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Existence of positive solutions for $(p(x), q(x))$ Laplacian system

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Abstract

We consider the system of differential equations

$$(P)\begin{cases} -\Delta_{p(x)}u = \lambda_1^{p(x)} g(x)a(u) + \mu_1^{p(x)} c(x)f(v) & \text{in } \Omega, \\ -\Delta_{q(x)}v = \lambda_2^{q(x)} g(x)b(v) + \mu_2^{q(x)} c(x)h(u) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2 boundary $\partial\Omega, 1 < p(x), q(x) \in C^1(\bar{\Omega})$ are functions, the operator $\Delta_{p(x)}u = div(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplacian λ_1, λ_2, μ_1 and μ_2 are positive parameters and g, c are continuous functions and f, h, a, b are C^1 nondecreasing functions satisfying $f(0), h(0), a(0), b(0) \geq 0$. We discuss the existence of positive solution via sub-super solutions.

Key Words: Positive solutions; $p(x)$-Laplacian Problems; sub-supersolution.
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1 Introduction

The study of differential equatons and variational problems with variable exponent has been a new and interesting topic. It arises from nonlinear elasticity theory, electrorheological fluids, etc.(see [3, 10, 13]). Many results have been obtained on this kind of problems, for example [1, 3, 4, 5, 6, 9]. In [2], the authors discussed the existence of at least one positive solution of the system

$$(\mathrm{I})\begin{cases} -\Delta_{p(x)}u = \lambda^{p(x)} F(x, u, v) & \text{in } \Omega, \\ -\Delta_{p(x)}v = \lambda^{p(x)} G(x, u, v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $p(x) \in C^1(\bar{\Omega})$ is a function, $F(x, u, v) = [g(x)a(u) + f(v)], G(x, u, v) = [g(x)b(v) + h(u)], \lambda$ is a positive parameter and $\Omega \subset R^N$ is a bounded domain. But in the present paper we extend the problem (I) to problem (P). In this paper, we mainly consider the existence of positive weak solutions for the problem (P) and we have proved the existence of at least one positive solution for the problem (P).

To study $p(x)$-Laplacian problems, we need some theory on the spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and properties of $p(x)$-Laplacian which we will use later (see [5, 11]). If $\Omega \subset \mathbb{R}^N$ is an open domain, we write

$$C_+(\Omega) = \{h : h \in C(\Omega), h(x) > 1 \ for \ x \in \Omega\}$$

$$h^+ = \sup_{x\in\Omega} h(x), \ h^- = \inf_{x\in\Omega} h(x), \quad for \ any \ h \in C(\bar{\Omega}).$$

Throughout the paper, we will assume that:

(H_1) $\Omega \subset \mathbb{R}^N$ is an open bounded domain with C^2 boundary $\partial\Omega$.

(H_2) $p(x), q(x) \in C^1(\bar{\Omega})$ are functions and $1 < p^- \leq p^+$ and $1 < q^- \leq q^+$.

(H_3) $a, b, f, h : [0, \infty] \to R$ are C^1, nondecreasing functions such that $f(0), h(0), a(0), b(0) \geq 0$ and $\lim_{u\to+\infty} a(u) = \lim_{u\to+\infty} b(u) = \lim_{u\to+\infty} f(u) = \lim_{u\to+\infty} h(u) = +\infty$.

(H_4) $\displaystyle\lim_{u\to+\infty} \frac{f[M(h(u))^{\frac{1}{(p^- -1)}}]}{u^{p^- -1}} = 0, \quad \forall M > 0.$

(H_5) $g, c : \bar{\Omega} \to [1, \infty]$ are continuous functions such that
$A_1 = \min_{x\in\bar{\Omega}} g(x), \quad A_2 = \max_{x\in\bar{\Omega}} g(x), \quad B_1 = \min_{x\in\bar{\Omega}} c(x), \quad B_2 = \max_{x\in\bar{\Omega}} c(x).$

(H_6) $\displaystyle\lim_{u\to+\infty} \frac{a(u)}{u^{p^- -1}} = 0, \quad \lim_{u\to+\infty} \frac{b(u)}{u^{p^- -1}} = 0.$

Definition 1. *If $(u, v) \in \Big(W_0^{1,p(x)}(\Omega), W_0^{1,q(x)}(\Omega)\Big)$, (u, v) is called a weak solution of (P) if it satisfies*

$$\begin{cases} \int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi dx = \int_\Omega \lambda_1^{p(x)} g(x)a(u) + \mu_1^{p(x)} c(x)f(v)\varphi dx, & \forall \ \varphi \in W_0^{1,p(x)}(\Omega), \\ \int_\Omega |\nabla v|^{q(x)-2}\nabla v \cdot \nabla\psi dx = \int_\Omega \lambda_2^{q(x)} g(x)b(v) + \mu_2^{q(x)} c(x)h(u)\psi dx, & \forall \ \psi \in W_0^{1,q(x)}(\Omega). \end{cases}$$

Lemma 1. *(Comparison Principle).*
Let $u, v \in W^{1,p(x)}(\Omega)$ satisfying $Au - Av \geq 0$ in $(W_0^{1,p(x)}(\Omega))^*$, $\varphi(x) = \min\{u(x) - v(x), 0\}$. If $\varphi(x) \in W_0^{1,p(x)}(\Omega)$(i.e.$u \geq v$ on $\partial\Omega$), then $u \geq v$ a.e. in Ω.

Here and hereafter, we will use the notation $d(x, \partial\Omega)$ to denote the distance of $x \in \Omega$ to the boundary of Ω.
Denote $d(x) = d(x, \partial\Omega)$ and $\partial\Omega_\epsilon = \{x \in \Omega \mid d(x, \partial\Omega) < \epsilon\}$. Since $\partial\Omega$ is C^2 regularly, then there exists a constant $\delta \in (0, 1)$ such that $d(x) \in C^2(\overline{\partial\Omega_{3\delta}})$, and $|\nabla d(x)| \equiv 1$.
Denote

$$v_1(x) = \begin{cases} \gamma d(x), & d(x) < \delta, \\ \gamma\delta + \int_\delta^{d(x)} \gamma(\frac{2\delta-t}{\delta})^{\frac{2}{p^- -1}} (\lambda_1^{p^+} A_1 + \mu_1^{p^+} B_1)^{\frac{2}{p^- -1}} dt, & \delta \leq d(x) < 2\delta, \\ \gamma\delta + \int_\delta^{2\delta} \gamma(\frac{2\delta-t}{\delta})^{\frac{2}{p^- -1}} (\lambda_1^{p^+} A_1 + \mu_1^{p^+} B_1)^{\frac{2}{p^- -1}} dt, & 2\delta \leq d(x). \end{cases}$$

Obviously, $0 \leq v_1(x) \in C^1(\bar{\Omega})$. Considering

$$-\Delta_{p(x)} w(x) = \eta \quad in\ \Omega, \quad w = 0 \quad on\ \partial\Omega, \tag{1}$$

we have the following result

Lemma 2. *(see [7]). If positive parameter η is large enough and w is the unique solution of (1), then we have*
(i) For any $\theta \in (0,1)$ there exists a positive constant C_1 such that

$$C_1 \eta^{\frac{1}{p^+ - 1 + \theta}} \leq \max_{x \in \bar{\Omega}} w(x);$$

(ii) There exists a positive constant C_2 such that

$$\max_{x \in \bar{\Omega}} w(x) \leq C_2 \eta^{\frac{1}{p^- - 1}}$$

2 Existence results

In the following, when there be no misunderstanding, we always use C_i to denote positive constants.

Theorem 1. *On the conditions of $(H_1) - (H_6)$, then (P) has a positive solution.*

Proof: We shall establish Theorem 1 by constructing a positive subsolution (Φ_1, Φ_2) and supersolution (z_1, z_2) of (P), such that $\Phi_1 \leq z_1$ and $\Phi_2 \leq z_2$.
According to the sub-supersolution method for $p(x)$-Laplacian equations (see [8]), then (P) has a positive solution.
Step 1. We construct a subsolution of (P).
Let $\sigma \in (0, \delta)$ is small enough. Denote

$$\phi_1(x) = \begin{cases} e^{kd(x)} - 1, \quad d(x) < \sigma, \\ e^{k\sigma} - 1 + \displaystyle\int_\sigma^{d(x)} ke^{k\sigma} \left(\frac{2\delta - t}{2\delta - \sigma}\right)^{\frac{2}{p^- - 1}} dt, \quad \sigma \leq d(x) < 2\delta, \\ e^{k\sigma} - 1 + \displaystyle\int_\sigma^{2\delta} ke^{k\sigma} \left(\frac{2\delta - t}{2\delta - \sigma}\right)^{\frac{2}{p^- - 1}} dt, \quad 2\delta \leq d(x). \end{cases}$$

$$\phi_2(x) = \begin{cases} e^{kd(x)} - 1, \quad d(x) < \sigma, \\ e^{k\sigma} - 1 + \displaystyle\int_\sigma^{d(x)} ke^{k\sigma} \left(\frac{2\delta - t}{2\delta - \sigma}\right)^{\frac{2}{q^- - 1}} dt, \quad \sigma \leq d(x) < 2\delta, \\ e^{k\sigma} - 1 + \displaystyle\int_\sigma^{2\delta} ke^{k\sigma} \left(\frac{2\delta - t}{2\delta - \sigma}\right)^{\frac{2}{q^- - 1}} dt, \quad 2\delta \leq d(x). \end{cases}$$

It is easy to see that $\phi_1, \phi_2 \in C^1(\bar{\Omega})$. Denote

$$\alpha = \min\left\{ \frac{\inf p(x) - 1}{4(\sup |\nabla p(x)| + 1)}, \frac{\inf q(x) - 1}{4(\sup |\nabla q(x)| + 1)}, 1 \right\}.$$

By computation

$$-\Delta_{p(x)}\phi_1 = \begin{cases} -k(ke^{kd(x)})^{p(x)-1}\left[(p(x)-1)+\left(d(x)+\frac{\ln k}{k}\right)\nabla p\nabla d+\frac{\Delta d}{k}\right], \quad d(x)<\sigma, \\ \left\{\frac{1}{2\delta-\sigma}\frac{2(p(x)-1)}{p^- -1}-\left(\frac{2\delta-d}{2\delta-\sigma}\right)\left[\left(\ln ke^{k\sigma}\left(\frac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2}{p^- -1}}\right)\nabla p\nabla d+\Delta d\right]\right\} \\ \times(ke^{k\sigma})^{p(x)-1}\left(\frac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2(p(x)-1)}{p^- -1}-1}, \ \sigma<d(x)<2\delta, \\ 0, \quad 2\delta<d(x). \end{cases}$$

From (H_3) and (H_4), there exists a positive constant $M>1$ such that

$$f(M-1)\geq 1, \ h(M-1)\geq 1, \ a(M-1)\geq 1, \ b(M-1)\geq 1.$$

Let $\sigma=\frac{1}{k}\ln M$, then

$$\sigma k=\ln M. \tag{2}$$

If k is sufficiently large, from (2), we have

$$-\Delta_{p(x)}\phi_1 \leq -k^{p(x)}\alpha, \quad d(x)<\sigma. \tag{3}$$

Let $k\alpha=(\lambda_1 A_1+\mu_1 B_1)$, then

$$k^{p(x)}\alpha \geq -(\lambda_1^{p(x)}A_1+\mu_1^{p(x)}B_1)$$

from (3), then we have

$$-\Delta_{p(x)}\phi_1 \leq \lambda_1^{p(x)}A_1+\mu_1^{p(x)}B_1 \leq \lambda_1^{p(x)}g(x)a(\phi_1)+\mu_1^{p(x)}c(x)f(\phi_2), \quad d(x)<\sigma. \tag{4}$$

Since $d(x)\in C^2(\overline{\partial\Omega_{3\delta}})$, then there exists a positive constant C_3 such that

$$\begin{aligned} -\Delta_{p(x)}\phi_1 \leq{}& (ke^{k\sigma})^{p(x)-1}\left(\frac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2(p(x)-1)}{p^- -1}-1} \\ &\cdot\left|\left\{\frac{2(p(x)-1)}{(2\delta-\sigma)(p^- -1)}-\left(\frac{2\delta-d}{2\delta-\sigma}\right)\left[\left(\ln ke^{k\sigma}\left(\frac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2}{p^- -1}}\right)\nabla p\nabla d+\Delta d\right]\right\}\right| \\ \leq{}& C_3(ke^{k\sigma})^{p(x)-1}\ln k, \quad \sigma<d(x)<2\delta. \end{aligned}$$

If k is sufficiently large, let $k\alpha=(\lambda_1 A_1+\mu_1 B_1)$, we have

$$C_3(ke^{k\sigma})^{p(x)-1}\ln k = C_3(kM)^{p(x)-1}\ln k \leq \lambda_1^{p(x)}A_1+\mu_1^{p(x)}B_1.$$

then

$$-\Delta_{p(x)}\phi_1 \leq \lambda_1^{p(x)}A_1+\mu_1^{p(x)}B_1, \quad \sigma<d(x)<2\delta.$$

Since $\phi_1(x)\geq 0$ and a , f are nondecreasing, then we have

$$-\Delta_{p(x)}\phi_1 \leq \lambda_1^{p(x)}g(x)a(\phi_1)+\mu_1^{p(x)}c(x)f(\phi_2), \quad \sigma<d(x)<2\delta. \tag{5}$$

Obviously

$$-\Delta_{p(x)}\phi_1 = 0 \le \lambda_1^{p(x)}A_1 + \mu_1^{p(x)}B_1 \le \lambda_1^{p(x)}g(x)a(\phi_1) + \mu_1^{p(x)}c(x)f(\phi_2), \quad 2\delta < d(x). \tag{6}$$

Combining (4), (5) and (6), we can conclude that

$$-\Delta_{p(x)}\phi_1 \le \lambda_1^{p(x)}g(x)a(\phi_1) + \mu_1^{p(x)}c(x)f(\phi_2), \quad a.e.\ on\ \Omega. \tag{7}$$

Similarly

$$-\Delta_{q(x)}\phi_2 \le \lambda_2^{q(x)}g(x)b(\phi_2) + \mu_2^{q(x)}c(x)h(\phi_1), \quad a.e.\ on\ \Omega. \tag{8}$$

From (7) and (8), we can see that (ϕ_1, ϕ_2) is a subsolution of (P).
Step 2. We construct a supersolution of (P).
We consider

$$\begin{cases} -\Delta_{p(x)}z_1 = (\lambda_1^{p^+}A_2 + \mu_1^{p^+}B_2)\mu & \text{in } \Omega, \\ -\Delta_{q(x)}z_2 = (\lambda_2^{q^+}A_2 + \mu_2^{q^+}B_2)h\Big(\beta\Big[(\lambda_1^{p^+}A_2 + \mu_1^{p^+}B_2)\mu\Big]\Big) & \text{in } \Omega, \\ z_1 = z_2 = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\beta = \beta((\lambda_1^{p^+}A_2 + \mu_1^{p^+}B_2)\mu) = \max_{x\in\bar{\Omega}} z_1(x)$. We shall prove that (z_1, z_2) is a supersolution for (P).
From Lemma 2, we have

$$\max_{x\in\bar{\Omega}} z_1(x) \le C_2\Big[(\lambda_1^{p^+}A_2 + \mu_1^{p^+}B_2)\mu\Big]^{\frac{1}{p^- - 1}}$$

and

$$\max_{x\in\bar{\Omega}} z_2(x) \le C_2\Big(\lambda_2^{q^+}A_2 + \mu_2^{q^+}B_2\Big)h\Big(\Big[(\lambda_1^{p^+}A_2 + \mu_1^{p^+}B_2)\mu\Big]\Big)^{\frac{1}{q^- - 1}}$$

For $\psi \in W_0^{1,q(x)}(\Omega)$ with $\psi \ge 0$, it is easy to see that

$$\int_\Omega |\nabla z_2|^{q(x)-2}\nabla z_2 \cdot \nabla\psi dx \ge \int_\Omega \lambda_2^{q^+}A_2 h\Big(\beta\Big[(\lambda_1^{p^+}A_2 + \mu_1^{p^+}B_2)\mu\Big]\Big)\psi dx + \int_\Omega \mu_2^{q^+}B_2 h(z_1)\psi dx. \tag{9}$$

Since $\lim_{u\to+\infty} \dfrac{f[M(h(u))^{\frac{1}{(p^- - 1)}}]}{u^{p^- - 1}} = 0$, when μ is sufficiently large, combining Lemma 2 and (H_6), then we have

$$h\Big(\beta\Big[(\lambda_1^{p^+}A_2 + \mu_1^{p^+}B_2)\mu\Big]\Big) \ge b(z_2) \tag{10}$$

Hence

$$\int_\Omega |\nabla z_2|^{q(x)-2}\nabla z_2 \cdot \nabla\psi dx \ge \int_\Omega \lambda_2^{q^+}g(x)b(z_2)\psi dx + \int_\Omega \mu_2^{q^+}c(x)h(z_1)\psi dx. \tag{11}$$

Also

$$\int_\Omega |\nabla z_1|^{p(x)-2}\nabla z_1 \cdot \nabla\varphi dx = \int_\Omega (\lambda_1^{p^+} A_2 + \mu_1^{p^+} B_2)\mu\varphi dx$$

By $(H_4), (H_6)$, when μ is sufficiently large, combining Lemma 2 and (H_6), we have

$$(\lambda_1^{p^+} A_2 + \mu_1^{p^+} B_2)\mu \geq \Big(\frac{1}{C_2}\beta\Big[(\lambda_1^{p^+} A_2 + \mu_1^{p^+} B_2)\mu\Big]\Big)^{p^- -1} \geq \lambda_1^{p^+} A_2 a(z_1) + \mu_1^{p^+} B_2 f(z_2)$$

Then

$$\int_\Omega |\nabla z_1|^{p(x)-2}\nabla z_1 \cdot \nabla\varphi dx \geq \int_\Omega \lambda_1^{p^+} g(x)a(z_1)\varphi dx + \int_\Omega \mu_1^{p^+} c(x)f(z_2)\varphi dx. \tag{12}$$

According to (11) and (12), we can conclude that (z_1, z_2) is a supersolution for (P).
It only remains to prove that $\phi_1 \leq z_1$ and $\phi_2 \leq z_2$.
In the definition of $v_1(x)$, let $\gamma = \frac{2}{\delta}(\max_{x\in\bar\Omega}\phi(x) + \max_{x\in\bar\Omega}|\nabla_1\phi(x)|)$. We claim that

$$\phi_1(x) \leq v_1(x), \quad \forall x \in \Omega. \tag{13}$$

From the definition of v_1, it is easy to see that

$$\phi_1(x) \leq 2\max_{x\in\bar\Omega}\phi_1(x) \leq v_1(x), \quad when\ d(x) = \delta,$$

and

$$\phi_1(x) \leq 2\max_{x\in\bar\Omega}\phi_1(x) \leq v_1(x), \quad when\ d(x) \geq \delta.$$

It only remains to prove that

$$\phi_1(x) \leq v_1(x), \quad when\ d(x) < \delta.$$

Since $v_1 - \phi_1 \in C^1(\overline{\partial\Omega_\delta})$, then there exists a point $x_0 \in \overline{\partial\Omega_\delta}$ such that

$$v_1(x_0) - \phi_1(x_0) = \min_{x_0\in\overline{\partial\Omega_\delta}}[v_1(x) - \phi_1(x)].$$

If $v_1(x_0) - \phi_1(x_0) < 0$, it is easy to see that $0 < d(x_0) < \delta$, and then

$$\nabla v_1(x_0) - \nabla\phi_1(x_0) = 0.$$

From the definition of v_1, we have

$$|\nabla v_1(x_0)| = \gamma = \frac{2}{\delta}(\max_{x\in\bar\Omega}\phi_1(x) + \max_{x\in\bar\Omega}|\nabla\phi_1(x)|) > |\nabla\phi_1(x_0)|.$$

It is a contradiction to $\nabla v_1(x_0) - \nabla\phi_1(x_0) = 0$. Thus (13) is valid.
Obviously, there exists a positive constant C_4 such that

$$\gamma \leq C_4(\lambda_1 + \mu_1).$$

Since $d(x) \in C^2(\overline{\partial\Omega_{3\delta}})$, according to the proof of Lemma 2, then there exist positive constant C_5, C_6 such that

$$-\Delta_{p(x)} v_1(x) \le C_* \gamma^{p(x)-1+\theta} \le C_5 \lambda^{p(x)-1+\theta} + C_6 \mu_1^{p(x)-1+\theta}, \quad a.e.\ in\ \Omega,\ where\ \theta \in (0,1).$$

When $\eta \ge \lambda_1^{p^+} + \mu_1^{p^+}$ is large enough, we have

$$-\Delta_{p(x)} v_1(x) \le \eta.$$

According to the comparison principle, we have

$$v_1(x) \le w(x), \quad \forall x \in \Omega. \tag{14}$$

From (13) and (14), when $\eta \ge \lambda_1^{p^+} + \mu_1^{p^+}$ is sufficiently large, we have

$$\phi_1(x) \le v_1(x) \le w(x), \quad \forall x \in \Omega. \tag{15}$$

According to the comparison principle, when μ is large enough, we have

$$v_1(x) \le w(x) \le z_1(x), \quad \forall x \in \Omega.$$

Combining the definition of $v_1(x)$ and (15), it is easy to see that

$$\phi_1(x) \le v_1(x) \le w(x) \le z_1(x), \quad \forall x \in \Omega.$$

When $\mu \ge 1$ and $\lambda_1 + \mu_1$ is large enough, from Lemma 2, we can see that $\beta[(\lambda_1^{p^+} A_2 + \mu_1^{p^+} B_2)\mu]$ is large enough, then $(\lambda_1^{p^+} A_2 + \mu_1^{p^+} B_2) h\Big(\beta\Big[(\lambda_1^{p^+} A_2 + \mu_1^{p^+} B_2)\mu\Big]\Big)$ is large enough. Similarly, we have $\phi_2 \le z_2$.
This completes the proof. $\square$

3 Asymptotic behavior of positive solutions

In this section, when parameters $\lambda_1, \mu_1, \lambda_2, \mu_2 \to +\infty$, we will discuss the asymptotic behavior of maximum of solutions about parameters $\lambda_1, \mu_1, \lambda_2, \mu_2$ and the asymptotic behavior of solutions near boundary about parameters $\lambda_1, \mu_1, \lambda_2, \mu_2$.

Theorem 2. *On the conditions of $(H_1) - (H_6)$, if (u, v) is a solution of (P) which has been given in Theorem 1, then*

(i) There exist positive constants C_1 and C_2 such that

$$C_1(\lambda_1 + \mu_1) \le \max_{x \in \overline{\Omega}} u(x) \le C_2 \Big((\lambda_1^{p^+} A_2 + \mu_1^{p^+} B_2)\mu\Big)^{\frac{1}{p^- - 1}} \tag{16}$$

$$C_1(\lambda_1 + \mu_1) \le \max_{x \in \overline{\Omega}} v(x) \le C_2 \Big\{(\lambda_2^{q^+} A_2 + \mu_2^{q^+} B_2) h\Big(C_2 \Big[\lambda_1^{p^+} A_2 + \mu_1^{p^+} B_2)\mu\Big]^{\frac{1}{q^- - 1}}\Big)\Big\}^{\frac{1}{q^- - 1}} \tag{17}$$

(ii) for any $\theta \in (0,1)$, there exist positive constants C_3 and C_4 such that

$$C_3(\lambda_1+\mu_1)d(x) \le u(x) \le C_4\Big((\lambda_1^{p^+}A_2+\mu_1^{p^+}B_2)\mu\Big)^{\frac{1}{p^--1}}(d(x))^{\theta}, \quad \textit{as } d(x)\to 0, \tag{18}$$

$$C_3(\lambda_1+\mu_1)d(x) \le v(x) \le C_4\Big\{(\lambda_2^{q^+}A_2+\mu_2^{q^+}B_2)h\Big(C_2\Big[(\lambda_1^{p^+}A_2+\mu_1^{p^+}B_2)\mu\Big]^{\frac{1}{q^--1}}\Big)\Big\}^{\frac{1}{q^--1}}(d(x))^{\theta}, \tag{19}$$

as $d(x)\to 0$,

where μ satisfies (10).

Proof: (i) Obvipusly, when $2\delta \le d(x)$, we have

$$\begin{aligned} u(x) &\ge \phi_1(x) = e^{k\sigma}-1+\int_\sigma^{2\delta} ke^{k\sigma}\Big(\frac{2\delta-t}{2\delta-\sigma}\Big)^{\frac{2}{p^--1}} \ge \frac{(\lambda_1A_1+\mu_1B_1)}{\alpha}\int_\sigma^{2\delta} M\Big(\frac{2\delta-t}{2\delta-\sigma}\Big)^{\frac{2}{p^--1}}dt \\ v(x) &\ge \phi_2(x) = e^{k\sigma}-1+\int_\sigma^{2\delta} ke^{k\sigma}\Big(\frac{2\delta-t}{2\delta-\sigma}\Big)^{\frac{2}{q^--1}} \ge \frac{(\lambda_1A_1+\mu_1B_1)}{\alpha}\int_\sigma^{2\delta} M\Big(\frac{2\delta-t}{2\delta-\sigma}\Big)^{\frac{2}{q^--1}}dt \end{aligned}$$

then there exists a positive constant C_1 such that

$$C_1(\lambda_1+\mu_1) \le \max_{x\in\bar\Omega} u(x) \quad and \quad C_1(\lambda_1+\mu_1) \le \max_{x\in\bar\Omega} v(x)$$

It is easy to see

$$u(x) \le z_1(x) \le \max_{x\in\bar\Omega} z_1(x) \le C_2\Big((\lambda_1^{p^+}A_2+\mu_1^{p^+}B_2)\mu\Big)^{\frac{1}{p^--1}}$$

then

$$\max_{x\in\bar\Omega} u(x) \le C_2\Big((\lambda_1^{p^+}A_2+\mu_1^{p^+}B_2)\mu\Big)^{\frac{1}{p^--1}}$$

Similarly

$$\max_{x\in\bar\Omega} v(x) \le C_2\Big\{(\lambda_2^{q^+}A_2+\mu_2^{q^+}B_2)h\Big(C_2\Big[(\lambda_1^{p^+}A_2+\mu_1^{p^+}B_2)\mu\Big]^{\frac{1}{q^--1}}\Big)\Big\}^{\frac{1}{q^--1}}$$

Thus (16) and (17) are valid.
(ii) Denote

$$v_3(x) = \alpha(d(x))^{\theta}, \quad d(x) \le \rho,$$

where $\theta \in (0,1)$ is a positive constant, $\rho \in (0,\delta)$ is small enough. Obviously, $v_3(x) \in C^1(\Omega_\rho)$. By computation

$$-\Delta_{p(x)}v_3(x) = -(\alpha\theta)^{p(x)-1}(\theta-1)(p(x)-1)(d(x))^{(\theta-1)(p(x)-1)-1}(1+\Pi(x)), \quad d(x)<\rho,$$

where

$$\Pi(x) = d\frac{(\nabla p\nabla d)\ln\alpha\theta}{(\theta-1)(p(x)-1)} + d\frac{(\nabla p\nabla d)\ln d}{(p(x)-1)} + d\frac{\Delta d}{(\theta-1)(p(x)-1)}.$$

Let $\alpha = \frac{1}{\rho}C_2\Big((\lambda_1^{p^+}A_2 + \mu_1^{p^+}B_2)\mu\Big)^{\frac{1}{p^- -1}}$, where $\rho > 0$ is small enough, it is easy to see that

$$(\alpha)^{p(x)-1} \geq (\lambda_1^{p^+}A_2 + \mu_1^{p^+}B_2)\mu \quad and \quad |\Pi(x)| \leq \frac{1}{2}.$$

when $\rho > 0$ is small enough, then we have

$$-\Delta_{p(x)}v_3(x) \geq (\lambda_1^{p^+}A_2 + \mu_1^{p^+}B_2)\mu.$$

Obviously $v_3(x) \geq z_1(x)$ on $\partial\Omega_\rho$. According to the comparison principle, we have $v_3(x) \geq z_1(x)$ on Ω_ρ. Thus

$$u(x) \leq C_4((\lambda_1^{p^+}A_2 + \mu_1^{p^+}B_2)\mu)^{\frac{1}{p^- -1}}(d(x))^\theta, \quad as \quad d(x) \to 0.$$

Let $\alpha = \dfrac{1}{\rho}C_2\Big\{(\lambda_2^{q^+}A_2 + \mu_2^{q^+}B_2)h\Big(C_2\Big[(\lambda_1^{p^+}A_2 + \mu_1^{p^+}B_2)\mu\Big]^{\frac{1}{q^- -1}}\Big)\Big\}^{\frac{1}{q^- -1}}$, when $\rho > 0$ is small enough, it is easy to see that

$$(\alpha)^{p(x)-1} \geq (\lambda_2^{q^+}A_2 + \mu_2^{q^+}B_2)h\Big(C_2\Big[(\lambda_1^{p^+} + \mu_1^{p^+}B_2)\mu\Big]^{\frac{1}{q^- -1}}\Big).$$

Similarly, when $\rho > 0$ is small enough, we have

$$v(x) \leq C_4\Big\{(\lambda_2^{q^+}A_2 + \mu_2^{q^+}B_2)h\Big(C_2\Big[(\lambda_1^{p^+}A_2 + \mu_1^{p^+}B_2)\mu\Big]^{\frac{1}{q^- -1}}\Big)\Big\}^{\frac{1}{q^- -1}} \qquad as\ d(x) \to 0$$

Obviously, when $d(x) < \sigma$, we have

$$u(x) \geq \phi_1(x) = e^{kd(x)} - 1 \geq C_3(\lambda_1 + \mu_1)d(x).$$

$$v(x) \geq \phi_2(x) = e^{kd(x)} - 1 \geq C_3(\lambda_1 + \mu_1)d(x).$$

Thus (18) and (19) are valid. This completes the proof. □

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