Bull. Math. Soc. Sci. Math. Roumanie
Tome 57(105) No. 1, 2014, 59–71

On the approximation of the Thue-Morse generating sequence

by
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Abstract

Let $T(x) = 1+x+x^3+x^6+\dots$ be the generating function of the Thue-Morse sequence. We show that for any coprime nonzero integers $a \in \mathbb{Z}$ and $b \in \mathbb{N}$ satisfying $b > a^2$ the irrationality exponent of $T(a/b)$ does not exceed $(2\log b - 2\log|a|)/(\log b - 2\log|a|)$. We also prove that infinitely many partial quotients of the number $T(\pm 1/b)$, where $b \geqslant 2$ is an integer, lie in the set $\{c-1, c\}$ for some integer $c = c(\pm 1, b) \geqslant 2$. For instance, the continued fraction of $T(-1/3)$ has infinitely many partial quotients smaller than or equal to 3. In passing, we obtain the following Lagrange type result: if for an irrational number α whose continued fraction expansion has only finitely many partial quotients smaller than or equal to $t-1$, where $t \geqslant 2$ is an integer, and some coprime integers p, q, where q is large enough, we have $|\alpha - p/q| < (t-1)/tq^2$ then p/q is a convergent to α.

Key Words: Thue-Morse sequence, irrationality measure, continued fraction, Lagrange's theorem.
2010 Mathematics Subject Classification: Primary 11J04;
Secondary 11J70, 11J82.

1 Introduction

Let ξ be a real irrational number. Recall that the *irrationality exponent* (or *irrationality measure*) of ξ is the supremum $\mu(\xi)$ of real numbers μ such that the inequality $|\xi - p/q| < q^{-\mu}$ has infinitely many solutions in rational numbers p/q, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. For almost every real number ξ its irrationality exponent is equal to 2, but for almost every 'concrete' irrational number ξ this is very difficult to prove and, usually, only upper bounds on the irrationality exponent $\mu(\xi)$ are known. A very helpful tool in approximation of a number by rational fractions is its continued fraction expansion, but once again there are not so many numbers ξ for which this expansion is known or at least one can say something nontrivial about it. Recall that the mth convergent to the continued fraction expansion of

$$\xi = a_0 + \frac{1}{a_1 + \frac{1}{a_2+\dots}} = [a_0, a_1, a_2, \dots],$$

where $a_0 \in \mathbb{Z}$ and $a_1, a_2, \dots \in \mathbb{N}$, is the rational number $p_m/q_m = [a_0, a_1, \dots, a_m]$, where $p_m \in \mathbb{Z}$, $q_m \in \mathbb{N}$ and $\gcd(p_m, q_m) = 1$. The numbers $a_1, a_2, a_3, \dots$ are called partial quotients of this continued fraction.

In this paper we shall investigate some diophantine properties of the Thue-Morse generating function at rational points. More precisely, let $\mathbf{t} = (t_n)_{n=0}^{\infty}$ be the Thue-Morse sequence

$$0110100110010110100101100110 1001\dots,$$

where $t_n = 0$ if the sum of binary digits of n is even and $t_n = 1$ otherwise. Let also

$$T(x) := \sum_{i=0}^{\infty} t_i x^{i-1} = 1 + x + x^3 + x^6 + x^7 + x^{10} + \dots$$

be the infinite series associated with $\mathbf{t}$. We shall only consider the function $T(x)$ for $x \in \mathbb{R}$ satisfying $0 < |x| < 1$. The sequence $\mathbf{t}$ is a famous one and appears in many unrelated subjects (see, e. g., [4] and [5]). Apparently, Mahler was the first who in [14] investigated the arithmetic properties of values of functions satisfying some functional equations at algebraic points. In particular, $T(x)$ is such a function, since

$$P(x) := \prod_{j=0}^{\infty} (1 - x^{2^j}) = \frac{1}{1-x} - 2xT(x) \tag{1.1}$$

and $P(x) = (1-x)P(x^2)$. The diophantine properties of the numbers $T(1/b)$, where $b \geqslant 2$ is an integer, have been recently investigated by Adamczewski, Bugeaud, Cassaigne and Rivoal in [1], [2], [6].

In this direction we prove that

Theorem 1. *For any nonzero integers $a \in \mathbb{Z}$ and $b \in \mathbb{N}$ satisfying $\gcd(a, b) = 1$ and $b > a^2$ the irrationality exponent of $T(a/b)$ does not exceed*

$$(2 \log b - 2 \log |a|)/(\log b - 2 \log |a|).$$

For $a = 1$ one obtains $\mu(T(1/b)) = 2$ for each $b \geqslant 2$ which is the result of Bugeaud [6] (note that we also have $\mu(T(-1/b)) = 2$). Theorem 1 follows from the same construction as in [6] which in turn is based on a result of Allouche, Peyrière, Wen and Wen [3]. For $a = \pm 1$ one can get some nontrivial information about the continued fraction expansion of the number $T(a/b)$:

Theorem 2. *Let $a = \pm 1$ and let $b \geqslant 2$ be an integer. If $(a, b) \neq (-1, 2), (-1, 3)$ then for each sufficiently large n the number $b^{2^n - 2}(b - a)(b^{2^{n-1}} + 1)$ is a denominator of some convergent to $T(a/b)$ and the continued fraction expansion of $T(a/b)$ has infinitely many partial quotients in the set $\{c - 1, c\}$, where $c = c(a, b) := \lfloor b/P(b^{-2})(1 - a/b)^3) \rfloor$ and $P(x)$ is defined in (1.1). Furthermore, the continued fraction of $T(-1/3)$ has infinitely many partial quotients smaller than or equal to 3.*

Here is a table of values for the number

$$C(a, b) := \frac{b}{P(b^{-2})(1 - a/b)^3}$$

when $a = 1$ and $2 \leqslant b \leqslant 7$ with two correct digits

a	1	1	1	1	1	1
b	2	3	4	5	6	7
$C(a,b)$	22.84	11.53	10.15	10.18	10.67	11.35
$c(a,b) = \lfloor C(a,b) \rfloor$	22	11	10	10	10	11

and that for $a = -1$ and $3 \leqslant b \leqslant 7$ (with two correct digits):

a	-1	-1	-1	-1	-1
b	3	4	5	6	7
$C(a,b)$	1.44	2.19	3.01	3.88	4.78
$c(a,b) = \lfloor C(a,b) \rfloor$	1	2	3	3	4

Note that $C(\pm 1, b) \sim b$ as $b \to \infty$. For instance, we have $C(1, 100) = 103.07\ldots$ and $C(-1, 100) = 97.06\ldots$.

The first part of Theorem 2 for $a = 1$ proves that the fraction $p_n(b)/q_n(b)$ considered in Theorem 5.1 of [2] is in its reduced form which was left open in [2]. Of course, this also implies Conjecture 5.2 of [2], although a better (in fact, best possible) result is already given in [6]. In the proof of Theorem 2 we shall use arguments similar to those used in [7], where the continued fraction of the constant $T(1/2)/4$ was considered.

To cover the case $a/b = -1/3$ of Theorem 2 we shall use the following result:

Theorem 3. *Suppose α is an irrational number whose continued fraction expansion has only finitely many partial quotients smaller than or equal to $t - 1$, where $t \geqslant 2$ is an integer. If for some coprime integers p, q, where $q > 0$ is large enough, we have*

$$\left| \alpha - \frac{p}{q} \right| < \frac{t-1}{tq^2} \tag{1.2}$$

then p/q is a convergent to α.

Recall that the classical Lagrange's theorem asserts that if $|\alpha - p/q| < 1/2q^2$ then p/q is a convergent to α, so Theorem 3 weakens its assumption for each $t \geqslant 3$.

We will also show that the constant $(t-1)/t$ in Theorem 3 is best possible, namely, there exists an irrational number α whose continued fraction expansion has partial quotients greater than or equal to t such that the inequality (1.2) holds for infinitely many rational fractions p/q which are not convergents to α. (More precisely, we will take $(p, q) = (p_m + p_{m-1}, q_m + q_{m-1})$, where p_m/q_m is the mth convergent to α.) Because of this, the pair $(a, b) = (-1, 2)$ is an exception in Theorem 2. Indeed, since $C(-1, 2) < 1$, by the formula (6.2) below, for $T(-1/2)$ and the rational fractions $p/q = p_n(-1, 2)/q_n(-1, 2)$ we only have $|T(-1/2) - p/q| < c_0/q^2$ with some $c_0 > 1$. Thus, using the methods of this paper, we cannot say anything about the partial quotients of $T(-1/2)$, since Theorem 3 cannot be applied to $\alpha = T(-1/2)$ and some $t \in \mathbb{N}$.

In the next section we give the proof of Theorem 3. In Section 3 we give some identities and analytical estimates. Section 4 is devoted to some arithmetical results. Finally, in Sections 5 and 6 we complete the proofs of Theorems 1 and 2, respectively.

2 Continued fractions

Below, we shall use the following standard lemma (see, e. g., [15]):

Lemma 1. *Let* $\alpha = [a_0, a_1, a_2, \dots]$ *be an irrational number with convergents* $(p_m/q_m)_{m=0}^{\infty}$. *Then for each* $m \geqslant 0$ *we have*

$$q_m\alpha - p_m = \frac{(-1)^m}{q_m\alpha_{m+1} + q_{m-1}} = \frac{(-1)^m}{([a_{m+1}, a_{m+2}, a_{m+3}, \dots] + [0, a_m, a_{m-1}, \dots, a_1])q_m},$$

where $\alpha_{m+1} := [a_{m+1}, a_{m+2}, a_{m+3}, \dots]$.

The following result was first proved by Fatou (see [11], [12], p. 16 and also [9], [10], [16] for more recent work on this problem):

Lemma 2. *Suppose* α *is an irrational number with convergents* $(p_m/q_m)_{m=0}^{\infty}$. *If for two coprime integers* $p \in \mathbb{Z}$ *and* $q \geqslant 2$ *we have*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$$

then $p = p_m + \theta p_{m-1}$ *and* $q = q_m + \theta q_{m-1}$ *for some* $m \in \mathbb{N}$ *and some* $\theta \in \{-1, 0, 1\}$.

Proof of Theorem 3: Assume that p/q is not a convergent to α, namely, $q \neq q_m$ for $m \in \mathbb{N}$ (otherwise there is nothing to prove). We need to show that in two cases $(p, q) = (p_{m+1} - p_m, q_{m+1} - q_m)$ and $(p, q) = (p_m + p_{m-1}, q_m + q_{m-1})$ with $m \in \mathbb{N}$ that are possible by Lemma 2 the stronger bound (1.2) on $|\alpha - p/q|$ of the theorem leads to a contradiction.

Indeed, as

$$p_{m+1} + (1 - a_{m+1})p_m = a_{m+1}p_m + p_{m-1} + (1 - a_{m+1})p_m = p_m + p_{m-1}$$

(and, similarly, $q_{m+1} + (1 - a_{m+1})q_m = q_m + q_{m-1}$), in order to combine these two cases into one we may assume that for some nonnegative integer m

$$p = p_{m+1} + \ell p_m, \qquad q = q_{m+1} + \ell q_m$$

with $\ell = -1$ or $\ell = 1 - a_{m+1}$, where $a_{m+1} \geqslant 2$. Then, by Lemma 1, $q_{m+1} = a_{m+1}q_m + q_{m-1}$ and $\alpha_{m+1} = a_{m+1} + 1/\alpha_{m+2}$, we find that

$$\begin{aligned} q\alpha - p = (q_{m+1} + \ell q_m)\alpha - (p_{m+1} + \ell p_m) &= \frac{(-1)^{m+1}}{q_{m+1}\alpha_{m+2} + q_m} + \frac{\ell(-1)^m}{q_m\alpha_{m+1} + q_{m-1}} \\ &= \frac{(-1)^{m+1}}{q_{m+1}\alpha_{m+2} + q_m} + \frac{\ell(-1)^m}{a_{m+1}q_m + q_{m-1} + q_m/\alpha_{m+2}} \\ &= \frac{(-1)^{m+1}(1 - \ell\alpha_{m+2})}{q_{m+1}\alpha_{m+2} + q_m}. \end{aligned}$$

Now, from $|q\alpha - p| < (t-1)/tq = (t-1)/t(q_{m+1} + \ell q_m)$ (see (1.2)) it follows that

$$1 + |\ell|\alpha_{m+2} = |1 - \ell\alpha_{m+2}| = |q\alpha - p|(q_{m+1}\alpha_{m+2} + q_m) < \frac{(t-1)(q_{m+1}\alpha_{m+2} + q_m)}{t(q_{m+1} + \ell q_m)},$$

where $\ell = -1$ or $\ell = 1 - a_{m+1}$ and $a_{m+1} \geqslant 2$. Thus

$$t(|\ell|\alpha_{m+2} + 1)(q_{m+1} - |\ell|q_m) < (t-1)(q_{m+1}\alpha_{m+2} + q_m).$$

Dividing by q_m, we find that

$$t - 1 > (t(|\ell|\alpha_{m+2} + 1) - (t-1)\alpha_{m+2})q_{m+1}/q_m - t|\ell|(|\ell|\alpha_{m+2} + 1).$$

The coefficient for q_{m+1}/q_m on the right hand side is positive, so, using the inequality $q_{m+1}/q_m > a_{m+1}$, we further obtain

$$\begin{aligned} t - 1 &> (t(|\ell|\alpha_{m+2} + 1) - (t-1)\alpha_{m+2})a_{m+1} - t|\ell|(|\ell|\alpha_{m+2} + 1) \\ &= (t|\ell|a_{m+1} - (t-1)a_{m+1} - t|\ell|^2)\alpha_{m+2} + ta_{m+1} - t|\ell|. \end{aligned}$$

Next, let us write this inequality in the form

$$(|\ell| + 1)t - 1 - ta_{m+1} > (t|\ell|a_{m+1} - (t-1)a_{m+1} - t|\ell|^2)\alpha_{m+2}. \tag{2.1}$$

By the condition of the theorem, $a_{m+1} \geqslant t$, since $q = q_{m+1} + \ell q_m > q_m$ is large enough and so m is large enough.

For $\ell = -1$, the left hand side of (2.1) is $2t - 1 - ta_{m+1} \leqslant 2t - 1 - t^2 = -(t-1)^2 \leqslant 0$, whereas its right hand side is nonegative in view of

$$ta_{m+1} - (t-1)a_{m+1} - t = a_{m+1} - t \geqslant 0$$

and $\alpha_{m+2} > 0$. Therefore, (2.1) cannot hold. Similarly, for $\ell = 1 - a_{m+1}$, the the left hand side of (2.1) is equal to -1, but the right hand side is nonnegative in view of

$$\begin{aligned} t(a_{m+1} - 1)a_{m+1} - (t-1)a_{m+1} - t(a_{m+1} - 1)^2 &= t(a_{m+1} - 1) - (t-1)a_{m+1} \\ &= a_{m+1} - t \geqslant 0, \end{aligned}$$

so (2.1) cannot hold too. This completes the proof of Theorem 3.

Finally, to show that the constant $(t-1)/t$ of Theorem 3 is best possible we consider α whose continued fraction expansion is such that for infinitely many $m \in \mathbb{N}$ we have $a_{m+1} = t$ and the neighboring partial quotients a_m, a_{m+2} both tend to infinity as $m \to \infty$. Put $p = p_{m+1} - (a_{m+1} - 1)p_m = p_m + p_{m-1}$ and $q = q_m + q_{m-1}$ for those m. Note that $\gcd(p, q) = 1$, because $\gcd(p, q)$ divides $|pq_m - qp_m| = |p_{m-1}q_m - q_{m-1}p_m| = 1$. Therefore, p/q is not a convergent of α. As above, using $q_{m+1} = a_{m+1}q_m + q_{m-1} = tq_m + q_{m-1}$, we obtain

$$q\alpha - p = \frac{(-1)^{m+1}(1 + (a_{m+1} - 1)\alpha_{m+2})}{q_{m+1}\alpha_{m+2} + q_m} = \frac{(-1)^{m+1}(t - 1 + 1/\alpha_{m+2})}{(t + 1/\alpha_{m+2})(q - q_{m-1}) + q_{m-1}}.$$

Thus

$$q^2|\alpha - p/q| = \frac{t - 1 + 1/\alpha_{m+2}}{t + 1/\alpha_{m+2} + (1 - t - 1/\alpha_{m+2})q_{m-1}/q}$$

tends to $(t-1)/t$ as $m \to \infty$, since the quantities $q_{m-1}/q < q_{m-1}/q_m < 1/a_m$ and $1/\alpha_{m+2} < 1/a_{m+2}$ both tend to zero as $m \to \infty$.

3 Analytical results

Set

$$s_n(x) := \sum_{i=0}^{2^n-1} t_i x^{i-1} \quad \text{and} \quad \overline{s}_n(x) := \sum_{i=0}^{2^n-1} (1-t_i)x^{i-1}.$$

Then

$$s_n(x) + \overline{s}_n(x) = x^{-1} + 1 + x + \cdots + x^{2^n-2} = \frac{1-x^{2^n}}{x(1-x)}, \tag{3.1}$$

$$s_{n+1}(x) = s_n(x) + x^{2^n}\overline{s}_n(x), \tag{3.2}$$

$$s_n(x) - \overline{s}_n(x) = -\frac{1}{x}\prod_{j=0}^{n-1}(1-x^{2^j}), \tag{3.3}$$

and

$$T(x) = s_n(x) + x^{2^n}\overline{s}_n(x) + x^{2\cdot 2^n}\overline{s}_n(x) + x^{3\cdot 2^n}s_n(x) + \ldots \tag{3.4}$$

for each $n \in \mathbb{N}$. The pattern of s_n and $\overline{s}_n$ in $s_n\overline{s}_n\overline{s}_n s_n \ldots$ in (3.4) is the same as that of 0 and 1 in the original sequence $\mathbf{t}$. Using (3.1) we find that

$$R_n(x) := s_n(x) + (x^{2^n} + x^{2\cdot 2^n} + x^{3\cdot 2^n} + \ldots)\overline{s}_n(x) = \frac{1-2x^{2^n}}{1-x^{2^n}}s_n(x) + \frac{x^{2^n-1}}{1-x}. \tag{3.5}$$

Therefore, by (3.3),

$$\begin{aligned} T(x) - R_n(x) &= (s_n(x) - \overline{s}_n(x))(x^{3\cdot 2^n} + x^{5\cdot 2^n} + x^{6\cdot 2^n} + x^{8\cdot 2^n} + x^{9\cdot 2^n} + \ldots) \\ &= -(x^{3\cdot 2^n-1} + x^{5\cdot 2^n-1} + x^{6\cdot 2^n-1} + \ldots)\prod_{j=0}^{n-1}(1-x^{2^j}). \end{aligned}$$

Hence, putting

$$P_n(x) := \prod_{j=0}^{n-1}(1-x^{2^j}), \tag{3.6}$$

we obtain

$$|x|^{3\cdot 2^n-1}|P_n(x)| < |T(x) - R_n(x)| < \frac{|x|^{3\cdot 2^n-1}|P_n(x)|}{1-x^{2^n}} \tag{3.7}$$

for each $x \in \mathbb{R}$ satisfying $0 < |x| < 1$.

In [6] it was shown that

Lemma 3. *For each $k \in \mathbb{N}$ and each $m \geqslant 0$ there are polynomials $P_{k,m}(x) \in \mathbb{Z}[x]$ of degree at most $2^m k - 1$ and $Q_{k,m}(x) \in \mathbb{Z}[x]$ of degree at most $2^m k$ such that*

$$\left|P(x) - \frac{P_{k,m}(x)}{Q_{k,m}(x)}\right| \asymp |x|^{2^{m+1}k}$$

for every $x \neq 0$ satisfying $|x| \leqslant 1/2$, where the constants in $\asymp$ depend on k only. (Recall that $f \asymp g$ if there are two positive constants c_1 and c_2 such that $c_1 g \leqslant f \leqslant c_2 g$.) Furthermore, the integers $q_{k,m} := b^{2^m k} Q_{k,m}(a/b)$, where $a \in \mathbb{Z} \setminus \{0\}$, $b \in \mathbb{N}$ and $|a/b| \leqslant 1/2$, satisfy

$$q_{k,m} \asymp b^{2^m k},$$

where the constants in $\asymp$ depend on k only.

4 Arithmetical results

Put $F_n(a,b) := b^{2^n} + a^{2^n}$ and let $(k_n)_{n=1}^{\infty}$ be a sequence defined by $k_1 := 1$ and

$$k_{n+1} := (b^{2^{n-1}} - a^{2^{n-1}})k_n + a^{2^n - 1} = (F_{n-1}(a,b) - 2a^{2^{n-1}})k_n + a^{2^n-1} \tag{4.1}$$

for $n = 1, 2, \ldots$. Then

Lemma 4. *For any integers a and b satisfying $0 < |a| < b$ and $\gcd(a,b) = 1$ we have*

$$r_n := \frac{s_n(a/b)}{1-(a/b)^{2^n}} = \frac{b^2 k_n}{(b-a)F_{n-1}(a,b)}.$$

Furthermore, the fraction on the right hand side is in its reduced form for each $n \in \mathbb{N}$.

Proof: Put for brevity F_n for $F_n(a,b)$. Note that for $n = 1$ we have $s_1(x) = 1$. Hence

$$r_1 = \frac{1}{1-(a/b)^2} = \frac{b^2}{b^2-a^2} = \frac{b^2 k_1}{(b-a)(b+a)} = \frac{b^2 k_1}{(b-a)F_0}$$

is in its reduced form in view of $\gcd(b^2, b^2 - a^2) = 1$. Assume that the assertion of the lemma holds for some $n \in \mathbb{N}$. Then, by (3.1) and (3.2), we obtain

$$\begin{aligned} r_{n+1} &= \frac{s_{n+1}(a/b)}{1-(a/b)^{2^{n+1}}} = \frac{s_n(a/b) + (a/b)^{2^n}\overline{s}_n(a/b)}{1-(a/b)^{2^{n+1}}} \\ &= \frac{s_n(a/b)}{1+(a/b)^{2^n}} + \frac{(a/b)^{2^n-1}}{(1-a/b)(1+(a/b)^{2^n})} \\ &= \frac{r_n(1-(a/b)^{2^n})}{1+(a/b)^{2^n}} + \frac{(a/b)^{2^n-1}}{(1-a/b)(1+(a/b)^{2^n})} \\ &= \frac{r_n(F_n - 2a^{2^n})}{F_n} + \frac{a^{2^n-1}b^2}{(b-a)F_n}. \end{aligned}$$

Hence, using equality $F_n - 2a^{2^n} = (b^{2^{n-1}} - a^{2^{n-1}})F_{n-1}$ and the induction hypothesis on r_n, we find that

$$\begin{aligned} (b-a)F_n r_{n+1} &= r_n(b-a)(F_n - 2a^{2^n}) + a^{2^n-1}b^2 \\ &= \frac{b^2 k_n}{(b-a)F_{n-1}}(b-a)(b^{2^{n-1}} - a^{2^{n-1}})F_{n-1} + a^{2^n-1}b^2 = b^2 k_{n+1} \end{aligned}$$

in view of (4.1). This yields $r_{n+1} = b^2 k_{n+1}/(b-a)F_n$, which is the required inequality for $n+1$.

It remains to prove that $\gcd(b^2 k_{n+1}, (b-a)F_n) = 1$. Observe that

$$\begin{aligned}2ak_{n+1} - F_n &= 2ak_n(F_{n-1} - 2a^{2^{n-1}}) + 2a^{2^n} - F_n\\ &= 2ak_n(F_{n-1} - 2a^{2^{n-1}}) - (b^{2^n} - a^{2^n})\\ &= (2ak_n - F_{n-1})(F_{n-1} - 2a^{2^{n-1}})\\ &+ F_{n-1}(F_{n-1} - 2a^{2^{n-1}}) - (b^{2^n} - a^{2^n})\\ &= (2ak_n - F_{n-1})(F_{n-1} - 2a^{2^{n-1}})\\ &= (2ak_n - F_{n-1})(b-a)F_0F_1\dots F_{n-2}.\end{aligned}$$

Applying this inequality for n, etc. we see that the right hand side is equal to $(2ak_1 - F_0)(b-a)^n F_0^{n-1} F_1^{n-2} \dots F_{n-2}$. Hence

$$2ak_{n+1} - F_n = -(b-a)^{n+1} F_0^{n-1} F_1^{n-2} \dots F_{n-2}. \tag{4.2}$$

Assume that there is a prime number p which divides both $b^2 k_{n+1}$ and $(b-a)F_n$. Clearly, if $p|b$ then p does not divide neither $b-a$ nor F_n. Next, suppose that $p|k_{n+1}$.

Observe that

$$\gcd(b^{2^m} - a^{2^m}, F_n) = 1 \tag{4.3}$$

for any integers $0 \leqslant m \leqslant n$, because a and b are coprime. In particular, (4.3) implies that $b-a$ and F_n are coprime. Assume first that $p|(b-a)$. Then equality (4.2) combined with $p|k_{n+1}$ yields $p|F_n$, a contradiction. Otherwise, when $p|F_n$, equality (4.2) (combined with (4.3)) shows that p must divide some F_m with $0 \leqslant m \leqslant n-2$. Then p also divides $F_m(b^{2^m} - a^{2^m}) = b^{2^{m+1}} - a^{2^{m+1}}$, which contradicts to (4.3). □

Lemma 5. *For any integers a and b satisfying $0 < |a| < b$ and $\gcd(a,b) = 1$ the fraction*

$$R_n(a/b) = \frac{(b^{2^n} - 2a^{2^n})k_n + a^{2^n-1}F_{n-1}}{(b-a)b^{2^n-2}F_{n-1}} \tag{4.4}$$

is in its reduced form for each $n \in \mathbb{N}$.

Proof: Indeed, by (3.5) and Lemma 4, we have

$$\begin{aligned}R_n(a/b) &= (1 - 2(a/b)^{2^n})r_n + \frac{(a/b)^{2^n-1}}{1-a/b} = \frac{(b^{2^n} - 2a^{2^n})b^2k_n}{(b-a)b^{2^n}F_{n-1}} + \frac{a^{2^n-1}}{(b-a)b^{2^n-2}}\\ &= \frac{(b^{2^n} - 2a^{2^n})k_n + a^{2^n-1}F_{n-1}}{(b-a)b^{2^n-2}F_{n-1}},\end{aligned}$$

which implies (4.4). What is left is to show that for each $n \in \mathbb{N}$

$$\gcd(\ell_n, (b-a)b^{2^n-2}F_{n-1}) = 1, \tag{4.5}$$

where $\ell_n := (b^{2^n} - 2a^{2^n})k_n + a^{2^n-1}F_{n-1}$. In all what follows we assume that $n \geqslant 2$, because for $n = 1$ the numbers $\ell_1 = b^2 - a^2 + ab$ and $(b-a)b^{2-2}F_0 = b^2 - a^2$ are coprime in view of $\gcd(a, b) = 1$.

For a contradiction assume that for some positive integer $n \geqslant 2$ a prime number p divides both $\ell_n = (b^{2^n} - 2a^{2^n})k_n + a^{2^n-1}F_{n-1}$ and $(b-a)b^{2^n-2}F_{n-1}$. We will consider three cases

(i) $p|(b-a)$;

(ii) $p|b$;

(iii) $p|F_{n-1}$.

In case (i), the number $\ell_n = (b^{2^n} - 2a^{2^n})k_n + a^{2^n-1}F_{n-1}$ modulo p is equal to

$$-a^{2^n}k_n + 2a^{2^n+2^{n-1}-1} = a^{2^n+2^{n-1}-1}$$

in view of $k_n \equiv a^{2^{n-1}-1} \pmod{(b-a)}$. So ℓ_n is not divisible by p.

In case (ii), ℓ_n modulo p is equal to

$$-2a^{2^n}k_n + a^{2^n+2^{n-1}-1} = a^{2^n}(a^{2^{n-1}-1} - 2k_n).$$

As $p \nmid a$, it remains to prove that $t_n := 2k_n - a^{2^{n-1}-1}$ is not divisible by p. From (4.1) we derive that

$$\begin{aligned} t_{n+1} = 2k_{n+1} - a^{2^n-1} &= (b^{2^{n-1}} - a^{2^{n-1}})(t_n + a^{2^{n-1}-1}) + a^{2^n-1} \\ &= (b^{2^{n-1}} - a^{2^{n-1}})t_n + b^{2^{n-1}}a^{2^{n-1}-1}. \end{aligned}$$

Hence $t_2 = 2b - a \equiv -a \pmod{p}$ and $t_{n+1} \equiv -a^{2^{n-1}}t_n \pmod{p}$ for each $n \geqslant 2$. This implies $p \nmid t_n$.

Finally, in case (iii), the number ℓ_n modulo p is equal to $-a^{2^n}k_n$, because p divides $b^{2^n} - a^{2^n} = F_{n-1}(b^{2^{n-1}} - a^{2^{n-1}})$. By Lemma 4, the numbers k_n and F_{n-1} are coprime. Since F_{n-1} and a are coprime too, we conclude that $p \nmid \ell_n$. □

The following lemma is a corollary of Lemma 4.1 in [2]:

Lemma 6. *Let $\delta > 0$, $\theta \geqslant 1$ and ξ be real numbers. If there is a sequence p_n/q_n of rational numbers and some positive constants c_0, c_1, c_2 such that*

$$q_n < q_{n+1} \leqslant c_0 q_n^{\theta}$$

and

$$\frac{c_1}{q_n^{1+\delta}} \leqslant \left|\xi - \frac{p_n}{q_n}\right| \leqslant \frac{c_2}{q_n^{1+\delta}}$$

for each $n \geqslant n_0$ then

$$\mu(\xi) \leqslant \frac{(\delta+1)\theta}{\delta}.$$

5 Proof of Theorem 1

Set $\alpha := P(a/b)$. By (1.1), we have $\mu(T(a/b)) = \mu(\alpha)$. Evidently, $b > a^2$ implies $|a/b| \leqslant 1/2$. So, by Lemma 3 applied to $x = a/b$, we obtain

$$|\alpha - p_{k,m}/q_{k,m}| \asymp |a/b|^{2^{m+1}k}, \tag{5.1}$$

where $p_{k,m} := b^{2^m k} P_{k,m}(a/b) \in \mathbb{Z}$, $q_{k,m} := b^{2^m k} Q_{k,m}(a/b) \in \mathbb{N}$, $a \in \mathbb{Z} \setminus \{0\}$, and

$$q_{k,m} \asymp b^{2^m k}. \tag{5.2}$$

Now, as in [6], fix $K \in \mathbb{N}$ and consider an increasing sequence $\mathcal{Q} = (Q_{K,n})_{n=1}^{\infty} \subset \mathbb{N}$ composed of all integers $q_{k,m}$ with k odd in the range $1 \leqslant k \leqslant 2^K - 1$ and $m \geqslant m_0(K)$. It is easy to see that if

$$\{a_1 < a_2 < a_3 < \dots\} = \{2^m k \mid k = 1, \dots, 2^K - 1,\ m = m_0(K), m_0(K) + 1, \dots\},$$

then $a_{n+1} \leqslant (1 + 2^{1-K}) a_n$ for each sufficiently large n (see Lemma 2 in [6]). By (5.2), the nth element of the sequence $\mathcal{Q}$, namely, $Q_{K,n}$, is approximately b^{a_n}. Hence

$$Q_{K,n} < Q_{K,n+1} < Q_{K,n}^{1+2^{-K+2}} \tag{5.3}$$

for each sufficiently large n. Utilizing $|a/b| = |a|/b = b^{\log|a|/\log b - 1}$ and (5.2), we obtain

$$|a/b|^{2^{m+1}k} \asymp q_{k,m}^{-2(1-\log|a|/\log b)}.$$

Thus, by (5.1), there are $P_{K,n} \in \mathbb{Z}$ (corresponding to $Q_{K,n}$) such that

$$|\alpha - P_{K,n}/Q_{K,n}| \asymp Q_{K,n}^{-2(1-\log|a|/\log b)},$$

where the constant in $\asymp$ depends on K only. Now, by (5.3) and Lemma 6 with $\delta := 1 - 2\log|a|/\log b > 0$ and $\theta := 1 + 2^{-K+2}$, we deduce that

$$\mu(\alpha) \leqslant \frac{(\delta+1)\theta}{\delta} = (1 + 2^{-K+2}) \frac{2\log b - 2\log|a|}{\log b - 2\log|a|}.$$

Letting $K \to \infty$, we arrive to the required result.

6 Proof of Theorem 2

Suppose $a = \pm 1$, $b \geqslant 2$ and $n \geqslant 2$. Inserting $x = a/b$ into (3.7) we obtain

$$b^{-3 \cdot 2^n + 1} |P_n(a/b)| < |T(a/b) - R_n(a/b)| < \frac{b^{-3 \cdot 2^n + 1} |P_n(a/b)|}{1 - b^{-2^n}}.$$

By (3.6),

$$P_n(a/b) = (1-a/b)\prod_{j=1}^{n-1}(1-b^{-2^j}) = \frac{(1-a/b)\prod_{j=1}^{\infty}(1-b^{-2^j})}{\prod_{j=n}^{\infty}(1-b^{-2^j})}$$
$$= \frac{(1-a/b)P(b^{-2})}{P(b^{-2^n})},$$

thus

$$\frac{b^{-3\cdot 2^n+1}(1-a/b)P(b^{-2})}{P(b^{-2^n})} < \Big|T\Big(\frac{a}{b}\Big) - R_n\Big(\frac{a}{b}\Big)\Big| < \frac{b^{-3\cdot 2^n+1}(1-a/b)P(b^{-2})}{(1-b^{-2^n})P(b^{-2^n})}. \tag{6.1}$$

By Lemma 5, the denominator of $R_n(a,b) = p_n(a,b)/q_n(a,b)$ is

$$q_n(a,b) := (b-a)b^{2^n-2}F_{n-1} = b^{2^n-2}(b-a)(b^{2^{n-1}}+1).$$

As $q_n(a,b) \sim b^{3\cdot 2^{n-1}-1}(1-a/b)$ for $n\to\infty$, using (6.1), we find that

$$q_n(a,b)^2\Big|T(a/b) - \frac{p_n(a,b)}{q_n(a,b)}\Big| < \frac{1}{C(a,b)-\epsilon} \tag{6.2}$$

for each $\epsilon > 0$ and each $n \geqslant n(\epsilon)$. This is less than $1/2$ for each pair (a,b) (see the table in Section 1), where $a = \pm 1$ and $(a,b) \neq (-1,2),(-1,3)$. So $p_n(a,b)/q_n(a,b)$ is a convergent to $T(a/b)$ for each sufficiently large n, namely, $p_n(a,b)/q_n(a,b) = p_m/q_m$, where $m = m(n)$ and p_m/q_m is the mth convergent to $T(a/b) = [a_0, a_1, a_2, \dots]$.

On the other hand, as

$$q_n(a,b)^2 = b^{3\cdot 2^n-2}(1-a/b)^2(1+b^{-2^{n-1}})^2 > b^{3\cdot 2^n-2}(1-a/b)^2P(b^{-2^n}),$$

from (6.1) we find that

$$q_m^2\Big|T(a/b) - \frac{p_m}{q_m}\Big| = q_n(a,b)^2\Big|T(a/b) - \frac{p_n(a,b)}{q_n(a/b)}\Big| > \frac{(1-a/b)^3P(b^{-2})}{b} = \frac{1}{C(a,b)}.$$

Combining this with (6.2) and applying Lemma 1 we obtain

$$C(a,b) - \epsilon < [a_{m+1}, a_{m+2}, a_{m+3}, \dots] + [0, a_m, a_{m-1}, \dots, a_1] < C(a,b)$$

for each of those m. Since

$$a_{m+1} < [a_{m+1}, a_{m+2}, a_{m+3}, \dots] + [0, a_m, a_{m-1}, \dots, a_1] < a_{m+1} + 2,$$

it follows that $C(a,b) - 2 - \epsilon < a_{m+1} < C(a,b)$. As $C(a,b) \notin \mathbb{Q}$, by selecting ϵ small enough, we obtain $a_{m+1} \in \{c-1, c\}$, where $c = \lfloor C(a,b) \rfloor$, which completes the proof of the theorem for $(a,b) \neq (-1,2),(-1,3)$.

In case $a/b = -1/3$, we have $C(-1,3) = 81/64P(1/9) = 1.44\dots$. Assume that $T(-1/3) = [a_0, a_1, a_2, \dots]$ has only finitely many partial quotients smaller than or equal to 3. Then, by Theorem 3 with $t = 4$ and (6.2) with $a = -1$, $b = 3$, we find that $p_n(-1,3)/q_n(-1,3)$ is a convergent to $T(-1/3)$ for n large enough, i.e. $p_n(-1,3)/q_n(-1,3)$ is equal to p_m/q_m, where $m = m(n)$ and p_m/q_m is the mth convergent to $T(-1/3)$. As above, using Lemma 1, we derive that

$$a_{m+1} < [a_{m+1}, a_{m+2}, a_{m+3}, \dots] + [0, a_m, a_{m-1}, \dots, a_1] < C(-1,3) = 1.44\dots.$$

It follows that $a_{m+1} = 1$ for each of those m, which contradicts to our assumption.
Acknowledgement. I thank the referee for pointing out some references related to Theorem 3. This work is supported by grant no. MIP-068/2013/LSS-110000-740 from the Research Council of Lithuania.

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Received: 16.05.2013,

Revised: 20.07.2013,

Accepted: 10.01.2014.

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