *such that for almost all* $x \in \bar{\Omega}$ *and every* $\xi \in \mathbb{R}^N$, *we have* $\kappa|\xi|^p \leq pA(x,\xi)$.

Example 3.1. We present some examples of functions $A(x,\xi)$ which correspond to the map $a(x,\xi)$ and satisfy the above hypotheses.

- $A(x,\xi) := \dfrac{|\xi|^p}{p}$ with $p > 1$. Then

$$a(x,\xi) := \nabla_\xi A(x,\xi) = |\xi|^{p-2}\xi.$$

  In this setting, the resulting differential operator is the usual p-Laplacian

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u);$$

- $A(x,\xi) := \dfrac{a_1(x)}{p}|\xi|^p + \dfrac{a_2(x)}{p}|\xi|^r$, with $a_1, a_2 \in L^\infty(\Omega)_+$, $a_1(x) \geq c_0 > 0$ for almost every $x \in \bar{\Omega}$ and $1 < r < p$;
- $A(x,\xi) := \dfrac{a_1(x)}{p}|\xi|^p + \dfrac{1}{r}\log(1+|\xi|^r)$, with $a_1 \in L^\infty(\Omega)_+$, $a_1(x) \geq c_0 > 0$ for almost every $x \in \bar{\Omega}$ and $1 < r \leq p$;
- $A(x,\xi) := \dfrac{1}{p}((1+|\xi|^2)^{p/2})$, with $p > 1$. Thus

$$a(x,\xi) = (1+|\xi|^2)^{(p-2)/2}\xi.$$

  The resulting differential operator is the generalized mean curvature operator

$$\operatorname{div}((1+|\nabla u|^2)^{(p-2)/2}\nabla u);$$

- $A(x,\xi) := \dfrac{M(x)\xi \cdot \xi}{2}$, with $M \in L^\infty(\bar{\Omega}; \mathbb{R}^{N\times N})$ and $M(x) \geq c_0 I_N$ for almost every $x \in \bar{\Omega}$, with $c_0 > 0$ and I_N being the identity N-matrix.

Remark 3.2. The operator $a(x,\xi) := \nabla_\xi A(x,\xi)$ satisfies the (S_+) property; see [9, Proposition 3.1]. This means that for every sequence $\{u_n\} \subset W^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$ (weakly) in $W^{1,p}(\Omega)$ and

$$\limsup_{n\to\infty} \int_\Omega a(x, \nabla u_n(x)) \cdot \nabla(u_n - u)(x)dx \leq 0,$$

then $u_n \to u$ (strongly) in $W^{1,p}(\Omega)$.

From now on, let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that

(h_∞) $\displaystyle\lim_{|t|\to\infty} \frac{f(t)}{|t|^{p-1}} = 0.$

A typical case when (h_∞) holds is

($h_{q\rho}$) *There exist* $q \in (0, p-1)$ *and* $\rho > 0$ *such that* $|f(t)| \leq \rho|t|^q$ *for every* $t \in \mathbb{R}$.

In order to obtain our multiplicity result, in addition to condition (h_∞), we also require that:

($h_m^{\mu_0}$) *There exists* $\mu_0 \in (0, \infty)$ *such that the set of global minima of the function*

$$s \mapsto \tilde{F}_{\mu_0}(s) := \Lambda s^p - \mu_0 F(s),$$

has at least $m \geq 2$ *connected components.*

Note that ($h_m^{\mu_0}$) implies that the function $s \mapsto \tilde{F}_{\mu_0}(s)$ has at least $m-1$ local maxima.

We are interested in the existence of multiple weak solutions for the following Neumann problem

$$\begin{cases} -\operatorname{div}(a(x,\nabla u)) + |u|^{p-2}u = (\lambda k(x) + \mu_0) f(u) & \text{in } \Omega \\ \dfrac{\partial u}{\partial n_a} = 0 & \text{on } \partial\Omega. \end{cases} \tag{N_{λ,μ_0}}$$

For the sake of completeness we recall that, fixing $\lambda > 0$, a *weak solution* of problem (N_{λ,μ_0}) is a function $u \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} \int_\Omega a(x,\nabla u(x)) \cdot \nabla v(x)\, dx &= -\int_\Omega |u(x)|^{p-2} u(x) v(x)\, dx \\ &+ \lambda \int_\Omega k(x) f(u(x)) v(x)\, dx \\ &+ \mu_0 \int_\Omega f(u(x)) v(x)\, dx, \end{aligned}$$

for every $v \in W^{1,p}(\Omega)$.

Set $\Phi : W^{1,p}(\Omega) \to \mathbb{R}$ given by

$$\Phi(u) := \int_\Omega A(x,\nabla u(x))dx + \frac{1}{p}\int_\Omega |u(x)|^p dx,$$

and

$$\mathcal{N}_{\mu_0}(u) := \Phi(u) - \mu_0 \int_\Omega F(u(x))dx,$$

as well as

$$\mathcal{G}(u) := -\int_\Omega k(x) F(u(x))dx,$$

for every $u \in W^{1,p}(\Omega)$. Here, as usual, we put

$$F(s) := \int_0^s f(t)dt,$$

for every $s \in \mathbb{R}$.

With the above notations and assumptions, it is easy to prove that $\mathcal{N}_{\mu_0}$ and $\mathcal{G}$ are C^1-functionals with Gâteaux derivatives given by

$$\begin{aligned}\langle \mathcal{N}'_{\mu_0}(u), v\rangle = \int_\Omega a(x, \nabla u(x)) \cdot \nabla v(x)dx \quad &+ \quad \int_\Omega |u(x)|^{p-2}u(x)v(x)dx \\ &- \quad \mu_0 \int_\Omega f(u(x))v(x)dx,\end{aligned}$$

and

$$\langle \mathcal{G}'(u), v\rangle = -\int_\Omega k(x)f(u(x))v(x)dx,$$

for every $v \in W^{1,p}(\Omega)$.
Thus, the critical points of $J_\lambda := \mathcal{N}_{\mu_0} + \lambda\mathcal{G}$ are exactly the weak solutions of problem (P_λ).

Finally, denote

$$\Lambda := \frac{\min\{\kappa, 1\}}{p}.$$

Standard arguments ensure the validity of the following preliminary regularity result on the functionals $\mathcal{N}_{\mu_0}$ and $\mathcal{G}$.

Lemma 3.3. *Let us assume that condition* (h_∞) *holds. Then the above functionals* $\mathcal{N}_{\mu_0}$ *and* $\mathcal{G}$ *are sequentially weakly lower semicontinuous.*

Proof: Due to condition (α_2) the functional Φ is convex. Since Φ is strongly continuous it is also weakly lower semicontinuous. On the other hand, since condition (h_∞) holds, there exists a positive constant c such that $|f(t)| \leq c(1 + |t|^{p-1})$, for every $t \in \mathbb{R}$. Finally, due to the fact that the embedding $X \hookrightarrow L^p(\Omega)$ is compact, we obtain that the functionals

$$u \mapsto -\int_\Omega F(u(x))dx, \quad \text{and} \quad u \mapsto -\int_\Omega k(x)F(u(x))dx,$$

are sequentially weakly lower semicontinuous by arguing in standard way. □

Further, the C^1-functional J_λ satisfies the (PS)-condition as proved in the next result.

Lemma 3.4. *Assume that condition* (h_∞) *holds. Then the functional* J_λ *is coercive and satisfies the* (PS)*-condition for every real parameter* λ.

Proof: Let us fix $\lambda \in \mathbb{R}$ and consider

$$0 < \beta < \frac{1}{\mu_0 + |\lambda|\|k\|_\infty}.$$

By condition (h_∞), there exists δ_λ such that

$$|f(t)| \leq \frac{\beta p\Lambda}{c_p^p}|t|^{p-1},$$

for every $|t| \geq \delta_\lambda$. By integration we have

$$|F(s)| \leq \frac{\beta\Lambda}{c_p^p}|s|^p + \max_{|t|\leq\delta_\lambda}|f(t)||s|,$$

for every $s \in \mathbb{R}$.

Thus, by using the above inequality and bearing in mind relation (1), one has

$$\begin{aligned} J_\lambda(u) &\geq \Phi(u) - \mu_0\left|\int_\Omega F(u(x))dx\right| - |\lambda||\mathcal{G}(u)| \\ &\geq \Lambda(1-\beta(\mu_0+|\lambda|\|k\|_\infty))\|u\|^p \\ &- c_1(\mu_0+|\lambda|\|k\|_\infty)\max_{|t|\leq\delta_\lambda}|f(t)|\|u\|. \end{aligned}$$

Then the functional J_λ is bounded from below and, since $p > 1$, $J_\lambda(u) \to +\infty$ whenever $\|u\| \to +\infty$. Hence J_λ is coercive.

Now, fix $\alpha \in \mathbb{R}$ and let us prove that J_λ satisfies the condition $(\mathrm{PS})_\alpha$. For this goal, let $\{u_n\} \subset W^{1,p}(\Omega)$ be a Palais-Smale sequence, i.e.

$$J_\lambda(u_n) \to \alpha, \qquad \text{and} \qquad \|J'_\lambda(u_n)\|_{W^{-1,p}} \to 0.$$

Taking into account the coercivity of J_λ, the sequence $\{u_n\}$ is necessarily bounded in $W^{1,p}(\Omega)$. Since $W^{1,p}(\Omega)$ is reflexive, we may extract a subsequence that for simplicity we call again $\{u_n\}$, such that $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$.

We will prove that u_n strongly converges to $u \in W^{1,p}(\Omega)$. Exploiting the derivative $J'_\lambda(u_n)(u_n - u)$, we obtain

$$\begin{aligned} \int_\Omega a(x,\nabla u_n(x))\cdot\nabla(u_n-u)(x)dx &= \langle J'_\lambda(u_n), u_n-u\rangle \\ &- \int_\Omega |u_n(x)|^{p-2}u_n(x)(u_n-u)(x)dx \\ &+ \mu_0\int_\Omega f(u_n(x))(u_n-u)(x)dx \\ &+ \lambda\int_\Omega k(x)f(u_n(x))(u_n-u)(x)dx. \end{aligned}$$

Since $\|J'_\lambda(u_n)\|_{W^{-1,p}} \to 0$ and the sequence $\{u_n - u\}$ is bounded in $W^{1,p}(\Omega)$, taking into account that $|\langle J'_\lambda(u_n), u_n - u\rangle| \leq \|J'_\lambda(u_n)\|_{W^{-1,p}}\|u_n - u\|$, one has

$$\langle J'_\lambda(u_n), u_n - u\rangle \to 0.$$

Further, by the asymptotic condition (h_∞), there exists a real positive constant c such that $|f(t)| \leq c(1 + |t|^{p-1})$, for every $t \in \mathbb{R}$. Then

$$\begin{aligned} &\int_\Omega |f(u_n(x))||u_n(x)-u(x)|dx \\ &\leq c\left(\int_\Omega |u_n(x)-u(x)|dx + \int_\Omega |u_n(x)|^{p-1}|u_n(x)-u(x)|dx\right) \\ &\leq c((\mathrm{meas}(\Omega))^{1/p'} + \|u_n\|_{L^p}^{p-1})\|u_n-u\|_{L^p(\Omega)}. \end{aligned}$$

Now, the embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact, hence $u_n \to u$ strongly in $L^p(\Omega)$. So we obtain

$$\int_\Omega |f(u_n(x))||u_n(x) - u(x)|dx \to 0.$$

Analogously, one has

$$\int_\Omega k(x)|f(u_n(x))||u_n(x) - u(x)|dx \to 0.$$

Moreover, considering the inequality

$$\begin{aligned}\int_\Omega ||u_n(x)|^{p-2}u_n(x)(u_n(x) - u(x))|dx &= \int_\Omega |u_n(x)|^{p-1}|u_n(x) - u(x)|dx \\ &\leq \|u_n\|^{p-1}_{L^p(\Omega)}\|u_n - u\|_{L^p(\Omega)},\end{aligned}$$

and $u_n \to u$ strongly in $L^p(\Omega)$, we have

$$\int_\Omega ||u_n(x)|^{p-2}u_n(x)(u_n(x) - u(x))|dx \to 0.$$

We can conclude that

$$\limsup_{n\to\infty}\langle a(x, u_n), u_n - u\rangle \leq 0,$$

where $\langle a(x, u_n), u_n - u\rangle$ denotes

$$\int_\Omega a(x, \nabla u_n(x)) \cdot \nabla(u_n - u)(x)dx.$$

But as observed in Remark 3.2, the operator has the (S_+) property. So, in conclusion, $u_n \to u$ strongly in $W^{1,p}(\Omega)$.

Hence, J_λ is bounded from below and fulfills (PS), for every positive parameter λ. □

Remark 3.5. We observe that by the above lemma, the functional

$$J_0 = \mathcal{N}_{\mu_0}(u) := \Phi(u) - \mu_0\int_\Omega F(u(x))dx, \quad (u \in W^{1,p}(\Omega))$$

is coercive.

Proposition 3.6. *The set of all global minima of the functional $\mathcal{N}_{\mu_0}$ has at least m connected components in the weak topology on $W^{1,p}(\Omega)$.*

Proof: First, for every $u \in W^{1,p}(\Omega)$ we have

$$\begin{aligned}\mathcal{N}_{\mu_0}(u) &= \Phi(u) - \mu_0\int_\Omega F(u(x))dx \\ &\geq \Lambda\int_\Omega |\nabla u(x)|^p dx + \int_\Omega \tilde{F}_{\mu_0}(u(x))dx \\ &\geq \left(\inf_{s\in\mathbb{R}} \tilde{F}_{\mu_0}(s)\right)\text{meas}(\Omega).\end{aligned}$$

Moreover, if we consider $u(x) = u_{\tilde{s}}(x) = \tilde{s}$ for almost every $x \in \Omega$, where $\tilde{s} \in \mathbb{R}$ is a minimum point of the function $s \mapsto \tilde{F}_{\mu_0}(s)$, then we have the equality from the previous estimate (note that $\Phi(0) = 0$ by using the last part of condition (α_2)). Thus,

$$\inf_{u \in W^{1,p}(\Omega)} \mathcal{N}_{\mu_0}(u) = \left(\inf_{s \in \mathbb{R}} \tilde{F}_{\mu_0}(s) \right) \operatorname{meas}(\Omega).$$

Further, if $u \in W^{1,p}(\Omega)$ is not a constant function, we have

$$\begin{aligned} \mathcal{N}_{\mu_0}(u) &\geq \Lambda \int_\Omega |\nabla u(x)|^p dx + \int_\Omega \tilde{F}_{\mu_0}(u(x)) dx \\ &> \left(\inf_{s \in \mathbb{R}} \tilde{F}_{\mu_0}(s) \right) \operatorname{meas}(\Omega). \end{aligned}$$

Consequently, between the sets

$$\operatorname{Min}(\mathcal{N}_{\mu_0}) = \left\{ u \in W^{1,p}(\Omega) : \mathcal{N}_{\mu_0}(u) = \inf_{u \in W^{1,p}(\Omega)} \mathcal{N}_{\mu_0}(u) \right\},$$

and

$$\operatorname{Min}(\tilde{F}_{\mu_0}) = \left\{ s \in \mathbb{R} : \tilde{F}_{\mu_0}(s) = \inf_{s \in \mathbb{R}} \tilde{F}_{\mu_0}(s) \right\},$$

there is a one-to-one correspondence.

Indeed, let θ be the function which associates to every number $s \in \mathbb{R}$ the equivalence class of those functions which are almost everywhere equal to s in Ω.

Then $\theta : \operatorname{Min}(\tilde{F}_{\mu_0}) \to \operatorname{Min}(\mathcal{N}_{\mu_0})$ is actually a homeomorphism between $\operatorname{Min}(\tilde{F}_{\mu_0})$ and $\operatorname{Min}(\mathcal{N}_{\mu_0})$, where the set $\operatorname{Min}(\mathcal{N}_{\mu_0})$ is considered with the relativization of the weak topology on $W^{1,p}(\Omega)$.

On account of the hypothesis $(\mathrm{h}_m^{\mu_0})$, the set $\operatorname{Min}(\tilde{F}_{\mu_0})$ contains at least $m \geq 2$ connected components. Therefore the same is true for the set $\operatorname{Min}(\mathcal{N}_{\mu_0})$, which completes the proof. □

Our main result is as follows.

Theorem 3.7. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that conditions (h_∞) and $(\mathrm{h}_m^{\mu_0})$ hold. Then*

a) For every $\eta > 0$, there exists a number $\tilde{\lambda}_\eta > 0$ such that for every $\lambda \in (0, \tilde{\lambda}_\eta)$ problem (N_λ) has at least $m+1$ weak solutions $u_\lambda^{1,\eta}, \ldots, u_\lambda^{m+1,\eta} \in W^{1,p}(\Omega)$; and

b) If $(\mathrm{h}_{q\rho})$ holds then for each $\lambda \in (0, \tilde{\lambda}_\eta)$ there is a set $I_\lambda \subset \{1, \ldots, m+1\}$ with $\operatorname{card}(I_\lambda) = m$ such that

$$\|u_\lambda^{i,\eta}\| < t_{\eta q\rho}, \qquad (i \in I_\lambda)$$

where $t_{\eta q\rho} > 0$ is the greatest solution of the equation

$$\Lambda t^p - \rho\mu_0 \frac{\operatorname{meas}(\Omega)^{((p-1)-q)/p}}{q+1} t^{q+1} - \eta = 0, \quad (t > 0).$$

Proof: Let us choose $H = W^{1,p}(\Omega)$, and

$$\mathcal{N} := \mathcal{N}_{\mu_0} = \Phi(u) - \mu_0 \int_\Omega F(u(x))dx,$$

as well as

$$\mathcal{G}(u) := -\int_\Omega k(x)F(u(x))dx,$$

for every $u \in W^{1,p}(\Omega)$, in Theorem 2.1.

Due to Proposition 3.6, Lemmas 3.3 and 3.4 all the hypotheses of Theorem 2.1 are satisfied. Note that $\mathcal{N}(0) = 0$, so $\inf_{u\in H} \mathcal{N}(u) \leq 0$. Therefore, for every

$$\eta > 0 \geq \inf_{u\in H} \mathcal{N}(u),$$

there is a number $\tilde{\lambda}_\eta > 0$ such that for every $\lambda \in (0, \tilde{\lambda}_\eta)$ the function $\mathcal{N}_{\mu_0} + \lambda\mathcal{G}$ has at least $m+1$ critical points; let us denote them by $u_\lambda^{1,\eta}, \ldots, u_\lambda^{m+1,\eta} \in H$. Clearly, they are solutions of problem (N_λ), which proves the first claim.

We know in addition that m elements from $u_\lambda^{1,\eta}, \ldots, u_\lambda^{m+1,\eta}$ belong to the set $\mathcal{N}_{\mu_0}^{-1}((-\infty, \eta))$. Let $\tilde{u}$ be such an element, i.e.,

$$\mathcal{N}_{\mu_0}(\tilde{u}) = \Phi(\tilde{u}) - \mu_0 \int_\Omega F(\tilde{u}(x))dx < \eta.$$

Hence, one has

$$\Lambda\|\tilde{u}\|^p - \mu_0 \int_\Omega F(\tilde{u}(x))dx < \eta. \tag{2}$$

Assume that $(\mathrm{h}_{q\rho})$ holds. Then $|F(t)| \leq \dfrac{\rho}{q+1}|t|^{q+1}$ for every $t \in \mathbb{R}$.

By using the Hölder inequality, one has

$$\int_\Omega |\tilde{u}(x)|^{q+1}dx \leq \mathrm{meas}(\Omega)^{((p-1)-q)/p}\|\tilde{u}\|^{q+1}. \tag{3}$$

On account of (2) and (3) it follows that

$$\Lambda\|\tilde{u}\|^p - \rho\mu_0 \frac{\mathrm{meas}(\Omega)^{((p-1)-q)/p}}{q+1}\|\tilde{u}\|^{q+1} < \eta. \tag{4}$$

Now, observe that, since $\eta > 0$ and $q \in (0, p-1)$, it is easy to see that the following algebraic equation

$$\Lambda t^p - \rho\mu_0 \frac{\mathrm{meas}(\Omega)^{((p-1)-q)/p}}{q+1} t^{q+1} - \eta = 0, \tag{5}$$

always has a positive solution.

Finally, bearing in mind (4), the number $\|\tilde{u}\|$ is less than the greatest solution $t_{\eta q\rho} > 0$ of the equation (5). The proof is complete. $\square$

In conclusion we present a direct and easy application of Theorem 3.7 for an elliptic Neumann problem involving the Laplace operator.

Example 3.8. Let $k \in L^\infty(\Omega)_+$ and $f : \mathbb{R} \to \mathbb{R}$ be the continuous function defined by $f(t) := \min\{t_+ - \sin(\pi t_+), 2(m-1)\}$, where $m \geq 2$ is fixed and $t_+ = \max\{t, 0\}$. Consider the following Neumann problem

$$\begin{cases} -\Delta u + u = (\lambda k(x) + 1) f(u) & \text{in} \quad \Omega \\ \dfrac{\partial u}{\partial n} = 0 & \text{on}\ \partial\Omega. \end{cases} \tag{$\widetilde{N}_{\lambda,1}$}$$

Owing to Theorem 3.7, for every $\eta > 0$, there exists a number $\tilde{\lambda}_\eta > 0$ such that for every $\lambda \in (0, \tilde{\lambda}_\eta)$ problem $(\widetilde{N}_{\lambda,1})$ has at least $m+1$ weak solutions $u_\lambda^{1,\eta}, \ldots, u_\lambda^{m+1,\eta} \in W^{1,2}(\Omega)$. Indeed, clearly, (h_∞) holds, while for $\mu_0 = 1$, the assumption (h_m^1) is also fulfilled. Indeed, the function $t \mapsto \tilde{F}_1(t)$ has precisely m global minima; they are $0, 2, \ldots, 2(m-1)$. Moreover, $\min_{t\in\mathbb{R}} \tilde{F}_1(t) = 0$.

Remark 3.9. We emphasize that there are several multiplicity results for nonlinear Neumann problems driven by the p-Laplacian differential operator. We mention, among others, the works [2, 4, 8, 18]. With exception of [4] and [18], in all the cited papers, it is assumed that $p > N$ and the authors exploit the fact that, in this context, the Sobolev space $W^{1,p}(\Omega)$ is compactly embedded in $C^0(\bar{\Omega})$.

Remark 3.10. For completeness we also cite a recent interesting paper of Colasuonno, Pucci, and Varga [3] which contains some multiplicity results on elliptic problems with either Dirichlet or Robin boundary conditions and involving a general operator in divergence form. Moreover, some contributions for nonlinear problems involving a general operator not in divergence form are contained in [16, 17, 23]. Finally, our abstract methods can be also used studying fractional laplacian equations. See, for instance, the manuscript [24] and references therein for related topics.

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