Existence of positive solutions for \((p(x), q(x))\) Laplacian system

by

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Abstract

We consider the system of differential equations

\[
(P) \begin{cases} 
-\Delta_{p(x)} u = \lambda_1^{p(x)} g(x) a(u) + \mu_1^{p(x)} c(x) f(v) & \text{in } \Omega, \\
-\Delta_{q(x)} v = \lambda_2^{q(x)} g(x) b(v) + \mu_2^{q(x)} c(x) h(u) & \text{in } \Omega, \\
u = v = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \(\Omega \subset \mathbb{R}^N\) is a bounded domain with \(C^2\) boundary \(\partial \Omega, 1 < p(x), q(x) \in C^1(\overline{\Omega})\) are functions, the operator \(\Delta_{p(x)} u = \text{div}(\nabla u |p(x)|^{-2} \nabla u)\) is called \(p(x)\)-Laplacian \(\lambda_1, \lambda_2, \mu_1\) and \(\mu_2\) are positive parameters and \(g, c\) are continuous functions and \(f, h, a, b\) are \(C^1\) nondecreasing functions satisfying \(f(0), h(0), a(0), b(0) \geq 0\). We discuss the existence of positive solution via sub-super solutions.

Key Words: Positive solutions; \(p(x)\)-Laplacian Problems; sub-supersolution.

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1 Introduction

The study of differential equations and variational problems with variable exponent has been a new and interesting topic. It arises from nonlinear elasticity theory, electrorheological fluids, etc. (see [3, 10, 13]). Many results have been obtained on this kind of problems, for example [1, 3, 4, 5, 6, 9]. In [2], the authors discussed the existence of at least one positive solution of the system

\[
(I) \begin{cases} 
-\Delta_{p(x)} u = \lambda^{p(x)} F(x, u, v) & \text{in } \Omega, \\
-\Delta_{p(x)} v = \lambda^{p(x)} G(x, u, v) & \text{in } \Omega, \\
u = v = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \(p(x) \in C^1(\overline{\Omega})\) is a function, \(F(x, u, v) = [g(x) a(u) + f(v)], G(x, u, v) = [g(x) b(v) + h(u)]\), \(\lambda\) is a positive parameter and \(\Omega \subset \mathbb{R}^N\) is a bounded domain. But in the present paper we extend the problem (I) to problem (P). In this paper, we mainly consider the existence of positive weak
solutions for the problem (P) and we have proved the existence of at least one positive solution for the problem (P).

To study $p(x)$-Laplacian problems, we need some theory on the spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and properties of $p(x)$-Laplacian which we will use later (see [5, 11]). If $\Omega \subset \mathbb{R}^N$ is an open domain, we write

\[ C_{+}(\Omega) = \{ h : h \in C(\Omega), h(x) > 1 \text{ for } x \in \Omega \} \]

\[ h^+ = \sup_{x \in \Omega} h(x), \quad h^- = \inf_{x \in \Omega} h(x), \quad \text{for any } h \in C(\Omega). \]

Throughout the paper, we will assume that:

(H1) $\Omega \subset \mathbb{R}^N$ is an open bounded domain with $C^2$ boundary $\partial \Omega$.

(H2) $p(x), q(x) \in C^1(\Omega)$ are functions and $1 < p^- \leq p^+ \leq 1 < q^- \leq q^+$.

(H3) $a, b, f, h : [0, \infty] \to \mathbb{R}$ are nondecreasing functions such that $f(0), h(0), a(0), b(0) \geq 0$ and $\lim_{u \to +\infty} a(u) = \lim_{u \to +\infty} b(u) = lim_{u \to +\infty} f(u) = lim_{u \to +\infty} h(u) = +\infty$.

(H4) $\lim_{u \to +\infty} \frac{f[M(h(u))]^{(p^-)-1}}{u^{p^-}-1} = 0$, \quad $\forall M > 0$.

(H5) $g, c : \bar{\Omega} \to [1, \infty]$ are continuous functions such that

\[ A_1 = \min_{x \in \Omega} g(x), \quad A_2 = \max_{x \in \Omega} g(x), \quad B_1 = \min_{x \in \Omega} c(x), \quad B_2 = \max_{x \in \Omega} c(x). \]

(H6) $\lim_{u \to +\infty} \frac{a(u)}{u^{p^-}-1} = 0$, \quad $\lim_{u \to +\infty} \frac{b(u)}{u^{q^-}-1} = 0$.

**Definition 1.** If $(u, v) \in \left( W^{1,p(x)}_0(\Omega), W^{1,q(x)}_0(\Omega) \right)$, $(u, v)$ is called a weak solution of (P) if it satisfies

\[
\begin{cases}
\int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi dx = \int_{\Omega} \lambda_1^{p(x)} g(x)u(a(x) + \mu_1^{p(x)} c(x) f(v)) \varphi dx, & \forall \varphi \in W^{1,p(x)}_0(\Omega), \\
\int_{\Omega} |\nabla v|^{q(x)-2} \nabla v \cdot \nabla \psi dx = \int_{\Omega} \lambda_2^{q(x)} g(x)b(v) + \mu_2^{q(x)} c(x) h(u) \psi dx, & \forall \psi \in W^{1,q(x)}_0(\Omega).
\end{cases}
\]

**Lemma 1.** (Comparison Principle).

Let $u, v \in W^{1,p(x)}(\Omega)$ satisfying $Au - Av \geq 0$ in $(W^{1,p(x)}_0(\Omega))^\ast$, $\varphi(x) = \min\{u(x) - v(x), 0\}$. If $\varphi(x) \in W^{1,p(x)}_0(\Omega)$ (i.e. $u \geq v$ on $\partial \Omega$), then $u \geq v$ a.e. in $\Omega$.

Here and hereafter, we will use the notation $d(x, \partial \Omega)$ to denote the distance of $x \in \Omega$ to the boundary of $\Omega$.

Denote $d(x) = d(x, \partial \Omega)$ and $\partial \Omega_\epsilon = \{ x \in \Omega \mid d(x, \partial \Omega) < \epsilon \}$. Since $\partial \Omega$ is $C^2$ regularly, then there exists a constant $\delta \in (0, 1)$ such that $d(x) \in C^2(\partial \Omega_\delta)$, and $|\nabla d(x)| \equiv 1$.

Denote

\[ v_1(x) = \begin{cases} \gamma d(x), & d(x) < \delta, \\
\gamma \delta + \int_0^{d(x)} \left( \frac{2\delta - t}{\delta} \right)^{p^- - 1} (\lambda_1^{p^+} A_1 + \mu_1^{p^+} B_1) \frac{2}{d(x) - t} dt, & \delta \leq d(x) < 2\delta, \\
\gamma \delta + \int_0^{2\delta} \left( \frac{2\delta - t}{\delta} \right)^{p^- - 1} (\lambda_1^{p^+} A_1 + \mu_1^{p^+} B_1) \frac{2}{d(x) - t} dt, & 2\delta \leq d(x). \end{cases} \]
Existence of positive solutions

Obviously, $0 \leq v_1(x) \in C^1(\bar{\Omega})$. Considering

$$-\Delta_{p(x)} w(x) = \eta \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \partial \Omega, \quad (1)$$

we have the following result

**Lemma 2.** (see [7]). If positive parameter $\eta$ is large enough and $w$ is the unique solution of (1), then we have

(i) For any $\theta \in (0, 1)$ there exists a positive constant $C_1$ such that

$$C_1 \eta^{\frac{1}{p-1+\theta}} \leq \max_{x \in \Omega} w(x);$$

(ii) There exists a positive constant $C_2$ such that

$$\max_{x \in \Omega} w(x) \leq C_2 \eta^{\frac{1}{p-1}}$$

2 Existence results

In the following, when there be no misunderstanding, we always use $C_i$ to denote positive constants.

**Theorem 1.** On the conditions of $(H_1)-(H_6)$, then (P) has a positive solution.

**Proof:** We shall establish Theorem 1 by constructing a positive subsolution $(\Phi_1, \Phi_2)$ and supersolution $(z_1, z_2)$ of (P), such that $\Phi_1 \leq z_1$ and $\Phi_2 \leq z_2$.

According to the sub-supersolution method for $p(x)$-Laplacian equations (see [8]), then (P) has a positive solution.

Step 1. We construct a subsolution of (P).

Let $\sigma \in (0, \delta)$ is small enough. Denote

$$\phi_1(x) = \begin{cases} e^{kd(x)} - 1, & d(x) < \sigma, \\ e^{k\sigma} - 1 + \int_\sigma^{d(x)} ke^{k\sigma}(\frac{2\delta - t}{2\delta - \sigma})^{\frac{q-1}{q}} dt, & \sigma \leq d(x) < 2\delta, \\ e^{k\sigma} - 1 + \int_\sigma^{2\delta} ke^{k\sigma}(\frac{2\delta - t}{2\delta - \sigma})^{\frac{q-1}{q}} dt, & 2\delta \leq d(x). \end{cases}$$

$$\phi_2(x) = \begin{cases} e^{kd(x)} - 1, & d(x) < \sigma, \\ e^{k\sigma} - 1 + \int_\sigma^{d(x)} ke^{k\sigma}(\frac{2\delta - t}{2\delta - \sigma})^{\frac{q-1}{q}} dt, & \sigma \leq d(x) < 2\delta, \\ e^{k\sigma} - 1 + \int_\sigma^{2\delta} ke^{k\sigma}(\frac{2\delta - t}{2\delta - \sigma})^{\frac{q-1}{q}} dt, & 2\delta \leq d(x). \end{cases}$$

It is easy to see that $\phi_1, \phi_2 \in C^1(\bar{\Omega})$. Denote

$$\alpha = \min \left\{ \frac{\inf p(x) - 1}{4(<\sup \left| \nabla p(x) \right| + 1)}, \frac{\inf q(x) - 1}{4(<\sup \left| \nabla q(x) \right| + 1)} \right\}.$$
By computation

\[-\Delta_{p(x)} \phi_1 = \begin{cases} 
-k(ke^{k \alpha} p(x))^{-1} [(p(x) - 1) + (d(x) + \frac{\ln k}{c}) H_p \nabla d + \frac{\Delta d}{c}], & d(x) < \sigma, \\
\left\{ \frac{1}{2 \delta - \sigma} \right\} 2(p(x) - 1) - \left( \frac{2 \delta - d}{2 \delta - \sigma} \right) \left[ \ln k e^{k \alpha} \left( \frac{2 \delta - d}{2 \delta - \sigma} \right)^{\frac{2}{p-1} - 1} \right] \nabla_p \nabla d + \Delta d \right\}, \\
0, & 2 \delta < d(x). 
\end{cases}\]

From (H3) and (H4), there exists a positive constant \( M > 1 \) such that

\[ f(M - 1) \geq 1, \ h(M - 1) \geq 1, \ a(M - 1) \geq 1, \ b(M - 1) \geq 1. \]

Let \( \sigma = \frac{1}{c} \ln M \), then

\[ \sigma = \ln M. \] (2)

If \( k \) is sufficiently large, from (2), we have

\[ -\Delta_{p(x)} \phi_1 \leq -k^{p(x)} \alpha, \quad d(x) < \sigma. \] (3)

Let \( k \alpha = (\lambda_1 A_1 + \mu_1 B_1) \), then

\[ k^{p(x)} \alpha \geq -(\lambda_1^{p(x)} A_1 + \mu_1^{p(x)} B_1) \]

from (3), then we have

\[ -\Delta_{p(x)} \phi_1 \leq \lambda_1^{p(x)} A_1 + \mu_1^{p(x)} B_1 \leq \lambda_1^{p(x)} g(x) a(\phi_1) + \mu_1^{p(x)} c(x) f(\phi_2), \quad d(x) < \sigma. \] (4)

Since \( d(x) \in C^2(\overline{D_{2 \delta}}) \), then there exists a positive constant \( C_3 \) such that

\[ -\Delta_{p(x)} \phi_1 \leq (ke^{k \alpha} p(x))^{-1} \left( \frac{2 \delta - d}{2 \delta - \sigma} \right)^{\frac{2}{p-1} - 1} \left\{ \left[ \left( \frac{2(p(x) - 1)}{2 \delta - \sigma} \right) - \left( \frac{2 \delta - d}{2 \delta - \sigma} \right) \left[ \ln k e^{k \alpha} \left( \frac{2 \delta - d}{2 \delta - \sigma} \right)^{\frac{2}{p-1} - 1} \right] \nabla_p \nabla d + \Delta d \right\} \right\} \]

\[ \leq C_3(k e^{k \alpha} p(x))^{-1} \ln k, \quad \sigma < d(x) < 2 \delta. \]

If \( k \) is sufficiently large, let \( k \alpha = (\lambda_1 A_1 + \mu_1 B_1) \), we have

\[ C_3(k e^{k \alpha} p(x))^{-1} \ln k = C_3(k M)^{p(x) - 1} \ln k \leq \lambda_1^{p(x)} A_1 + \mu_1^{p(x)} B_1. \]

then

\[ -\Delta_{p(x)} \phi_1 \leq \lambda_1^{p(x)} A_1 + \mu_1^{p(x)} B_1, \quad \sigma < d(x) < 2 \delta. \]

Since \( \phi_1(x) \geq 0 \) and \( a, f \) are nondecreasing, then we have

\[ -\Delta_{p(x)} \phi_1 \leq \lambda_1^{p(x)} g(x) a(\phi_1) + \mu_1^{p(x)} c(x) f(\phi_2), \quad \sigma < d(x) < 2 \delta. \] (5)
Existence of positive solutions

Obviously

\[-\Delta^{p(x)} \phi_1 = 0 \leq \lambda_1^{p(x)} A_1 + \mu_1^{p(x)} B_1 \leq \lambda_1^{p(x)} g(x) a(\phi_1) + \mu_1^{p(x)} c(x) f(\phi_2), \quad 2\delta < d(x).\]  \hspace{1cm} (6)

Combining (4), (5) and (6), we can conclude that

\[-\Delta^{p(x)} \phi_1 \leq \lambda_1^{p(x)} g(x) a(\phi_1) + \mu_1^{p(x)} c(x) f(\phi_2), \quad \text{a.e. on } \Omega.\]  \hspace{1cm} (7)

Similarly

\[-\Delta^{q(x)} \phi_2 \leq \lambda_2^{q(x)} g(x) b(\phi_2) + \mu_2^{q(x)} c(x) h(\phi_1), \quad \text{a.e. on } \Omega.\]  \hspace{1cm} (8)

From (7) and (8), we can see that \((\phi_1, \phi_2)\) is a subsolution of \((P)\).

Step 2. We construct a supersolution of \((P)\).

We consider

\[
\begin{align*}
-\Delta^{p(x)} z_1 &= (\lambda_1^{p(x)} A_2 + \mu_1^{p(x)} B_2) \mu, & \text{in } \Omega, \\
-\Delta^{q(x)} z_2 &= (\lambda_2^{q(x)} A_2 + \mu_2^{q(x)} B_2) h \left( \beta \left( \lambda_1^{p(x)} A_2 + \mu_1^{p(x)} B_2 \right) \mu \right), & \text{in } \Omega, \\
z_1 &= z_2 = 0, & \text{on } \partial \Omega,
\end{align*}
\]

where \(\beta = \beta((\lambda_1^{p(x)} A_2 + \mu_1^{p(x)} B_2) \mu) = \max_{x \in \Omega} z_1(x)\). We shall prove that \((z_1, z_2)\) is a supersolution for \((P)\).

From Lemma 2, we have

\[
\max_{x \in \Omega} z_1(x) \leq C_2 \left[ (\lambda_1^{p(x)} A_2 + \mu_1^{p(x)} B_2) \mu \right]^{\frac{1}{p(x)-1}}
\]

and

\[
\max_{x \in \Omega} z_2(x) \leq C_2 \left( \lambda_2^{q(x)} A_2 + \mu_2^{q(x)} B_2 \right) h \left( \left[ (\lambda_1^{p(x)} A_2 + \mu_1^{p(x)} B_2) \mu \right] \right)^{\frac{1}{q(x)-1}}
\]

For \(\psi \in W_0^{1,q(x)}(\Omega)\) with \(\psi \geq 0\), it is easy to see that

\[
\int_{\Omega} |\nabla z_2|^{q(x)-2} \nabla z_2 \cdot \nabla \psi dx \geq \int_{\Omega} \lambda_2^{q(x)} A_2 h \left( \beta \left( (\lambda_1^{p(x)} A_2 + \mu_1^{p(x)} B_2) \mu \right) \right) \psi dx + \int_{\Omega} \mu_2^{q(x)} B_2 h(z_1) \psi dx. \hspace{1cm} (9)
\]

Since \(\lim_{\mu \to +\infty} \frac{f[M(h(u))]^{\frac{1}{u^p-1}}}{u^{p-1}} = 0\), when \(\mu\) is sufficiently large, combining Lemma 2 and \((H_0)\), then we have

\[
h \left( \beta \left( (\lambda_1^{p(x)} A_2 + \mu_1^{p(x)} B_2) \mu \right) \right) \geq b(z_2) \hspace{1cm} (10)
\]

Hence

\[
\int_{\Omega} |\nabla z_2|^{q(x)-2} \nabla z_2 \cdot \nabla \psi dx \geq \int_{\Omega} \lambda_2^{q(x)} g(x) b(z_2) \psi dx + \int_{\Omega} \mu_2^{q(x)} c(x) h(z_1) \psi dx. \hspace{1cm} (11)
\]
Also
\[
\int_{\Omega} |\nabla z_1|^{p(x)-2} \nabla z_1 \cdot \nabla \varphi \, dx = \int_{\Omega} (\lambda_1^{p^+} A_2 + \mu_1^{p^+} B_2) \mu \varphi \, dx
\]

By \((H_4),(H_6)\), when \(\mu\) is sufficiently large, combining Lemma 2 and \((H_6)\), we have
\[
(\lambda_1^{p^+} A_2 + \mu_1^{p^+} B_2) \mu \geq \left( \frac{1}{C_2} \beta \left( (\lambda_1^{p^+} A_2 + \mu_1^{p^+} B_2) \mu \right) \right)^{p^-} \geq \lambda_1^{p^+} A_2 a(z_1) + \mu_1^{p^+} B_2 f(z_2)
\]

Then
\[
\int_{\Omega} |\nabla z_1|^{p(x)-2} \nabla z_1 \cdot \nabla \varphi \, dx \geq \int_{\Omega} \lambda_1^{p^+} g(x) a(z_1) \varphi \, dx + \int_{\Omega} \mu_1^{p^+} c(x) f(z_2) \varphi \, dx. \tag{12}
\]

According to (11) and (12), we can conclude that \((z_1, z_2)\) is a supersolution for \((P)\).

It only remains to prove that \(\phi_1 \leq z_1\) and \(\phi_2 \leq z_2\).

In the definition of \(v_1(x)\), let \(\gamma = \frac{2}{\delta} (\max_{x \in \Omega} |\varphi(x)| + \max_{x \in \Omega} |\nabla_1 \varphi(x)|)\). We claim that
\[
\phi_1(x) \leq v_1(x), \quad \forall x \in \Omega. \tag{13}
\]

From the definition of \(v_1\), it is easy to see that
\[
\phi_1(x) \leq 2 \max_{x \in \Omega} \phi_1(x) \leq v_1(x), \quad \text{when } d(x) = \delta,
\]

and
\[
\phi_1(x) \leq 2 \max_{x \in \Omega} \phi_1(x) \leq v_1(x), \quad \text{when } d(x) \geq \delta.
\]

It only remains to prove that
\[
\phi_1(x) \leq v_1(x), \quad \text{when } d(x) < \delta.
\]

Since \(v_1 - \phi_1 \in C^1(\overline{\Omega \setminus S})\), then there exists a point \(x_0 \in \overline{\Omega \setminus S}\) such that
\[
v_1(x_0) - \phi_1(x_0) = \min_{x_0 \in \overline{\Omega \setminus S}} [v_1(x) - \phi_1(x)].
\]

If \(v_1(x_0) - \phi_1(x_0) < 0\), it is easy to see that \(0 < d(x_0) < \delta\), and then
\[
\nabla v_1(x_0) - \nabla \phi_1(x_0) = 0.
\]

From the definition of \(v_1\), we have
\[
|\nabla v_1(x_0)| = \gamma = \frac{2}{\delta} (\max_{x \in \Omega} \phi_1(x) + \max_{x \in \Omega} |\nabla \phi_1(x)|) > |\nabla \phi_1(x_0)|.
\]

It is a contradiction to \(\nabla v_1(x_0) - \nabla \phi_1(x_0) = 0\). Thus (13) is valid.

Obviously, there exists a positive constant \(C_4\) such that
\[
\gamma \leq C_4 (\lambda_1 + \mu_1).
\]
Existence of positive solutions

Since $d(x) \in C^2(\partial\Omega_{3\delta})$, according to the proof of Lemma 2, then there exist positive constant $C_5, C_6$ such that

$$-\Delta_p v_1(x) \leq C_5 \gamma^{p(x)-1+\theta} \leq C_6 \lambda_1^{p(x)-1+\theta} + C_6 \mu_1^{p(x)-1+\theta}, \quad \text{a.e. in } \Omega, \text{ where } \theta \in (0, 1).$$

When $\eta \geq \lambda_1^p + \mu_1^p$ is large enough, we have

$$-\Delta_p v_1(x) \leq \eta.$$

According to the comparison principle, we have

$$v_1(x) \leq w(x), \quad \forall x \in \Omega. \quad (14)$$

From (13) and (14), when $\eta \geq \lambda_1^p + \mu_1^p$ is sufficiently large, we have

$$\phi_1(x) \leq v_1(x) \leq w(x), \quad \forall x \in \Omega. \quad (15)$$

According to the comparison principle, when $\mu$ is large enough, we have

$$v_1(x) \leq w(x) \leq z_1(x), \quad \forall x \in \Omega.$$

Combining the definition of $v_1(x)$ and (15), it is easy to see that

$$\phi_1(x) \leq v_1(x) \leq w(x) \leq z_1(x), \quad \forall x \in \Omega.$$

When $\mu \geq 1$ and $\lambda_1 + \mu_1$ is large enough, from Lemma 2, we can see that $\beta[(\lambda_1^p A_2 + \mu_1^p B_2) \mu]$ is large enough, then $(\lambda_1^p A_2 + \mu_1^p B_2) h(\beta[(\lambda_1^p A_2 + \mu_1^p B_2) \mu])$ is large enough. Similarly, we have $\phi_2 \leq z_2$.

This completes the proof.

3 Asymptotic behavior of positive solutions

In this section, when parameters $\lambda_1, \mu_1, \lambda_2, \mu_2 \to +\infty$, we will discuss the asymptotic behavior of maximum of solutions about parameters $\lambda_1, \mu_1, \lambda_2, \mu_2$ and the asymptotic behavior of solutions near boundary about parameters $\lambda_1, \mu_1, \lambda_2, \mu_2$.

**Theorem 2.** On the conditions of $(H_1) - (H_6)$, if $(u, v)$ is a solution of $(P)$ which has been given in Theorem 1, then

(i) There exist positive constants $C_1$ and $C_2$ such that

$$C_1(\lambda_1 + \mu_1) \leq \max_{x \in \Omega} u(x) \leq C_2 \left( (\lambda_1^p A_2 + \mu_1^p B_2) \mu \right)^{\frac{1}{p-1}} \quad (16)$$

$$C_1(\lambda_1 + \mu_1) \leq \max_{x \in \Omega} v(x) \leq C_2 \left( (\lambda_2^q A_2 + \mu_2^q B_2) h \left( C_2 \left[ (\lambda_1^p A_2 + \mu_1^p B_2) \mu \right]^{\frac{1}{p-1}} \right) \right)^{\frac{1}{q-1}} \quad (17)$$
(ii) for any $\theta \in (0, 1)$, there exist positive constants $C_3$ and $C_4$ such that

\[ C_3(\lambda_1 + \mu_1) d(x) \leq u(x) \leq C_4 \left( (\lambda_1^{p^+} A_2 + \mu_1^{p^+} B_2) \mu \right)^{\frac{1}{p-1}} (d(x))^\theta, \quad \text{as } d(x) \to 0, \quad (18) \]

\[ C_3(\lambda_1 + \mu_1) d(x) \leq v(x) \leq C_4 \left( (\lambda_2^{q^+} A_2 + \mu_2^{q^+} B_2) h \left( C_2 \left[ (\lambda_1^{p^+} A_2 + \mu_1^{p^+} B_2) \mu \right]^{\frac{1}{p-1}} \right) \right)^{\frac{1}{q-1}} (d(x))^\theta, \quad (19) \]

as $d(x) \to 0$,

where $\mu$ satisfies (10).

**Proof:**  (i) Obviusly, when $2\delta \leq d(x)$, we have

\[ u(x) \geq \phi_1(x) = e^{k\sigma} - 1 + \int_{\sigma}^{2\delta} ke^{k\sigma} \left( \frac{2\delta - t}{2\delta - \sigma} \right)^{\frac{2}{p-1}} \geq \frac{(\lambda_1 A_1 + \mu_1 B_1)}{\alpha} \int_{\sigma}^{2\delta} M \left( \frac{2\delta - t}{2\delta - \sigma} \right)^{\frac{2}{p-1}} dt \]

\[ v(x) \geq \phi_2(x) = e^{k\sigma} - 1 + \int_{\sigma}^{2\delta} ke^{k\sigma} \left( \frac{2\delta - t}{2\delta - \sigma} \right)^{\frac{2}{q-1}} \geq \frac{(\lambda_1 A_1 + \mu_1 B_1)}{\alpha} \int_{\sigma}^{2\delta} M \left( \frac{2\delta - t}{2\delta - \sigma} \right)^{\frac{2}{q-1}} dt \]

then there exists a positive constant $C_1$ such that

\[ C_1(\lambda_1 + \mu_1) \leq \max_{x \in \Omega} u(x) \quad \text{and} \quad C_1(\lambda_1 + \mu_1) \leq \max_{x \in \Omega} v(x) \]

It is easy to see

\[ u(x) \leq z_1(x) \leq \max_{x \in \Omega} z_1(x) \leq C_2 \left( (\lambda_1^{p^+} A_2 + \mu_1^{p^+} B_2) \mu \right)^{\frac{1}{p-1}} \]

then

\[ \max_{x \in \Omega} u(x) \leq C_2 \left( (\lambda_1^{p^+} A_2 + \mu_1^{p^+} B_2) \mu \right)^{\frac{1}{p-1}} \]

Similarly

\[ \max_{x \in \Omega} v(x) \leq C_2 \left( (\lambda_2^{q^+} A_2 + \mu_2^{q^+} B_2) h \left( C_2 \left[ (\lambda_1^{p^+} A_2 + \mu_1^{p^+} B_2) \mu \right]^{\frac{1}{p-1}} \right) \right)^{\frac{1}{q-1}} \]

Thus (16) and (17) are valid.

(ii) Denote

\[ v_3(x) = \alpha (d(x))^\theta, \quad d(x) \leq \rho, \]

where $\theta \in (0, 1)$ is a positive constant, $\rho \in (0, \delta)$ is small enough. Obviously, $v_3(x) \in C^1(\Omega_\rho)$.

By computation

\[ -\Delta_p v_3(x) = -(\alpha \theta)^{p(x)-1}(\theta - 1)(p(x) - 1)(d(x))^{(\theta-1)(p(x)-1)-1}(1 + \Pi(x)), \quad d(x) < \rho, \]

where

\[ \Pi(x) = \frac{d}{\rho} \frac{\nabla p \nabla d \ln \alpha \theta}{(\theta - 1)(p(x) - 1)} + d \frac{\nabla p \nabla d \ln d}{(p(x) - 1)} + d \frac{\Delta d}{(\theta - 1)(p(x) - 1)}. \]
Existence of positive solutions

Let \( \alpha = \frac{1}{\rho} C_2 \left( (\lambda_1^p A_2 + \mu_1^p B_2) \mu \right)^{\frac{1}{p-1}} \), where \( \rho > 0 \) is small enough, it is easy to see that

\[
(\alpha)^{p(x)-1} \geq (\lambda_1^p A_2 + \mu_1^p B_2) \mu \quad \text{and} \quad |\Pi(x)| \leq \frac{1}{2},
\]

when \( \rho > 0 \) is small enough, then we have

\[-\Delta_{p(x)} v_3(x) \geq (\lambda_1^p A_2 + \mu_1^p B_2) \mu.\]

Obviously \( v_3(x) \geq z_1(x) \) on \( \partial \Omega_\rho \). According to the comparison principle, we have \( v_3(x) \geq z_1(x) \) on \( \Omega_\rho \). Thus

\[
u(x) \leq C_4 \left( (\lambda_1^p A_2 + \mu_1^p B_2) \mu \right)^{\frac{1}{q-1}} (d(x))^q, \quad \text{as} \quad d(x) \to 0.
\]

Let \( \alpha = \frac{1}{\rho} C_2 \left( (\lambda_2^q A_2 + \mu_2^q B_2) h \left( C_2 \left[ (\lambda_1^p A_2 + \mu_1^p B_2) \mu \right]^{\frac{1}{p-1}} \right) \right)^{\frac{1}{q-1}} \), when \( \rho > 0 \) is small enough, it is easy to see that

\[(\alpha)^{p(x)-1} \geq (\lambda_2^q A_2 + \mu_2^q B_2) h \left( C_2 \left[ (\lambda_1^p A_2 + \mu_1^p B_2) \mu \right]^{\frac{1}{p-1}} \right)^{\frac{1}{q-1}}.
\]

Similarly, when \( \rho > 0 \) is small enough, we have

\[
v(x) \leq C_4 \left( (\lambda_2^q A_2 + \mu_2^q B_2) h \left( C_2 \left[ (\lambda_1^p A_2 + \mu_1^p B_2) \mu \right]^{\frac{1}{p-1}} \right) \right)^{\frac{1}{q-1}} \quad \text{as} \quad d(x) \to 0.
\]

Obviously, when \( d(x) < \sigma \), we have

\[
u(x) \geq \varphi_1(x) = e^{kd(x)} - 1 \geq C_3 (\lambda_1 + \mu_1) d(x).
\]

\[
u(x) \geq \varphi_2(x) = e^{kd(x)} - 1 \geq C_3 (\lambda_1 + \mu_1) d(x).
\]

Thus (18) and (19) are valid. This completes the proof.

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References


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