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Total vertex irregularity strength of certain classes of unicyclic graphs*

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Abstract

A total vertex irregular k-labeling ϕ of a graph G is a labeling of the vertices and edges of G with labels from the set $\{1, 2, \dots, k\}$ in such a way that for any two different vertices x and y their weights $wt(x)$ and $wt(y)$ are distinct. Here, the weight of a vertex x in G is the sum of the label of x and the labels of all edges incident with the vertex x. The minimum k for which the graph G has a vertex irregular total k-labeling is called the *total vertex irregularity strength* of G.

We have determined an exact value of the total vertex irregularity strength of certain classes of unicyclic graphs.

Key Words: Vertex irregular total k-labeling, total vertex irregularity strength, stars, kite graphs, paths, cycles.
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1 Introduction

Let us consider a simple (without loops and multiple edges) undirected graph $G = (V, E)$. For a graph G we define a labeling $\phi : V \cup E \to \{1, 2, \dots, k\}$ to be a total vertex irregular k-labeling of the graph G if for every two different vertices x and y of G one has $wt(x) \neq wt(y)$ where the weight of a vertex x in the labeling ϕ is $wt(x) = \phi(x) + \sum_{y \in N(x)} \phi(xy)$, where $N(x)$ is the set of neighbors of x. Bača et al. [2] defined a new graph invariant, called the *total vertex irregularity strength* of G, $tvs(G)$, that is the minimum k for which the graph G has a vertex irregular total k-labeling.
The original motivation for the definition of the total vertex irregularity strength came from irregular assignments and the irregularity strength of graphs introduced by Chartrand et al. [4], and studied by numerous authors e.g. [3, 5, 6].
In [2] several bounds and exact values of $tvs(G)$ were determined for different types of graphs

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(in particular for stars, cliques and prisms). Among others, the authors proved the following theorem

Theorem 1. *Let G be a (p,q)-graph with minimum degree $\delta = \delta(G)$ and maximum degree $\Delta = \Delta(G)$. Then*

$$\lceil (p+\delta)/(\Delta+1) \rceil \le tvs(G) \le p + \Delta - 2\delta + 1. \tag{1}$$

These results were then improved by Przybylo in [7] for sparse graphs and for graphs with large minimum degree. In the latter case the bounds $tvs(G) < 32\,\frac{p}{\delta} + 8$ in general and $tvs(G) < 8\,\frac{p}{r} + 3$ for r-regular (p,q)-graphs were proved to hold. In [1] Anholcer et al. established a new upper bound of the form

$$tvs(G) \le 3\frac{p}{\delta} + 1. \tag{2}$$

The main aim of this paper is to find an exact value of the total vertex irregularity strength of certain classes of unicyclic graphs which is much closer to the lower bound in (1) than to the upper bound in (2).

2 Main Result

Let C_n be a cycle with vertices $v_1, v_2, \ldots, v_n$ and S_{m_i}, $i = 1, 2, \ldots, n$, be a star with the central vertex u_i and leaves u^i_j, $1 \le j \le m_i$.

If the star S_{m_i} is adjoined to each vertex v_i, $i = 1, 2, \ldots, n$, by identifying v_i and u_i for $i = 1, 2, \ldots, n$, we obtained a unicyclic graph denoted by $C_n \triangle S_{m_i}$.

Let $V(C_n \triangle S_{m_i}) = \{v_i : 1 \le i \le n\} \cup \{u^i_j : 1 \le i \le n, 1 \le j \le m_i\}$ and $E(C_n \triangle S_{m_i}) = \{v_i v_{i+1} : 1 \le i \le n-1\} \cup \{v_n v_1\} \cup \{v_i u^i_j : 1 \le i \le n, 1 \le j \le m_i\}$ be the vertex set and the edge set, respectively.

From Theorem 1 it follows that

$$tvs(C_n \triangle S_{m_i}) \ge \left\lceil (\sum_{i=1}^{n} m_i + n + 1)/(3 + \max\{m_1, \ldots, m_n\}) \right\rceil.$$

Next theorem gives a new lower bound for $C_n \triangle S_{m_i}$.

Theorem 2. *Let $n \ge 3$ and $C_n \triangle S_{m_i}$ be the unicyclic graph with $2 \le m_1 \le m_2 \le \cdots \le m_n$. Then $tvs(C_n \triangle S_{m_i}) \ge \left\lceil \left(1 + \sum_{i=1}^{n} m_i\right)/2 \right\rceil$.*

Proof. The unicyclic graph $C_n \triangle S_{m_i}$ has $\sum_{i=1}^{n} m_i$ vertives of degree 1 and vertices of degree $m_i + 2$, $1 \le i \le n$.

To prove the lower bound we consider the weights of the vertices. The smallest weight among all vertices of $C_n \triangle S_{m_i}$ is at least 2, so the largest weight of a vertex of degree 1 is at least $1 + \sum_{r=1}^{n} m_r$. Since the weight of any vertex of degree 1 is the sum of two positive integers, so at least one label is at least $\left\lceil \left(1 + \sum_{r=1}^{n} m_r\right)/2 \right\rceil$.

Moreover, the largest value among the weights of vertices of degree 1 and m_i+2, $1 \le i \le n$, is at least $1+i+\sum_{r=1}^{n} m_r$, $1 \le i \le n$, and this weight for fix i is the sum of at most m_i+3 integers. Hence the largest label contributing to this weight must be at least $\left\lceil \left(1+i+\sum_{r=1}^{n} m_r\right)/(m_i+3)\right\rceil$.

Consequently, the largest label of a vertex or an edge of $C_n \triangle S_{m_i}$ is at least
$\max\left\{\left\lceil\left(1+\sum_{r=1}^{n} m_r\right)/2\right\rceil, \left\lceil\left(2+\sum_{r=1}^{n} m_r\right)/(m_1+3)\right\rceil, \left\lceil\left(3+\sum_{r=1}^{n} m_r\right)/(m_2+3)\right\rceil,\right.$
$\left.\dots, \left\lceil\left(n+1+\sum_{r=1}^{n} m_r\right)/(m_n+3)\right\rceil\right\} = \left\lceil\left(1+\sum_{r=1}^{n} m_r\right)/2\right\rceil$ for $n \ge 3$.

Thus $tvs(G) \ge \left\lceil\left(1+\sum_{r=1}^{n} m_r\right)/2\right\rceil$. □

The following theorem determines the exact value of the total vertex irregularity strength of $C_n \triangle S_{m_i}$.

Theorem 3. *Let $n \ge 3$ and $2 \le m_1 \le m_2 \le \dots \le m_n$. Then*

$$tvs(C_n \triangle S_{m_i}) = \left\lceil\left(1+\sum_{i=1}^{n} m_i\right)/2\right\rceil.$$

Proof. Suppose that $n \ge 3$, $2 \le m_1 \le m_2 \le \dots \le m_n$ and $k = \left\lceil\left(1+\sum_{r=1}^{n} m_r\right)/2\right\rceil$. According to Theorem 2 it is sufficient to prove the existence of a vertex irregular total k-labeling for the $C_n \triangle S_{m_i}$.

We define a labeling $\phi : V(C_n \triangle S_{m_i}) \cup E(C_n \triangle S_{m_i}) \to \{1,2,\dots,k\}$ in the following way

$$\begin{aligned} \phi(v_i) &= k \quad \text{for} \quad 1 \le i \le n, \\ \phi(v_i v_{i+1}) &= k \quad \text{for} \quad 1 \le i \le n-1, \\ \phi(v_n v_1) &= k, \end{aligned}$$

$$\begin{aligned} \phi(u_j^i) &= \left\lceil\left(j+\sum_{r=1}^{i-1} m_r\right)/2\right\rceil \quad \text{for} \quad 1 \le i \le n \quad \text{and} \quad 1 \le j \le m_i, \\ \phi(v_i u_j^i) &= \left\lceil\left(1+j+\sum_{r=1}^{i-1} m_r\right)/2\right\rceil \quad \text{for} \quad 1 \le i \le n \quad \text{and} \quad 1 \le j \le m_i. \end{aligned}$$

The weights of vertices of $C_n \triangle S_{m_i}$ are as follows:
$wt(u_j^i) = 1+j+\sum_{r=1}^{i-1} m_r$ for $1 \le i \le n$ and $1 \le j \le m_i$,
$wt(v_i) = 3k+\sum_{j=1}^{m_i}\left\lceil\left(1+j+\sum_{r=1}^{i-1} m_r\right)/2\right\rceil$ for $1 \le i \le n$. Thus, the weights of vertices u_j^i, $1 \le i \le n$, $1 \le j \le m_i$, successively attain values $2,3,\dots,1+\sum_{r=1}^{n} m_r$ and the weights of vertices v_i, $1 \le i \le n$, receive distinct values from $3k+\sum_{j=1}^{m_1}\lceil(1+j)/2\rceil$ up to $3k+\sum_{j=1}^{m_n}\left\lceil\left(1+j+\sum_{r=1}^{n-1} m_r\right)/2\right\rceil$.

The labeling ϕ is the desired vertex irregular total k-labeling and provides the upper bound on $tvs(C_n \triangle S_{m_i})$. This concludes the proof. $\square$

An $(n,t)-kite$ is a cycle of length n with a t-edge path (the tail) attached to one vertex. The following theorem gives the exact value of the total vertex irregularity strength for (n,t)-kites.

Theorem 4. *Every (n,t)-kite with $n \geq 3$ and $t \geq 1$ satisfies*

$$tvs((n,t)-kite) = \lceil (n+t)/3 \rceil .$$

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of a cycle and $u_1, u_2, \dots, u_t$ be the vertices on a path. Let $E((n,t)-kite) = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_n v_1\} \cup \{u_j u_{j+1} : 1 \leq j \leq t-1\} \cup \{u_t v_s : s \in \{1,\dots,n\}\}$ be the edge set of (n,t)-kite. Thus the (n,t)-kite has the vertex u_1 of degree 1, $n+t-2$ vertices of degree 2 and the vertex v_s of degree 3. The smallest weight among all vertices of (n,t)-kite is at least 2. The largest weight of vertices of degree 1 and 2 is at least $n+t$ and this weight is the sum of at most three integers. Hence the largest label contributing to this weight must be at least $\lceil (n+t)/3 \rceil$. Moreover, the largest value among the weights of vertices of degree 2 and 3 is at least $n+t+1$ and this weight is the sum of at most four integers, so at least one label is at least $\lceil (n+t+1)/4 \rceil$. Consequently, the largest label of one of vertex or edge of (n,t)-kite is at least $\max\{1, \lceil (n+t)/3 \rceil, \lceil (n+t+1)/4 \rceil\} = \lceil (n+t)/3 \rceil$ for $n \geq 3$ and $t \geq 1$. Thus $tvs((n,t)-kite) \geq \lceil (n+t)/3 \rceil$.

Put $k = \lceil (n+t)/3 \rceil$. To show that k is an upper bound for total vertex irregularity strength of (n,t)-kite we describe a total k-labeling $\psi : V((n,t)-kite) \cup E((n,t)-kite) \to \{1,2,\dots,k\}$ as follows

$$\psi(u_j) = \begin{cases} 1 & \text{if } j = 1 \\ \lceil (j-1)/3 \rceil & \text{if } 2 \leq j \leq t \end{cases}$$

$$\psi(u_j u_{j+1}) = \lceil (j+1)/3 \rceil \quad \text{for } 1 \leq j \leq t-1.$$

For $t \equiv 0 \pmod 3$

$$\psi(v_i) = \begin{cases} \lceil (t-1)/3 \rceil - 1 + \lceil (i+1)/2 \rceil & \text{if } 1 \leq i \leq n+1-k+\lceil (t-1)/3 \rceil \\ n+2+\lceil (t-1)/3 \rceil - i & \text{if } n+2-k+\lceil (t-1)/3 \rceil \leq i \leq n \end{cases}$$

$$\psi(v_i v_{i+1}) = \begin{cases} \lceil (t-1)/3 \rceil + \lceil i/2 \rceil & \text{if } 1 \leq i \leq n-k+\lceil (t-1)/3 \rceil \\ \lceil (t-1)/3 \rceil + \lceil i/2 \rceil - 1 & \text{if } i = n+1-k+\lceil (t-1)/3 \rceil \\ & \text{and } n+t \equiv 0 \pmod 3 \\ \lceil (t-1)/3 \rceil + \lceil i/2 \rceil & \text{if } i = n+1-k+\lceil (t-1)/3 \rceil \\ & \text{and } n+t \equiv 1,2 \pmod 3 \\ n+1+\lceil (t-1)/3 \rceil - i & \text{if } n+2-k+\lceil (t-1)/3 \rceil \leq i \leq n-1 \end{cases}$$

$$\psi(v_n v_1) = 1 + \lceil (t-1)/3 \rceil$$

$$\psi(u_t v_s) = \lceil (t+1)/3 \rceil \quad \text{for } s = \begin{cases} n-k+2+\lceil (t-1)/3 \rceil & \text{if } n \equiv 1 \pmod 3 \\ n-k+1+\lceil (t-1)/3 \rceil & \text{otherwise.} \end{cases}$$

For $t \equiv 1 \pmod 3$

$$\psi(v_i) = \begin{cases} \lceil (t-1)/3 \rceil + \lceil i/2 \rceil & \text{if } \ 1 \le i \le n+1-k+\lceil (t-1)/3 \rceil \\ n+2+\lceil (t-1)/3 \rceil - i & \text{if } \ n+2-k+\lceil (t-1)/3 \rceil \le i \le n \end{cases}$$

$$\psi(v_i v_{i+1}) = \begin{cases} \lceil (t-1)/3 \rceil + \lceil (i+1)/2 \rceil & \text{if } \ 1 \le i \le n-k+\lceil (t-1)/3 \rceil \\ \lceil (t-1)/3 \rceil + \lceil (i+1)/2 \rceil - 1 & \text{if } \ i = n+1-k+\lceil (t-1)/3 \rceil \\ & \text{and } \ n+t \equiv 0 \pmod 3 \\ \lceil (t-1)/3 \rceil + \lceil (i+1)/2 \rceil & \text{if } \ i = n+1-k+\lceil (t-1)/3 \rceil \\ & \text{and } \ n+t \equiv 1,2 \pmod 3 \\ n+1+\lceil (t-1)/3 \rceil - i & \text{if } n+2-k+\lceil (t-1)/3 \rceil \le i \le n-1 \end{cases}$$

$$\psi(v_n v_1) = 1 + \lceil (t-1)/3 \rceil$$

$$\psi(u_t v_s) = \lceil (t+1)/3 \rceil \quad \text{for } \ s = \begin{cases} n-k+2+\lceil (t-1)/3 \rceil & \text{if } \ n \equiv 0 \pmod 3 \\ n-k+1+\lceil (t-1)/3 \rceil & \text{otherwise.} \end{cases}$$

For $t \equiv 2 \pmod 3$

$$\psi(v_i) = \begin{cases} \lceil (t-1)/3 \rceil - 1 + \lceil (i+1)/2 \rceil & \text{if } \ 1 \le i \le n-k-1+\lceil (t-1)/3 \rceil \\ \lceil (t-1)/3 \rceil + \lceil i/2 \rceil - 1 & \text{if } i = n-k+\lceil (t-1)/3 \rceil \\ & \text{and } \ n+t \equiv 0 \pmod 3 \\ \lceil (t-1)/3 \rceil - 1 + \lceil (i+1)/2 \rceil & \text{if } i = n-k+\lceil (t-1)/3 \rceil \\ & \text{and } \ n+t \equiv 1,2 \pmod 3 \\ n+1+\lceil (t-1)/3 \rceil - i & \text{if } n+1-k+\lceil (t-1)/3 \rceil \le i \le n \end{cases}$$

$$\psi(v_i v_{i+1}) = \begin{cases} \lceil (t-1)/3 \rceil + \lceil i/2 \rceil & \text{if } \ 1 \le i \le n-1-k+\lceil (t-1)/3 \rceil \\ n+\lceil (t-1)/3 \rceil - i & \text{if } \ n-k+\lceil (t-1)/3 \rceil \le i \le n-1 \end{cases}$$

$$\psi(v_n v_1) = \lceil (t-1)/3 \rceil$$

$$\psi(u_t v_s) = \lceil (t+1)/3 \rceil \quad \text{for } \ s = \begin{cases} n-k+1+\lceil (t-1)/3 \rceil & \text{if } \ n \equiv 2 \pmod 3 \\ n-k+\lceil (t-1)/3 \rceil & \text{otherwise.} \end{cases}$$

Observe that under the labeling ψ the weights of the vertices of (n,t)-kite are:

$$\begin{aligned} \{wt(u_j) : 1 \le j \le t\} &= \{2, 3, \dots, t+1\} \\ \{wt(v_i) : 1 \le i \le n, i \ne s\} &= \{t+2, t+3, \dots, t+n\} \end{aligned}$$

and

$$wt(v_s) = \begin{cases} t+n+\lceil (t+1)/3 \rceil & \text{for } \ n+t \equiv 0 \pmod 3 \\ t+n+1+\lceil (t+1)/3 \rceil & \text{for } \ n+t \equiv 1,2 \pmod 3). \end{cases}$$

Thus the labeling ψ is the desired vertex irregular total k-labeling and the proof is complete. □

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