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The semi-convergence of GSI method for singular saddle point problems

by
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Abstract

Recently, Miao and Wang considered the GSI method for solving nonsingular saddle point problems and studied the convergence of the GSI method. In this paper, we prove the semi-convergence of the GSI method when it is applied to solve the singular saddle point problems.

Key Words: GSI method, Singular saddle point problem, Semi-convergence, Iterative method.
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1 Introduction

Consider the saddle point problems of the form

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ q \end{pmatrix}, \tag{1.1}$$

where $A \in \mathbb{R}^{m\times m}$ is symmetric positive definite, $B \in \mathbb{R}^{m\times n}$ is a matrix of rank r, $b \in \mathbb{R}^m$ and $q \in \mathbb{R}^n$ are given vectors, with $m \geq n$. When $r = n$, note that the coefficient matrix is nonsingular and the saddle point problem (1.1) has a unique solution. When $r < n$, the coefficient matrix is singular, in such case, we assume that the saddle point problem (1.1) is consistent [22].

The saddle point problem (1.1) appears in many engineering and scientific computing applications such as constrained optimization, the finite element method to Stokes equations, fluid dynamics and weighted linear squares problem [6, 13, 17, 19]. (1.1) is also termed as a Karsh-Kuhn-Tucker (KKT) system, or an augmented system, or an equilibrium system [10, 11, 12].

For its property of large and sparsity, (1.1) is suitable for being solved by the iterative methods. For nonsingular saddle point problem (1.1), there are many efficient iterative methods have been studied in the literature [13, 17, 14, 19, 7, 2, 3, 1, 18], see [4] for a comprehensive survey. Recently, Miao and Wang studied the generalized stationary iterative (GSI) method

[20] for nonsingular saddle point problem (1.1), see [18]. Note that the GSI method, studied in [18], includes SOR-like method [13] and GAOR method [14] as special cases.

In most cases, the matrix B is full column rank in scientific computing and engineering applications, but not always. If $r < n$, (1.1) is a singular saddle point problem. Zheng, Bai and Yang [22] show that the GSOR method [3] can be used to solve the singular saddle point problem (1.1), and it is semi-convergent. Li et al. [15, 16] give the semi-convergent analyses of the GSSOR method [21] and the inexact Uzawa methods [9, 8] for the singular saddle point problem (1.1).

In this paper, the GSI method for solving singular saddle point problem (1.1) is further investigated and the semi-convergence conditions are proposed, which generalize the result of Miao and Wang [18] for the nonsingular saddle point problems to the singular saddle point problems.

Throughout this paper, for $A \in \mathbb{R}^{m\times m}$, A^T, $\sigma(A)$ and $\rho(A)$ denote the transpose, the spectral set and the spectral radius of the matrix A, respectively. I_n is the identity matrix with order n.

2 The semi-convergence of the GSI method

Firstly, some basic concepts and lemmas are given for latter use. For a matrix $C \in \mathbb{R}^{p\times p}$, we call $C = M - N$ a splitting if M is nonsingular. Let $T = M^{-1}N$, then solving linear systems $Cz = c$ is equivalent to considering the following iterative scheme

$$x_{k+1} = Tx_k + c,\ \ k = 0, 1, 2, \cdots . \tag{2.1}$$

When C is nonsingular, for any initial vector x_0 the iteration scheme (2.1) converges to the exact solution of the system of linear equations $Cz = c$ if and only if $\rho(T) < 1$. But for the singular matrix C, we have $1 \in \sigma(T)$ and $\rho(T) \geq 1$, so that one can require only the semi-convergence of the iterative scheme (2.1). By [5], the iterative scheme (2.1) is semi-convergent if and only if the following three conditions are satisfied:

(1) The spectral radius of the iterative matrix T is equal to one, i.e., $\rho(T) = 1$;

(2) The elementary divisors of the iterative matrix T associated with $\lambda = 1 \in \sigma(T)$ are linear, i.e., $\mathrm{rank}(I_p - T)^2 = \mathrm{rank}(I_p - T)$, here $\mathrm{rank}(\cdot)$ denotes the rank of the corresponding matrix;

(3) If $\lambda \in \sigma(T)$ with $|\lambda| = 1$, then $\lambda = 1$, i.e., $\vartheta(T) < 1$, where

$$\vartheta(T) = \max\{|\lambda|, \lambda \in \sigma(T), \lambda \neq 1\}$$

is called the semi-convergence factor of the iterative scheme (2.1).

We call a matrix T is semi-convergent provided it satisfies the above three conditions, and iterative method (2.1) is semi-convergent if T is a semi-convergent matrix. When C is singular, the semi-convergence property about the iteration scheme (2.1) are described in the following two lemmas.

Lemma 1. [5] *Let $C = M - N$ with M nonsingular, $T = M^{-1}N$. Then for any initial vector x_0 the iterative scheme (2.1) is semi-convergent to a solution $x_\star$ of the system of linear equations $Cz = c$ if and only if the matrix T is semi-convergent.*

Lemma 2. [22] *Let $H \in \mathbb{R}^{l\times l}$ with positive integers l. Then the partitioned matrix*

$$T = \begin{pmatrix} H & 0 \\ L & I_t \end{pmatrix}$$

is semi-convergent if and only if either of the following conditions holds true:

(1) $L = 0$ *and H is semi-convergent;*

(2) $\rho(H) < 1$.

Secondly, we review the GSI method presented in [18]. Following [13], we rewrite system (1.1) as

$$\left(\begin{array}{cc} A & B \\ -B^T & 0 \end{array} \right) \left(\begin{array}{c} x \\ y \end{array} \right) = \left(\begin{array}{c} b \\ -q \end{array} \right) \tag{2.2}$$

for the sake of simplicity. For the coefficient matrix of the linear system (2.2), we make the following splitting:

$$\left(\begin{array}{cc} A & B \\ -B^T & 0 \end{array} \right) \equiv \left(\begin{array}{cc} \alpha A & 0 \\ -\beta B^T & \alpha Q \end{array} \right) - \left(\begin{array}{cc} (\alpha-1)A & -B \\ (1-\beta)B^T & \alpha Q \end{array} \right), \tag{2.3}$$

where α and β are real parameters with $\alpha \neq 0$, $Q \in \mathbb{R}^{n\times n}$ is a nonsingular matrix.

With the splitting (2.3), the GSI iterative scheme [20] for augmented system (2.2) is defined as

$$\left(\begin{array}{c} x^{(k+1)} \\ y^{(k+1)} \end{array} \right) = T_{\alpha,\beta} \left(\begin{array}{c} x^{(k)} \\ y^{(k)} \end{array} \right) + (\alpha D - \beta L)^{-1} \left(\begin{array}{c} b \\ -q \end{array} \right), k = 0, 1, 2, \cdots \tag{2.4}$$

where

$$T_{\alpha,\beta} = \left(\begin{array}{cc} \left(1 - \frac{1}{\alpha}\right) I_m & \frac{1}{\alpha} A^{-1}B \\ \frac{2\alpha-\beta-1}{\alpha^2} Q^{-1}B^T & I_n - \frac{1}{\alpha^2} Q^{-1}B^T A^{-1} B \end{array} \right) \tag{2.5}$$

is the GSI iteration matrix.

Algorithm 1. GSI Method*:*

1. Given the initial vectors $x^{(0)}$, $y^{(0)}$, the relaxation parameters α, β and the nonsingular matrix Q.

2. For $i = 0,\ 1, \cdots$, until convergence, compute

$$\left\{ \begin{array}{l} x^{(k+1)} = (1 - \frac{1}{\alpha})x^{(k)} + \frac{1}{\alpha}A^{-1}(b - By^{(k)}), \\ y^{(k+1)} = y^{(k)} + \frac{1}{\alpha}Q^{-1}\left\{B^T[x^{(k+1)} + (1-\beta)x^{(k)}] - q\right\}. \end{array} \right.$$

Remark 1. *It is worth mentioning that the GSI method (2.4) becomes the SOR-like method [13] when $\alpha = \frac{1}{\omega}$ and $\beta = 1$, and the GAOR method [14] when $\alpha = \frac{1}{\omega}$ and $\beta = \frac{\gamma}{\omega}$.*

Finally, we discuss the semi-convergence of the GSI method. When the coefficient matrix of (1.1) is nonsingular, the convergence of GSI method is studied in [18]. When $r < n$, the coefficient matrix of (1.1) is singular, the following theorem describes the semi-convergence property when the GSI method is applied to solve the singular saddle point poblem (1.1).

Theorem 1. *Let $A \in \mathbb{R}^{m\times m}$, $Q \in \mathbb{R}^{n\times n}$ be symmetric positive definite matrices, and $B \in \mathbb{R}^{m\times n}$ be a matrix of rank r with $r < n$. Denote the largest eigenvalues of the matrix $Q^{-1}B^TA^{-1}B$ by $\mu_{\max}$. Then the GSI method is semi-convergent to a solution of the singular saddle point problem (1.1) if α and β satisfies*

$$\alpha > \max\{\frac{1}{2}, \frac{\sqrt{\mu_{\max}}}{2}\}$$

and

$$1 - \frac{\alpha}{\mu_{\max}} < \beta < \frac{1}{2} + \frac{\alpha(2\alpha-1)}{\mu_{\max}}. \tag{2.6}$$

Proof: By Lemma 1, we only need to demonstrate the semi-convergence of the iteration matrix $T_{\alpha,\beta}$, defined by equation (2.5), of the GSI method.

Let $B = U(B_r,\ 0)V^*$ be the singular value decomposition of B, where $B_r = (\Sigma_r,\ 0)^T \in \mathbb{R}^{m\times n}$ with $\Sigma_r = \mathrm{diag}(\sigma_1, \sigma_2, \cdots, \sigma_r)$, U, V are unitary matrices. Then

$$P = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$$

is an $(m+n)$-by-$(m+n)$ unitary matrix. Let $\widehat{T}_{\alpha,\beta} = P^*T_{\alpha,\beta}P$, then the matrix $T_{\alpha,\beta}$ has the same eigenvalues with matrix $\widehat{T}_{\alpha,\beta}$. Here, we have used $(\cdot)^*$ to denote the conjugate transpose of the corresponding complex matrix. Hence, we only need to demonstrate the semi-convergence of the matrix $\widehat{T}_{\alpha,\beta}$.

Define matrices

$$\widehat{A} = U^*AU, \quad \widehat{B} = U^*BV \quad \text{and} \quad \widehat{Q} = V^*QV.$$

Then it holds that $\widehat{B} = (B_r,\ 0)$ and

$$\widehat{Q}^{-1} = \begin{pmatrix} V_1^*Q^{-1}V_1 & V_1^*Q^{-1}V_2 \\ V_2^*Q^{-1}V_1 & V_2^*Q^{-1}V_2 \end{pmatrix},$$

here we have partitioned the unitary matrix V into the block form $V = (V_1,\ V_2)$ confirmly to the partition of the matrix B. Through direct operations, we have

$$\begin{aligned} U^*A^{-1}BV &= (U^*A^{-1}U)(U^*BV) \\ &= \widehat{A}^{-1}(B_r,\ 0) \\ &= (\widehat{A}^{-1}B_r,\ 0), \\ V^*Q^{-1}B^TU &= (V^*Q^{-1}V)(V^*B^TU) \\ &= \begin{pmatrix} V_1^*Q^{-1}V_1B_r^T \\ V_2^*Q^{-1}V_1B_r^T \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} V^*Q^{-1}B^TA^{-1}BV &= (V^*Q^{-1}V)(V^*B^TU)(U^*A^{-1}U)(U^*BV) \\ &= \widehat{Q}^{-1}(B_r,\ 0)^T\widehat{A}^{-1}(B_r,\ 0) \\ &= \widehat{Q}^{-1}\begin{pmatrix} B_r^T\widehat{A}^{-1}B_r & 0 \\ 0 & 0_{n-r} \end{pmatrix}. \end{aligned} \tag{2.7}$$

Hence,

$$\begin{aligned} \widehat{T}_{\alpha,\beta} &= P^*T_{\alpha,\beta}P \\ &= \begin{pmatrix} \left(1-\frac{1}{\alpha}\right)I_m & \frac{1}{\alpha}U^*A^{-1}BV \\ \frac{2\alpha-\beta-1}{\alpha^2}V^*Q^{-1}B^TU & I_n-\frac{1}{\alpha^2}V^*Q^{-1}B^TA^{-1}BV \end{pmatrix} \\ &= \begin{pmatrix} \widehat{H}_{\alpha,\beta} & 0 \\ \widehat{L}_{\alpha,\beta} & I_{n-r} \end{pmatrix}, \end{aligned}$$

where

$$\widehat{H}_{\alpha,\beta} = \begin{pmatrix} \left(1-\frac{1}{\alpha}\right)I_m & \frac{1}{\alpha}\widehat{A}^{-1}B_r \\ \frac{2\alpha-\beta-1}{\alpha^2}(V_1^*Q^{-1}V_1)B_r^T & I_r-\frac{1}{\alpha^2}(V_1^*Q^{-1}V_1)B_r^T\widehat{A}^{-1}B_r \end{pmatrix}$$

and

$$\widehat{L}_{\alpha,\beta} = \left(\frac{2\alpha-\beta-1}{\alpha^2}(V_2^*Q^{-1}V_1)B_r^T,\ -\frac{1}{\alpha^2}(V_2^*Q^{-1}V_1)B_r^T\widehat{A}^{-1}B_r\right).$$

As $\widehat{L}_{\alpha,\beta} \neq 0$, from Lemma 2 we know that the matrix $\widehat{T}_{\alpha,\beta}$ is semi-convergent if and only if $\rho(\widehat{H}_{\alpha,\beta}) < 1$. Hence, in what follows, we will give out the restrictions for the parameters α and β such that $\rho(\widehat{H}_{\alpha,\beta}) < 1$.

Consider the following nonsingular saddle point problem

$$\begin{pmatrix} \widehat{A} & B_r \\ -B_r^T & 0 \end{pmatrix}\begin{pmatrix} \widehat{x} \\ \widehat{y} \end{pmatrix} = \begin{pmatrix} \widehat{b} \\ -\widehat{q} \end{pmatrix}. \tag{2.8}$$

If the coefficient matrix of the nonsingular saddle point problem (2.8) be splitted as

$$\begin{pmatrix} \widehat{A} & B_r \\ -B_r^T & 0 \end{pmatrix} = \begin{pmatrix} \alpha\widehat{A} & 0 \\ -\beta B_r^T & \alpha\widehat{Q}_1 \end{pmatrix} - \begin{pmatrix} (\alpha-1)\widehat{A} & -B_r \\ (1-\beta)B_r^T & \alpha\widehat{Q}_1 \end{pmatrix}$$

with the preconditioning matrix $\widehat{Q}_1 = (V_1^*Q^{-1}V_1)^{-1}$, then the GSI method for solving (2.8) can be well defined, and the iteration matrix is $\widehat{H}_{\alpha,\beta}$. From Theorem 2.4 in [18], we know that $\rho(\widehat{H}_{\alpha,\beta}) < 1$ if α and β satisfies

$$\alpha > \max\{\frac{1}{2}, \frac{\sqrt{\mu_{\max}}}{2}\}$$

and

$$1-\frac{\alpha}{\mu_{\max}} < \beta < \frac{1}{2}+\frac{\alpha(2\alpha-1)}{\mu_{\max}},$$

here $\mu_{\max}$ is the largest eigenvalue of the matrix $\widehat{Q}_1^{-1} B_r^T \widehat{A}^{-1} B_r$. By (2.7), we can see that $\mu_{\max}$ is also the largest eigenvalue of the matrix $Q^{-1}B^TA^{-1}B$.

The proof is completed. □

The GSI method has the SOR-like method [13] and the GAOR method [14] as its special cases, thus from Theorem 1, we can derive the following corollaries directly.

Corollary 1. *Let A and Q be symmetric positive definite, and B is a matrix of rank r with $r < n$. Denote the largest eigenvalues of the matrix $Q^{-1}B^TA^{-1}B$ by $\mu_{\max}$. Then the SOR-like method is semi-convergent to a solution of the singular saddle point problems (1.1) if*

$$0 < \omega < \frac{4}{1 + \sqrt{1 + 4\mu_{\max}}}.$$

Corollary 2. *Let A and Q be symmetric positive definite, and B is a matrix of rank r with $r < n$. Denote the largest eigenvalues of the matrix $Q^{-1}B^TA^{-1}B$ by $\mu_{\max}$. Then the GAOR method is semi-convergent to a solution of the singular saddle point problems (1.1) if ω and γ satisfies*

$$0 < \omega < \min\{2, \frac{2}{\sqrt{\mu_{\max}}}\}$$

and

$$\omega - \frac{1}{\mu_{\max}} < \gamma < \frac{\omega}{2} + \frac{2 - \omega}{\omega \mu_{\max}}.$$

Note that $\min\{2, \frac{2}{\sqrt{\mu_{\max}}}\} \leq 2$, from Theorem 2.3 in [15], we known that the condition in Corollary 2 is the sufficient but necessary condition.

3 Conclusion

In this paper, we consider the GSI method for solving singular saddle point problems. The semi-convergence conditions are given, which generalize the result of Miao and Wang [18] for nonsingular saddle point problems to singular saddle point problems. Meanwhile, from the proof of Theorem 1, we note that the semi-convergence factor of the GSI method for the singular saddle point problem (1.1) is the convergence factor of the GSI method for the nonsingular saddle point problem (2.8).

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