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On totally disconnected generalised Sierpiński carpets

by
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Abstract

Generalised Sierpiński carpets are planar sets that generalise the well-known Sierpiński carpet and are defined by means of sequences of patterns. We study the structure of the sets at the kth iteration in the construction of the generalised carpet, for $k \geq 1$. Subsequently, we show that certain families of patterns provide total disconnectedness of the resulting generalised carpets. Moreover, analogous results hold even in a more general setting.

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1 Introduction

Sierpiński carpets [3, 8] are self-similar fractals in the plane that originate from the Sierpiński carpet [6, 7] and have been used, e.g., as models for porous materials [3, 8].

In a recent paper Cristea and Steinsky[2] presented necessary and sufficient conditions, under which planar sets that generalise the Sierpiński carpets, called *generalised Sierpiński carpets*, are connected.

In the present paper we use the construction, the definitions and notations from the mentioned paper [2]. We also refer to Hata[4] for connectedness properties of self similar fractals, and to Cristea[1] for connectedness properties of fractals, that can be viewed as a special case of the generalised Sierpiński carpets analysed here.

2 Definitions and construction

Let $x, y, q \in [0,1]$ such that $Q = [x, x+q] \times [y, y+q] \subseteq [0,1] \times [0,1]$. For any point $(z_x, z_y) \in [0,1] \times [0,1]$ we define the function $P_Q(z_x, z_y) = (qz_x + x, qz_y + y)$.

Let $m \geq 1$. $S^m_{i,j} = \{(x,y) \mid \frac{i}{m} \leq x \leq \frac{i+1}{m} \text{ and } \frac{j}{m} \leq y \leq \frac{j+1}{m}\}$ and $\mathcal{S}_m = \{S^m_{i,j} \mid 0 \leq i \leq m-1 \text{ and } 0 \leq j \leq m-1\}$. We call any nonempty $\mathcal{A} \subseteq \mathcal{S}_m$ an *m-pattern*. Let $\{\mathcal{A}_k\}_{k=1}^\infty$ be a sequence of non-empty patterns and $\{m_k\}_{k=1}^\infty$ be the corresponding *width-sequence, i.e., for all $k \geq 1$ we have $\mathcal{A}_k \subseteq \mathcal{S}_{m_k}$*. We let $\mathcal{W}_1 = \mathcal{A}_1$, and call it the *set of white squares of level* 1.

Then we define $\mathcal{B}_1 = \mathcal{S}_{m_1} \setminus \mathcal{W}_1$ as the *set of black squares of level* 1. For $k \geq 2$ we define the *set of white squares of level* k by $\mathcal{W}_k = \bigcup_{W \in \mathcal{A}_k, W_{k-1} \in \mathcal{W}_{k-1}} \{P_{W_{k-1}}(W)\}$.

For a sequence of patterns $\{\mathcal{A}_k\}_{k=1}^{\infty}$ with width sequence $\{m_k\}_{k=1}^{\infty}$ we introduce the notation $m(k) := \prod_{l=1}^{k} m_l$. In all the considerations to follow we will assume $m_k \geq 2$, for all $k \geq 1$. We note that $\mathcal{W}_k \subset \mathcal{S}_{m(k)}$, and we define the *set of black squares of level* k, $\mathcal{B}_k = \mathcal{S}_{m(k)} \setminus \mathcal{W}_k$. For $k \geq 1$, we define $L_k = \bigcup_{W \in \mathcal{W}_k} W$. Therefore, $\{L_k\}_{k=1}^{\infty}$ is a monotonically decreasing sequence of compact sets. We write $L_\infty = \bigcap_{k=1}^{\infty} L_k$, for the *limit set of the pattern sequence* $\{\mathcal{A}_k\}_{k=1}^{\infty}$. We call any such L_∞ a *generalised Sierpiński carpet* and L_n the n-th *approximation* of L_∞.

For any $0 \leq i \leq m(k) - 1$ we call $\cup_{j=0}^{m(k)-1}\{S_{i,j}^{m(k)}\}$ a *column of level* k. Moreover, we call $\cup_{j=0}^{m(k)-1}\{S_{0,j}^{m(k)}\}$ the *left column of level* k (in short the *left column of* $\mathcal{S}_{m(k)}$), and $\cup_{j=0}^{m(k)-1}\{S_{m(k)-1,j}^{m(k)}\}$ the *right column of level* k (in short the *right column of* $\mathcal{S}_{m(k)}$). Analogously, for any $0 \leq j \leq m(k) - 1$ we call $\cup_{i=0}^{m(k)-1}\{S_{i,j}^{m(k)}\}$ a *row of level* k. $\cup_{i=0}^{m(k)-1}\{S_{i,0}^{m(k)}\}$ is the *bottom row of level* k (in short the *bottom row of* $\mathcal{S}_{m(k)}$) and $\cup_{i=0}^{m(k)-1}\{S_{i,m(k)-1}^{m(k)}\}$ is the *top row of level* k (in short the *top row of* $\mathcal{S}_{m(k)}$).

For $\mathcal{W} \subseteq \mathcal{S}_m$ we define, $\mathcal{G}(\mathcal{W})$ to be the graph of $\mathcal{W}$, as in the mentioned paper [2]. We call any path in $\mathcal{G}(\mathcal{B}_k)$ a *black path of level* k. If $p = \{S_i\}_{i=1}^{r}$ is a path in $\mathcal{G}(\mathcal{W}_k)$ or $\mathcal{G}(\mathcal{B}_k)$ then we call $\Gamma(p) := \cup_{i=1}^{r} S_i$ *the corridor of the path* p.

3 Special families of m-patterns

For an m-pattern $\mathcal{A}$ we denote by $\mathcal{A}^c$ the set $\mathcal{S}_m \setminus \mathcal{A}$. For any $\mathcal{A} \subseteq \mathcal{S}_m$ we define $\mathcal{G}^s(\mathcal{A}) = (V(\mathcal{G}^s(\mathcal{A})), E(\mathcal{G}^s(\mathcal{A})))$ to be the graph whose set of vertices $V(\mathcal{G}^s(\mathcal{A}))$ consists of the squares $S_{i,j}^m$ that are elements of $\mathcal{A}$ and whose set of edges consists of unordered pairs of distinct squares that are elements of $\mathcal{A}$ and have a common side. Now, we introduce several particular types of patterns.

An m-pattern $\mathcal{A}$ is *of type* $\mathcal{V}$ *("vertically cutting")*, if $\mathcal{G}(\mathcal{A}^c)$ contains a connected component $\mathcal{G}(K)$ that corresponds to a subset K of $\mathcal{A}^c$, connects the top and the bottom row of $\mathcal{S}_m$, and has the property that there exist indices i_1, i_2 such that $i_1 \in \{i, S_{i,m-1}^m \in K\}$, $i_2 \in \{i, S_{i,0}^m \in K\}$ and $i_2 \in \{i_1 - 1, i_1, i_1 + 1\}$. We also denote by $\mathcal{V}$ the family of all patterns of type $\mathcal{V}$.

An m-pattern $\mathcal{A}$ is *of type* $\mathcal{H}$ *("horizontally cutting")*, if $\mathcal{G}(\mathcal{A}^c)$ contains a connected component $\mathcal{G}(K)$ that corresponds to a subset K of $\mathcal{A}^c$, connects the left and the right column of $\mathcal{S}_m$, and has the property that there exist indices j_1, j_2 such that $j_1 \in \{j \in S_{0,j}^m \in K\}$, $j_2 \in \{j, S_{m-1,j}^m \in K\}$ and $j_2 \in \{j_1 - 1, j_1, j_1 + 1\}$. We also denote by $\mathcal{H}$ the family of all patterns of type $\mathcal{H}$.

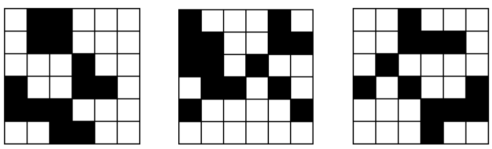

Figure 1: Patterns of type $\mathcal{V}$, $\mathcal{H}$, and both $\mathcal{V}$ and $\mathcal{H}$, respectively.

An m-pattern $\mathcal{A}$ is *of type* $\mathcal{D}_1$ *("diagonally cutting parallel to the first diagonal")* in the following two cases:

1. $\mathcal{G}(\mathcal{A}^c)$ contains a connected component $\mathcal{G}(K)$ corresponding to a subset K of $\mathcal{A}^c$, such that $\{S^m_{0,0},\ S^m_{m-1,m-1}\} \subseteq K$,

2. $\mathcal{G}(\mathcal{A}^c)$ contains a connected component $\mathcal{G}(K_1)$ that corresponds to a subset K_1 of $\mathcal{A}^c$ and connects the left column and the top row of $\mathcal{S}_m$, and a connected component $\mathcal{G}(K_2)$ that corresponds to a subset K_2 of $\mathcal{A}^c$ and connects the bottom row and the right column of $\mathcal{S}_m$, such that, on the one hand, there exist indices j_1, j_2 such that $j_1 \in \{j, S^m_{0,j} \in K_1\}$, $j_2 \in \{j, S^m_{m-1,j} \in K_2\}$ and $j_2 \in \{j_1 - 1, j_1, j_1 + 1\}$, and, on the other hand, there exist indices i_1, i_2 such that $i_1 \in \{i, S^m_{i,m-1} \in K_1\}$, $i_2 \in \{i, S^m_{i,0} \in K_2\}$, and $i_2 \in \{i_1 - 1, i_1, i_1 + 1\}$.

An m-pattern $\mathcal{A}$ is *of type* $\mathcal{D}_2$ *("diagonally cutting parallel to the second diagonal")* in the following two cases:

1. $\mathcal{G}(\mathcal{A}^c)$ contains a connected component $\mathcal{G}(K)$ corresponding to a subset K of $\mathcal{A}^c$, such that $\{S^m_{0,m-1},\ S^m_{m-1,0}\} \subseteq K$,

2. $\mathcal{G}(\mathcal{A}^c)$ contains a connected component $\mathcal{G}(K_1)$ that corresponds to a subset K_1 of $\mathcal{A}^c$ and connects the left column and the bottom row of $\mathcal{S}_m$, and a connected component $\mathcal{G}(K_2)$ that corresponds to a subset K_2 of $\mathcal{A}^c$ and connects the top row and the right column of $\mathcal{S}_m$, such that, on the one hand, there exist indices j_1, j_2 such that $j_1 \in \{j, S^m_{0,j} \in K_1\}$, $j_2 \in \{j, S^m_{m-1,j} \in K_2\}$ and $j_2 \in \{j_1 - 1, j_1, j_1 + 1\}$, and, on the other hand, there exist indices i_1, i_2 such that $i_1 \in \{i, S^m_{i,0} \in K_1\}$, $i_2 \in \{i, S^m_{i,m-1} \in K_2\}$ and $i_2 \in \{i_1 - 1, i_1, i_1 + 1\}$.

We also denote by $\mathcal{D}_2$ the family of all patterns of type $\mathcal{D}_2$.

An m-pattern $\mathcal{A}$ is *of type* $\mathcal{C}_1$, *("corner square on the first diagonal")* if $\{S^m_{0,0}, S^m_{m-1,m-1}\} \cap \mathcal{A}^c \neq \emptyset$, and *of type* $\mathcal{C}_2$, *("corner square on the second diagonal")* if $\{S^m_{0,m-1}, S^m_{m-1,0}\} \cap \mathcal{A}^c \neq \emptyset$. We denote by $\mathcal{C}_1$ and $\mathcal{C}_2$ the family of all patterns of type $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. We also denote by $\mathcal{D}_1$ the family of all patterns of type $\mathcal{D}_1$.

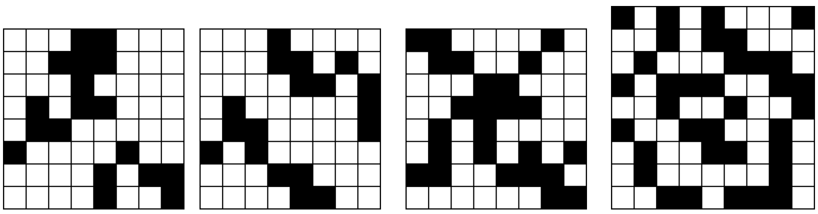

Figure 2: Patterns of type $\mathcal{D}_1$, $\mathcal{D}_2$, both $\mathcal{D}_1$ and $\mathcal{D}_2$, and of all types, respectively.

4 On the structure of generalised Sierpiński carpets given by the occurrence of special patterns

Throughout this section, we assume, when dealing with sequences of patterns $\{\mathcal{A}_k\}_{k=1}^{\infty}$, that these patterns define generalised Sierpiński carpets.

Proposition 1. *Let* $\{\mathcal{A}_k\}_{k=1}^{\infty}$ *be a sequence of patterns with width-sequence* $\{m_k\}_{k=1}^{\infty}$*. Let* $1 < k_1 < k_2, k_3$*, and* $\mathcal{A}_{k_1} \in \mathcal{V} \cup \mathcal{H}$*,* $\mathcal{A}_{k_2} \in \mathcal{C}_1$ *and* $\mathcal{A}_{k_3} \in \mathcal{C}_2$*.*

1. *If* $\mathcal{A}_{k_1} \in \mathcal{V}$*, then there exist* $m(k_1 - 1)$ *distinct paths in* $\mathcal{G}^s(\mathcal{B}_{k_3})$*, each of them connecting some square of* $\mathcal{B}_{k_3}$ *lying in the top row of* $\mathcal{S}_{m(k_3)}$ *with some square of* $\mathcal{B}_{k_3}$ *lying in the bottom row of* $\mathcal{S}_{m(k_3)}$*. Each of these paths is contained in a column of level* $k_1 - 1$*.*
2. *If* $\mathcal{A}_{k_1} \in \mathcal{H}$*, the analogous statements hold for paths in* $\mathcal{G}^s(\mathcal{B}_{k_3})$ *that connect squares that lie in the left column and in the right column of* $\mathcal{S}_{m(k_3)}$*.*

Proof: We just sketch the proof. We refer only to the case when $\mathcal{A}_{k_1} \in \mathcal{V}$, since the case $\mathcal{A}_{k_1} \in \mathcal{H}$ can be treated analogously. Let C be a column of level $k_1 - 1$. Based on the properties of the patterns of type $\mathcal{V}$, one can construct a path p in $\mathcal{G}(\mathcal{W}_{k_1})$ that has the corridor contained in C and that connects a square $S_1 \in V(\mathcal{G}(\mathcal{B}_{k_1}))$ that lies in the top row of $\mathcal{S}_{m(k_1)}$ with a square $S_1 \in V(\mathcal{G}(\mathcal{B}_{k_1}))$ that lies in the bottom row of $\mathcal{S}_{m(k_1)}$ (such that S_1 and S_2 lie in the same or in neighbouring columns of level k_1). The squares that are elements of p correspond to the vertices in a connected component $\mathcal{G}(K_{k_1})$ of $\mathcal{G}(\mathcal{B}_{k_1})$. If $\mathcal{G}^s(K_{k_1})$ is connected, let K_{k_3} be the set of all squares of level k_3 that are contained in some squares of level k_1 of $\mathcal{G}(K_{k_1})$. One can then construct a path p' in $\mathcal{G}^s(K_{k_3})$ that consists of squares of K_{k_3}, such that $\Gamma(p') \subseteq \Gamma(p)$. If $\mathcal{G}^s(K_{k_1})$ is not connected, then we denote by $B_k(p)$, for $k = k_2, k_3$, the set of all black squares of level k that lie in the same column C and have a common side with some square of level k_1 that belongs to the path p. Let now K_{k_3} be the set consisting of all black squares of level k_3 that are subsets of the squares in the path p, together with all the black squares in $B_{k_3}(p)$, and all the black squares of level k_3 that are contained in some black square of $B_{k_2}(p)$ and share a side with some black square of level k_3 which is a subset of a black square occurring in p. Then K_{k_3} contains a path p' of level k_3 in $\mathcal{G}^s(\mathcal{B}_{k_3})$, with $\Gamma(p) \subseteq \Gamma(p')$, that connects a square of level k_3 lying in the top row of $\mathcal{S}_{m(k_3)}$ with some square of level k_3 lying in the bottom row of $\mathcal{S}_{m(k_3)}$, and p' is contained in C. □

We call the paths occurring in Proposition 1 *vertical paths of level* k_3 and *horizontal paths of level* k_3 in the unit square, respectively.
Remark. Under the assumptions of case *1.* of Proposition 1, for each of the $m(k_1 - 1)$ columns of level $(k_1 - 1)$ there exists an empty corridor of level k_3 within that column. In case *2.* the analogous statement holds for each of the $m(k_1 - 1)$ rows of level $(k_1 - 1)$.
Proceeding analogously as above, one can prove the following result.

Proposition 2. *Let* $\{\mathcal{A}_k\}_{k=1}^{\infty}$ *be a sequence of patterns with width-sequence* $\{m_k\}_{k=1}^{\infty}$*. Let* $1 < k_1 < k_2, k_3$*, and* $\mathcal{A}_{k_1} \in \mathcal{D}_1 \cup \mathcal{D}_2$*,* $\mathcal{A}_{k_2} \in \mathcal{C}_1$ *and* $\mathcal{A}_{k_3} \in \mathcal{C}_2$*.*

1. *If* $\mathcal{A}_{k_1} \in \mathcal{D}_1$*, then the following statements hold. There exist* $m(k_1 - 1)$ *distinct paths in* $\mathcal{G}^s(\mathcal{B}_{k_3})$*, each of them connecting some square of* $\mathcal{B}_{k_3}$ *lying in the left column of* $\mathcal{S}_{m(k_3)}$ *with some square of* $\mathcal{B}_{k_3}$ *lying in the top row of* $\mathcal{S}_{m(k_3)}$*. There exist* $m(k_1 - 1)$ *distinct paths in* $\mathcal{G}^s(\mathcal{B}_{k_3})$*, each of them connecting some square of* $\mathcal{B}_{k_3}$ *lying in the bottom row of* $\mathcal{S}_{m(k_3)}$ *with some square of* $\mathcal{B}_{k_3}$ *lying in the right column of* $\mathcal{S}_{m(k_3)}$*.*
2. *The analogous statements hold for* $\mathcal{A}_{k_1} \in \mathcal{D}_2$ *and the corresponding paths in* $\mathcal{G}^s(\mathcal{B}_{k_3})$*.*

We call the paths occurring in Proposition 2 *diagonal paths of level k_3 and type $\mathcal{D}_1$*, or, respectively, *of type $\mathcal{D}_2$* in the unit square.

Remark. *(The translation property.)* Let $\mathcal{P}$ be the set of paths of level k_3 constructed for all columns of level $k_1 - 1$ as in (the proof of) Proposition 1. The intersection of the black squares of level k_3 belonging to a black vertical path of level k_3 in $\mathcal{P}$ with any square Q of level $k_1 - 1$ is the translated image of the intersection of the black squares of level k_3 of any black path of level k_3 in $\mathcal{P}$ with any square Q' of level $k_1 - 1$. The translation vetor is parallel to the Ox- or Oy- axis, and its length is $\frac{\alpha}{m(k_1-1)}$, $\alpha \in \mathbb{N}$.

One can show that under the assumptions of Proposition 2 there is a set $\mathcal{P}$ of diagonal paths of level k_3 (of type $\mathcal{D}_1$ or $\mathcal{D}_2$, depending on the type of the pattern $\mathcal{A}_{k_1}$) with analogous translation properties.

Proposition 3. *("Parallel" vertical curves for vertical patterns.) Under the assumptions of Proposition 1 let $\mathcal{A}_{k_1} \in \mathcal{V}$. Then there exists a set $\tilde{\mathcal{V}}(\mathcal{A}_{k_1})$ of curves that connect the top and the bottom side of the unit square with the following properties:*

1. *If $\pi \in \tilde{\mathcal{V}}(\mathcal{A}_{k_1})$ and $Q, Q' \in \mathcal{S}_{m(k_1-1)}$ lie in the same column of level $k_1 - 1$ such that $\pi \cap Q \neq \emptyset$ and $\pi \cap Q' \neq \emptyset$, then there exists a translation T by a vector of length $\frac{\alpha}{m(k_1-1)}$, $\alpha \in \mathbb{N}$, parallel to the Oy-axis, such that $\pi \cap Q' = T(\pi \cap Q)$.*
2. *If $\pi, \pi' \in \tilde{\mathcal{V}}(\mathcal{A}_{k_1})$ and $Q, Q' \in \mathcal{S}_{m(k_1-1)}$ lie in the same row of level $k_1 - 1$ such that $\pi \cap Q \neq \emptyset$ and $\pi \cap Q' \neq \emptyset$, then there exists a translation T by a vector of length $\frac{\alpha}{m(k_1-1)}$, $\alpha \in \mathbb{N}$, parallel to the Ox-axis, such that $\pi \cap Q' = T(\pi \cap Q)$.*
3. *If $\pi, \pi' \in \tilde{\mathcal{V}}(\mathcal{A}_{k_1})$, then there exists a translation T by a vector of length $\frac{\alpha}{m(k_1-1)}$, $\alpha \in \mathbb{N}$, parallel to the Ox-axis, such that $\pi' = T(\pi)$.*
4. *If $\pi \in \tilde{\mathcal{V}}(\mathcal{A}_{k_1})$, then it is contained in a column of level $k_1 - 1$.*

Proof: We give a sketch of the proof. For each path in $\mathcal{G}^s(\mathcal{B}_{k_3})$ constructed in the proof of Proposition 1, there exists a minimal path p^{min} from the top row to the bottom row of $\mathcal{S}_{m(k_3)}$, such that the $m(k_1 - 1)$ minimal paths have the properties stated in the above remark about the translation property of the paths. Let p be such a path and p^{min} the corresponding minimal sub-path. We construct a curve that lies inside the corridor p^{min} by taking the union of the line segments connecting, e.g., the midpoints of the top edge and the bottom edge in each square in p^{min}. □

The analogon of Proposition 3 holds for patterns of type $\mathcal{H}$. With a construction idea analogous to that of the curves in the proof of Proposition 3 one can prove the following result.

Proposition 4. *("Parallel diagonal curves for diagonal patterns".) Under the assumptions of Proposition 2 let $\mathcal{A}_{k_1} \in \mathcal{D}_1$. Then there exists a set $\tilde{\mathcal{D}}_1(\mathcal{A}_{k_1})$ of curves that connect the left and the top side or the bottom and the right side of the unit square with the following properties:*

1. *If $Q, Q' \in \mathcal{S}_{m(k_1-1)}$ lie in the same row (column) of level $k_1 - 1$, then there exists a translation T by a vector parallel to the Ox- (Oy)-axis, such that $\{Q' \cap \pi' \mid \pi' \in \tilde{\mathcal{D}_1}(\mathcal{A}_{k_1})\} = T\Big(\{Q \cap \pi \mid \pi \in \tilde{\mathcal{D}_1}(\mathcal{A}_{k_1})\}\Big)$.*

2. *If $\pi, \pi' \in \tilde{\mathcal{D}_1}(\mathcal{A}_{k_1})$, then there exists a translation T by a vector parallel to the Ox-axis, such that either $\pi' \subset T(\pi)$ or $T(\pi) \subset \pi'$.*

3. *If $\pi, \pi' \in \tilde{\mathcal{D}_1}(\mathcal{A}_{k_1})$, then there exists a translation T by a vector parallel to the Oy-axis, such that either $\pi' \subset T(\pi)$ or $T(\pi) \subset \pi'$.*

In each case the length of the vector defining T is $\frac{\alpha}{m(k_1-1)}$, $\alpha \in \mathbb{N}$.

5 Totally disconnected generalised Sierpiński carpets

Lemma 1. *Let L_∞ be a generalised carpet defined by a sequence of patterns $\{\mathcal{A}_k\}_{k=1}^{\infty}$ with width-sequence $\{m_k\}_{k=1}^{\infty}$. Let $1 < k_1 < k_2, k_3$, and $k_1 < k_4 < k_5, k_6$ such that $\mathcal{A}_{k_1} \in \mathcal{V}$, $\mathcal{A}_{k_2} \in \mathcal{C}_1$, $\mathcal{A}_{k_3} \in \mathcal{C}_2$, $\mathcal{A}_{k_4} \in \mathcal{H}$, $\mathcal{A}_{k_5} \in \mathcal{C}_1$ and $\mathcal{A}_{k_6} \in \mathcal{C}_2$. Then, for any two points $t = (t_1, t_2), z = (z_1, z_2)$ lying in the same connected component of L_{k_6}, $|t_1 - z_1| \leq \frac{2}{m(k_1-1)}$ and $|t_2 - z_2| \leq \frac{2}{m(k_4-1)}$.*

Proof: Let $\Omega_{k_6}(t, z)$ be the connected component in L_{k_6} that contains t and z. We give a proof by contradiction. We assume that there is a column C of level $k_1 - 1$ between t and z. As $\Omega_{k_6}(t, z)$ is a finite union of squares, it is path-connected. Thus, there is a curve c from t to z in $\Omega_{k_6}(t, z)$. Let C' denote the rectangle that is the union of all squares of level $k_1 - 1$ that belong to C. $c \cap C'$ is a curve from the left side of C' to the right side of C'.

From Proposition 3 it follows that there exists a curve $\pi \in \tilde{\mathcal{V}}(\mathcal{A}_{k_6})$ such that π is in C' and leads from the top side of C' to the bottom side of C'. We have $c \subseteq L_\infty$ and $\pi \subseteq [0,1] \times [0,1] \backslash L_\infty$, which is a contradiction to a known result, see e.g., Maehara[5, Lemma 2]. We obtain that t and z must lie within two consecutive columns of level $k_1 - 1$, and therefore $|t_1 - z_1| \leq \frac{2}{m(k_1-1)}$. Using an analogon of Proposition 3 for patterns of type $\mathcal{H}$ and the same arguments as before we infer $|t_2 - z_2| \leq \frac{2}{m(k_4-1)}$. □

The proofs of the following two lemmas are analogous to the above proof.

Lemma 2. *Let L_∞ be a generalised carpet defined by a sequence of patterns $\{\mathcal{A}_k\}_{k=1}^{\infty}$ with width-sequence $\{m_k\}_{k=1}^{\infty}$. Let $1 < k_1 < k_2, k_3$, and $k_1 < k_4 < k_5, k_6$ such that $\mathcal{A}_{k_1} \in \mathcal{D}_1 \cup \mathcal{D}_2$, $\mathcal{A}_{k_2} \in \mathcal{C}_1$ and $\mathcal{A}_{k_3} \in \mathcal{C}_2$, $\mathcal{A}_{k_4} \in \mathcal{H} \cup \mathcal{V}$,$\mathcal{A}_{k_5} \in \mathcal{C}_1$ and $\mathcal{A}_{k_6} \in \mathcal{C}_2$. Then, for any two points $t = (t_1, t_2), z = (z_1, z_2)$ lying in the same connected component of L_{k_6}, $|t_1 - z_1| \leq \frac{4}{m(k_1-1)}$ and $|t_2 - z_2| \leq \frac{2}{m(k_1-1)}$.*

Lemma 3. *Let L_∞ be a generalised carpet defined by a sequence of patterns $\{\mathcal{A}_k\}_{k=1}^{\infty}$ with width-sequence $\{m_k\}_{k=1}^{\infty}$. Let $1 < k_1 < k_2, k_3$ and $k_4 < k_5, k_6$ such that $\mathcal{A}_{k_1} \in \mathcal{D}_1$ and $\mathcal{A}_{k_4} \in \mathcal{D}_2$, $\mathcal{A}_{k_2} \in \mathcal{C}_1$ and $\mathcal{A}_{k_3} \in \mathcal{C}_2$, $\mathcal{A}_{k_5} \in \mathcal{C}_1$ and $\mathcal{A}_{k_6} \in \mathcal{C}_2$. Then, for any two points $t = (t_1, t_2), z = (z_1, z_2)$ lying in the same connected component of L_{k_6}, $|t_1 - z_1| \leq \frac{3}{m(k-1)}$ and $|t_2 - z_2| \leq \frac{3}{m(k-1)}$, where $k = \min(k_1, k_4)$.*

Theorem 1. *Let L_∞ be a generalised carpet defined by a sequence of patterns $\{\mathcal{A}_k\}_{k=1}^{\infty}$ with width-sequence $\{m_k\}_{k=1}^{\infty}$. If*

1. *there exist two distinct types of patterns, $\mathcal{T}_1, \mathcal{T}_2 \in \{\mathcal{V}, \mathcal{H}, \mathcal{D}_1, \mathcal{D}_2\}$ such that infinitely many patterns occurring in the sequence $\{\mathcal{A}_k\}_{k=1}^{\infty}$ are of type $\mathcal{T}_1$ and infinitely many patterns occurring in the sequence $\{\mathcal{A}_k\}_{k=1}^{\infty}$ are of type $\mathcal{T}_2$, and*
2. *infinitely many patterns occurring in the sequence $\{\mathcal{A}_k\}_{k=1}^{\infty}$ are of type $\mathcal{C}_1$ and infinitely many patterns occurring in the sequence $\{\mathcal{A}_k\}_{k=1}^{\infty}$ are of type $\mathcal{C}_2$,*

then L_∞ is totally disconnected with respect to the Euclidean topology.

Proof: As the first case, we assume that $\mathcal{T}_1 = \mathcal{V}$ and $\mathcal{T}_2 = \mathcal{H}$. Lemma 1 yields that any connected component of L_∞ consists of precisely one point. The second case is that $\mathcal{T}_1 \in \{\mathcal{D}_1, \mathcal{D}_2\}$ and $\mathcal{T}_2 \in \{\mathcal{H}, \mathcal{V}\}$. Here, we use Lemma 2 to obtain that any connected component of L_∞ consists of one point. In the third and final case, we have $\mathcal{T}_1 = \mathcal{D}_1$ and $\mathcal{T}_2 = \mathcal{D}_2$. By Lemma 3 we infer that any connected component of L_∞ consists of one point. □

The construction of generalised Sierpiński carpets, as it was given in Section 2, can be generalised, by allowing, at each inductive step k of the construction, not just the application of one pattern $\mathcal{A}_k \subset \mathcal{S}_{m_k}$ to all white squares that were created in the previous step, but, the application of a set of $n(k)$ distinct patterns $\{\mathcal{A}_k^i\}_{i=1}^{n(k)}$, $n(k) \geq 1$, $\mathcal{A}_k^i \subseteq \mathcal{S}_{m_k}$, with the possibility to apply distinct patterns of $\{\mathcal{A}_k^i\}_{i=1}^{n(k)}$ to distinct white squares of $\mathcal{W}_{k-1}$. In this case we call L_∞ a *non-uniform generalised Sierpiński carpet*. Thus, a non-uniform generalised Sierpiński carpet is defined by means of a sequence $\{\hat{\mathcal{A}}_k\}_{k=1}^{\infty}$, and its width sequence $\{m_k\}_{k=1}^{\infty}$, where $\hat{\mathcal{A}}_k$ is a set of $n(k)$ (with $n(k) \geq 1$) m_k-patterns, for all $k \geq 1$. Based on the above proof, one can show that Theorem 1 also holds in the case of non-uniform generalised Sierpiński carpets:

Theorem 2. *Let L_∞ be a non-uniform generalised carpet defined by a sequence of sets of patterns $\{\hat{\mathcal{A}}_k\}_{k=1}^{\infty}$ with width-sequence $\{m_k\}_{k=1}^{\infty}$. If*

1. *there exist two distinct types of patterns, $\mathcal{T}_1, \mathcal{T}_2 \in \{\mathcal{V}, \mathcal{H}, \mathcal{D}_1, \mathcal{D}_2\}$ such that infinitely many elements $\hat{\mathcal{A}}_k$ occurring in the sequence $\{\hat{\mathcal{A}}_k\}_{k=1}^{\infty}$ consist of only one pattern $\hat{\mathcal{A}}_k = \{\mathcal{A}_k\}$ and $\mathcal{A}_k \in \mathcal{T}_1$, and infinitely many elements occurring in the sequence $\{\hat{\mathcal{A}}_k\}_{k=1}^{\infty}$ consist of only one pattern $\hat{\mathcal{A}}_k = \{\mathcal{A}_k\}$ and $\mathcal{A}_k \in \mathcal{T}_2$, and*
2. *infinitely many elements of the sequence $\{\hat{\mathcal{A}}_k\}_{k=1}^{\infty}$ satisfy $\hat{\mathcal{A}}_k = \{\mathcal{A}_k^i\}_{i=1}^{n(k)}$, where $n(k) \geq 1$, $\mathcal{A}_k^i \in \mathcal{C}_1$, and $i = 1, \ldots, n(k)$, and infinitely many elements of the sequence $\{\hat{\mathcal{A}}_k\}_{k=1}^{\infty}$ satisfy $\hat{\mathcal{A}}_k = \{\mathcal{A}_k^i\}_{i=1}^{n(k)}$, where $n(k) \geq 1$, $\mathcal{A}_k^i \in \mathcal{C}_2$, and $i = 1, \ldots, n(k)$,*

then L_∞ is totally disconnected with respect to the Euclidean topology.

The results obtained here provide a method for constructing both self-similar and non-self-similar generalised carpets that are totally disconnected. Moreover, the construction of the generalised Sierpiński carpets described above makes it possible to obtain totally disconnected carpets of box-counting dimension less than or equal to 2.

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