Bull. Math. Soc. Sci. Math. Roumanie
Tome 57(105) No. 2, 2014, 163–179

Time-independent Schrödinger polyharmonic equation and applications

by
Imed Bachar

Abstract

We prove that the time-independent Schrödinger polyharmonic equation $(-\Delta)^m u + q(x)u = \psi(x) > 0$, $x \in D$, where D is an unbounded domain of $\mathbb{R}^n$ $(n \geq 2)$ has a positive solution provided that the function q belongs to a certain Kato class of functions $K^{\infty}_{m,n}(D)$. As applications, the existence and asymptotic behavior of positive solutions of some polyharmonic problems are established.

Key Words: Schrödinger polyharmonic equation, Green function, polyharmonic elliptic equation, positive solution.
2010 Mathematics Subject Classification: Primary 34B27, Secondary 35J40.

1 Introduction and statement of main results

Considerable attention has been given to the time-independent Schrödinger equation

$$-\Delta u + q(x)u = \psi(x), \ x \in \Omega \subseteq \mathbb{R}^n, \tag{1.1}$$

where Ω is an open subset of $\mathbb{R}^n$ and the potential q belongs to the Kato class $K^{n,1}_{loc}(\Omega)$. See, e.g., Aizenman and Simon [1], Chiarenza, Fabes and Garofalo [4], Fabes and Strook [9], Hinz and Kalf [14], Simader [17], Zhao [20-22] and the references therein. Following different approaches these authors have studied the existence and regularity of the solutions for the Dirichlet problem. In [12, Theorem 5.1], the authors considered the following Schrödinger polyharmonic equation:

$$(-\Delta)^m u + q(x)u = \psi(x), \ x \in \Omega \subseteq \mathbb{R}^n, \tag{1.2}$$

where Ω is the unit ball in $\mathbb{R}^n$ with $n \geq 1$ and $m \geq 1$. They have proved that if the coefficient q in (1.2) is continuous in $\overline{\Omega}$ and sufficiently small, ψ positive implies that the solution u of the Dirichlet problem for (1.2) is positive. The later result has been extended by the same authors in [13], by considering domains which are close to the unit ball and operators close to $(-\Delta)^m$. On the other hand in [5, Proposition 2.10], the authors studied the equation (1.2) in the case $m = 2$, $n > 4$ and where $\Omega = B(0,r)$ is the open ball of center 0 and radius r.

They have showed that the problem (1.2) subject to either Dirichlet boundary conditions or Navier boundary conditions admits a nonnegative Green function on $B(0,r)$ provided that the function q belongs to the Kato class $K_{loc}^{n,2}(B(0,r))$. For more related results we refer to [10]. For the convenience of the reader, we recall the definition of the functional class $K_{loc}^{n,m}(\Omega)$.

Definition 1.1. [5] *Given $n > 2m$ and Ω be an open subset of $\mathbb{R}^n$. The Kato class $K_{loc}^{n,m}(\Omega)$ is the set of functions $q \in L_{loc}^1(\Omega)$ such that for any compact set $K \subset \Omega$ the quantity*

$$\Phi q(r,K) = \sup_{x\in\mathbb{R}^n} \int_{B(x,r)} \frac{|q(y)|\chi_K(y)}{|x-y|^{n-2m}} dy$$

is finite (here, χ_K denotes the characteristic function of K) and

$$\lim_{r\to 0} \Phi q(r,K) = 0.$$

We emphasize that the proofs presented by these authors are based in the following 3-G Theorem satisfied by be the Green function $G_{m,n}^B$ for the m-polyharmonic operator $u \to (-\Delta)^m u$ with Dirichlet boundary conditions on the unit ball B in $\mathbb{R}^n$.

Theorem 1.2. [12, *Proposition* 4.1] *Given $n > 2m$. There exists a constant $C_{m,n} > 0$ such that for all $x, y, z \in B$,*

$$\frac{G_{m,n}^B(x,z)G_{m,n}^B(z,y)}{G_{m,n}^B(x,y)} \le C_{m,n}[|x-z|^{2m-n} + |z-y|^{2m-n}].$$

In the present paper, we shall prove that similar results remain valid for the equation (1.2) on the unbounded domain $D = \{x \in \mathbb{R}^n : |x| > 1\}$ $(n \ge 2)$, where the function q is assumed to belongs to the Kato class $K_{m,n}^{\infty}(D)$ (see Definition 1.3 below). As application we will answer the questions of existence and asymptotic behavior of positive solutions of some polyharmonic problems of the form:

$$\begin{cases} (-\Delta)^m u + f(.,u) = 0, \text{ in } D \quad \text{(in the sense of distributions)} \\ u > 0, \\ \lim\limits_{x\to\zeta\in\partial D} \frac{u(x)}{\left(|x|^2-1\right)^{m-1}} = \varphi(\zeta), \\ u(x) \simeq \rho_0(x), \text{ near } x = \infty, \end{cases} \tag{1.3}$$

where φ is a nonnegative continuous function on ∂D, m is a positive integer and

$$\rho_0(x) = \begin{cases} 1, & \text{for } n > 2m \\ \ln|x|, & \text{for } n = 2m \\ |x|^{2m-n}, & \text{for } n < 2m. \end{cases} \tag{1.4}$$

The notation $u(x) \simeq \rho_0(x)$, near $x = \infty$, means that for some constant $C > 0$,

$$\frac{1}{C}\rho_0(x) \le u(x) \le C\rho_0(x), \text{ when } x \text{ near } \infty.$$

Throughout this paper, we denote by $G^D_{m,n}$ a Green function of $(-\Delta)^m$ on D with Dirichlet boundary conditions $(\frac{\partial}{\partial\nu})^j u = 0$, $0 \le j \le m-1$.
In [3, Theorem 2.6], the authors proved the following 3-G Theorem: there exists a constant $C_{m,n} > 0$ such that for each $x, y, z \in D$,

$$\frac{G^D_{m,n}(x,z)G^D_{m,n}(z,y)}{G^D_{m,n}(x,y)} \le C_{m,n}\left[\left(\frac{\rho(z)}{\rho(x)}\right)^m G^D_{m,n}(x,z) + \left(\frac{\rho(z)}{\rho(y)}\right)^m G^D_{m,n}(y,z)\right], \tag{1.5}$$

where

$$\rho(z) = \begin{cases} \frac{|z|-1}{|z|} & \text{if } n \ge 2m \\ |z|^{1-\frac{n}{m}}(|z|-1) & \text{if } n < 2m. \end{cases} \tag{1.6}$$

This form of the 3-G Theorem has been exploited to introduce the Kato class $K^\infty_{m,n}(D)$ as follows :

Definition 1.3. *A Borel measurable function q in D belongs to the class $K^\infty_{m,n}(D)$ if q satisfies the following conditions*

$$\lim_{r\to 0}\left(\sup_{x\in D}\int_{D\cap B(x,r)}\left(\frac{\rho(y)}{\rho(x)}\right)^m G^D_{m,n}(x,y)|q(y)|dy\right) = 0, \tag{1.7}$$

$$\lim_{M\to\infty}\left(\sup_{x\in D}\int_{(|y|\ge M)}\left(\frac{\rho(y)}{\rho(x)}\right)^m G^D_{m,n}(x,y)|q(y)|dy\right) = 0, \tag{1.8}$$

where ρ is given by (1.6).

This class contains for example any function belonging to $L^s(D) \cap L^1(D)$ with $s > \frac{n}{2m} > 1$ (see Example 3.1).
We point out that the class $K^\infty_{m,n}(D)$ is well adapted to study various existence and multiplicity results for wide classes of polyharmonic boundary value problems including the case of equations with blow-up at infinity. In the later case, we develop a more careful analysis with respect to other recent papers in this field for $m = 1$ (see, e.g. [6, 11, 16]).
Next we shall often refer in this paper to $h_{m,n}$ the m-harmonic function defined in D by

$$h_{m,n}(x) := |x|^{2m-n} G^B_{m,n}(j(x), 0) = k_{m,n}\int_1^{|x|}\frac{(v^2-1)^{m-1}}{v^{n-1}}dv, \tag{1.9}$$

where $j : D \cup \{\infty\} \to B$ $j(x) = |x|^{-2}x$ is the inversion and $k_{m,n} = \frac{\Gamma(\frac{n}{2})}{2^{2m-1}\pi^{\frac{n}{2}}[(m-1)!]^2}$.
Observe that,

$$h_{m,n}(x) \simeq \rho_0(x), \text{ near } x = \infty. \tag{1.10}$$

We also let $H_D\varphi$ be the bounded continuous solution of the Dirichlet problem

$$\begin{cases} \Delta u = 0, & \text{in } D \\ u = \varphi & \text{on } \partial D \\ \lim\limits_{|x|\to\infty}\frac{u(x)}{h_{1,n}(x)} = 0, \end{cases} \tag{1.11}$$

where φ is a nonnegative nontrivial continuous on ∂D and $h_{1,n}$ is the harmonic function defined by (1.9).
Note that from [8, p.427], the function $H_D\varphi$ belongs to $C\left(\overline{D}\cup\{\infty\}\right)$ and satisfies

$$\lim_{|x|\to\infty} |x|^{n-2} H_D\varphi(x) = c > 0. \tag{1.12}$$

Our plan is organized as follows. In section 2, we will first study the existence and uniqueness of positive classical solution for the linear problem

$$\begin{cases} (-\Delta)^m u = F, \text{ in } D \\ u > 0, \\ \left(-\frac{\partial}{\partial\nu}\right)^j u = 0 \text{ on } \partial D \text{ for } j = 0, ..., m-2, \\ \left(-\frac{\partial}{\partial\nu}\right)^{m-1} u = \phi \text{ on } \partial D, \end{cases}$$

subject to an asymptotic behavior at ∞, where the functions F and ϕ are required to satisfy some convenient hypotheses.

In section 3, we collect some properties of functions belonging to $K^{\infty}_{m,n}(D)$. In particular, we derive from the 3-G Theorem (1.5) that for each $q \in K^{\infty}_{m,n}(D)$, we have

$$\alpha_q := \sup_{x,y\in D}\int_D \frac{G^D_{m,n}(x,z)G^D_{m,n}(z,y)}{G^D_{m,n}(x,y)} |q(z)|\, dz < \infty.$$

Next, we exploit again the inequality (1.5) to prove that on D the inverse of polyharmonic operators that are perturbed by a zero-order term, are positivity preserving. That is, if the coefficient $q \in K^{\infty}_{m,n}(D)$ with $\alpha_q \leq \frac{1}{2}$ and ψ is positive, then the equation

$$(-\Delta)^m u + q(x)\, u = \psi(x),\ x \in D \tag{1.13}$$

has a positive solution.

In section 4, we will establish two existence results for the problem (1.3), where the function f is closed to linear.
More precisely, first we consider the nonlinearity $f(x,t) = tg(x,t)$, we let

$$\omega(x) := h_{m,n}(x) + \left(|x|^2 - 1\right)^{m-1} H_D\varphi(x), \tag{1.14}$$

and we assume that

(H_1) g is a nonnegative measurable function on $D \times [0,\infty)$.
(H_2) For each $\lambda > 0$, there exists a positive function $q_\lambda = q \in K^{\infty}_{m,n}(D)$ with $\alpha_q \leq \frac{1}{2}$ such that for each $x \in D$, the map $t \longrightarrow t\left(q(x) - g(x, t\omega(x))\right)$ is continuous and nondecreasing on $[0,\lambda]$.

Using the pointwise estimates for the Green function and a perturbation arguments, we prove the following.

Theorem 1.4. *Under hypotheses* (H_1)-(H_2), *the problem*

$$\begin{cases} (-\Delta)^m u + ug(.,u) = 0, \text{ in } D \quad (\text{in the sense of distributions}) \\ u > 0, \\ \lim\limits_{x\to\zeta\in\partial D} \frac{u(x)}{\left(|x|^2-1\right)^{m-1}} = \varphi(\zeta), \\ u(x) \simeq \rho_0(x), \text{ near } x = \infty, \end{cases} \tag{1.15}$$

has at least one positive continuous solution u satisfying

$$(1-\alpha_q)\,\omega(x) \le u(x) \le \omega(x).$$

Moreover, for $n \ge 2m$ *we obtain* $\lim\limits_{|x|\to\infty} \frac{u(x)}{h_{m,n}(x)} = 1.$

This result extends Theorem 2 of [15] to the polyharmonic case.

To prove a second existence result for the problem (1.3), we fix a positive harmonic function h_0 in D, which is continuous and bounded in $\overline{D}$, we let

$$\omega_0(x) = h_{m,n}(x) + \left(|x|^2 - 1\right)^{m-1} h_0(x),$$

and we assume that:

(A_1) f is a nonnegative Borel measurable function on $D \times (0,\infty)$, which is continuous with respect to the second variable.

(A_2) There exists a positive function $q \in K^{\infty}_{m,n}(D)$ with $\alpha_q \le \frac{1}{2}$ such that $\forall x \in D$ and $\forall t \ge s \ge \omega_0(x)$ we have

$$\begin{cases} f(x,t) - f(x,s) \le q(x)(t-s) \text{ and} \\ 0 \le f(x,t) \le tq(x). \end{cases}$$

Then we prove the following theorem, which extends Theorem 1.2 of [18] to the polyharmonic case.

Theorem 1.5. *Assume* (A_1)-(A_2), *then there exists a constant* $c_1 > 1$ *such that if* $\widetilde{c} \ge c_1$ *and* $\varphi \ge c_1 h_0$ *on* ∂D, *then problem* (1.3) *has at least one positive continuous solution u satisfying for each* $x \in D$

$$\omega_0(x) \le u(x) \le \omega(x),$$

where $\omega(x) = \widetilde{c} h_{m,n}(x) + \left(|x|^2 - 1\right)^{m-1} H_D\varphi(x)$.
Moreover, for $n \ge 2m$ *we have*

$$\lim_{|x|\to\infty} \frac{u(x)}{h_{m,n}(x)} = \widetilde{c}.$$

A typical example of nonlinearity satisfying (A_1)-(A_2) :

$f(x,t) = p(x)t^{\gamma}$, for $\gamma \in (0,1]$ and some appropriate p admissible.
As usual, we denote by $\mathcal{B}^+(D)$, the set of of nonnegative Borel measurable functions in D.

For $x,y\in\mathbb{R}^n$, we let

$$\begin{cases} [x,y]^2 = |x-y|^2 + \left(|x|^2-1\right)\left(|y|^2-1\right), \\ \text{and} \\ \theta(x,y) = [x,y]^2 - |x-y|^2 = \left(|x|^2-1\right)\left(|y|^2-1\right). \end{cases}$$

For $\psi\in\mathcal{B}^+(D)$, we define

$$V\psi(x) := V_{m,n}\psi(x) = \int_D G^D_{m,n}(x,y)\psi(y)dy, \text{ for } x\in D.$$

and

$$\|\psi\| := \sup_{x\in D}\int_D (\frac{\rho(y)}{\rho(x)})^m G^D_{m,n}(x,y)\psi(y)dy.$$

For a continuous function φ on ∂D we denote by $P\varphi$ the function defined in D by

$$P\varphi(x) = \int_{\partial D} P(x,\xi)\varphi(\xi)\sigma(d\xi),$$

where $P(x,\xi) := \dfrac{|x|^2-1}{|x-\xi|^n}$ is the Poisson kernel on D and σ is the normalized measure on the unit sphere of $\mathbb{R}^n$.
We remark that $P\varphi$ is a harmonic function in D satisfying $\lim\limits_{x\to\xi\in\partial D} P\varphi(x) = \varphi(\xi)$.
We also denote by $\mathfrak{J}$ the set of nonnegative harmonic functions h defined in D by

$$h(x) = \int_{\partial D} P(x,\xi)\nu(d\xi),$$

where ν is a nonnegative measure on ∂D and $P(x,\xi)$ is the Poisson kernel on D.
Let f and g be two nonnegative functions on a set S.

We call $f\preceq g$ on S if and only if there exists a constant $C>0$ such that

$$f(x)\le Cg(x) \quad \text{for all } x\in S.$$

We say $f\simeq g$ on S if and only if there exists a constant $C>0$ such that

$$\frac{1}{C}g(x)\le f(x)\le Cg(x) \quad \text{for all } x\in S.$$

The letter C will denote a generic positive constant which may vary from line to line.

2 The linear boundary value problem

First we consider the polyharmonic prototype Dirichlet problem:

$$\begin{cases} (-\Delta)^m u = f^*, \text{ in } B, \\ u>0, \text{ in } B, \\ \left(-\frac{\partial}{\partial\nu}\right)^j u = 0 \quad \text{on } \partial B \quad \text{for } j=0,...,m-2, \\ \left(-\frac{\partial}{\partial\nu}\right)^{m-1} u = \phi \quad \text{on } \partial B, \end{cases} \tag{2.1}$$

where f^* is a positive function belonging to $C^{0,\gamma}(\overline{B})$ and ϕ is a positive function belonging to $C^{m+1,\gamma}(\partial B)$, for $0<\gamma<1$.

We recall the following existence results which is stated in [10].

Theorem 2.1. [10] *The problem* (2.1) *admits a unique classical solution u given by*

$$u(x)=\int_B G_{m,n}^B(x,y)f^*(y)dy+\int_{\partial B} L_{m,n}^B(x,\xi)\phi(\xi)d\omega(\xi), \quad x\in B \tag{2.2}$$

where the Poisson kernel $L_{m,n}^B$ is defined by

$$L_{m,n}^B(x,\xi)=\frac{\Gamma\left(\frac{n}{2}\right)}{2^m(m-1)!\pi^{\frac{n}{2}}}\frac{\left(1-|x|^2\right)^m}{|x-\xi|^n},\ \textit{with } x\in B,\ \xi\in\partial B. \tag{2.3}$$

Proposition 2.2. *The unique positive solution u of the problem* (2.1) , *satisfies*

$$(1-|x|)^m\preceq u(x)\preceq(1-|x|)^{m-1}, \quad \textit{on } B. \tag{2.4}$$

Proof: Let f^* be a positive function belonging to $C^{0,\gamma}(\overline{B})$ and ϕ be a positive function belonging to $C^{m+1,\gamma}(\partial B)$, for $0<\gamma<1$. It is clear that

$$(1-|x|)^m\preceq\int_{\partial B}L_{m,n}^B(x,\xi)\phi(\xi)d\omega(\xi)\preceq(1-|x|)^{m-1}. \tag{2.5}$$

Next, we aim at proving that

$$\int_B G_{m,n}^B(x,y)f^*(y)dy\simeq(1-|x|)^m, \text{ on } B, \tag{2.6}$$

where, $G_{m,n}^B$ is the Green function for the m-polyharmonic operator $u\to(-\Delta)^m u$ with Dirichlet boundary conditions on the unit ball B in $\mathbb{R}^n$.
To this end, we claim that on B^2 (that is $(x,y)\in B^2$), we have

$$(1-|x|)^m(1-|y|)^m\preceq G_{m,n}^B(x,y)\preceq\begin{cases}(1-|x|)^m & \text{if } m\geq n,\\ \frac{(1-|x|)^m}{|x-y|^{n-m}} & \text{if } m<n.\end{cases} \tag{2.7}$$

Indeed, from [12, Proposition 2.3], we have

$$G_{m,n}^B(x,y)\simeq\begin{cases}|x-y|^{2m-n}\min\left(1,\dfrac{((1-|x|)(1-|y|))^m}{|x-y|^{2m}}\right) & \text{for } n>2m,\\ Log\left(1+\dfrac{((1-|x|)(1-|y|))^m}{|x-y|^{2m}}\right) & \text{for } n=2m,\\ ((1-|x|)(1-|y|))^{m-\frac{n}{2}}\min\left(1,\dfrac{((1-|x|)(1-|y|))^{\frac{n}{2}}}{|x-y|^n}\right) & \text{for } n<2m.\end{cases}$$

Which implies that

$$G_{m,n}^{B}(x,y) \simeq \begin{cases} \dfrac{((1-|x|)(1-|y|))^{m}}{|\,x-y\,|^{n-2m}[x,y]^{2m}} & \text{for } n>2m, \\ \dfrac{((1-|x|)(1-|y|))^{m}}{[x,y]^{2m}} Log(1+\dfrac{[x,y]^{2}}{|x-y|^{2}}) & \text{for } n=2m, \\ \dfrac{((1-|x|)(1-|y|))^{m}}{[x,y]^{n}} & \text{for } n<2m. \end{cases} \tag{2.8}$$

So the lower inequality in (2.7) follows from (2.8) and the fact that for each $x,y\in B$, we have $|x-y|\leq [x,y]\preceq 1$.
Now, if $m\geq n$, then using the fact that for each $x,y\in B$, we have $(1-|y|)\leq [x,y]$, we deduce form (2.8) that

$$G_{m,n}^{B}(x,y) \simeq \frac{((1-|x|)(1-|y|))^{m}}{[x,y]^{n}} \preceq ((1-|x|))^{m}.$$

By similar argument we prove the upper inequality in (2.7) for the case $m<n$.
So using (2.7), we obtain

$$\begin{aligned} (1-|x|)^{m} \quad \preceq \int_{B} G_{m,n}^{B}(x,y)f^{*}(y)dy \quad &\preceq (1-|x|)^{m}\int_{B}\frac{f^{*}(y)}{|x-y|^{\max(n-m,0)}}dy \\ &\preceq (1-|x|)^{m}\int_{B(0,2)}\frac{1}{|z|^{\max(n-m,0)}}dz \\ &\preceq (1-|x|)^{m}. \end{aligned}$$

This proves (2.6).
Finally, the required inequality (2.4) follows from (2.2), (2.5) and (2.6). □

The m-Kelvin transform of a function u, is defined by

$$v(y) = |y|^{2m-n}\,u(\frac{y}{|y|^{2}}), \text{ for } y\in D. \tag{2.9}$$

By direct computation, $v(y)$ satisfies

$$\Delta^{m} v(y) = |y|^{-2m-n}(\Delta^{m} u)(\frac{y}{|y|^{2}}). \tag{2.10}$$

See [19, p. 221]. This fact and Theorem 2.1 and Proposition 2.2 immediately imply the following result.

Theorem 2.3. *Let F be a nonnegative function such that $x\to |x|^{-2m-n}F\left(\frac{x}{|x|^{2}}\right)\in C^{0,\gamma}(\overline{B})$ and ϕ is a positive function belonging to $C^{m+1,\gamma}(\partial B)$, for $0<\gamma<1$. Then the problem*

$$\begin{cases} (-\Delta)^{m} v = F, \ \text{in } D, \\ v>0, \ \text{in } D, \\ \left(-\frac{\partial}{\partial\nu}\right)^{j} v = 0 \quad \text{on } \partial D \quad \text{for } j=0,\dots,m-2, \\ \left(-\frac{\partial}{\partial\nu}\right)^{m-1} v = \phi \quad \text{on } \partial D, \\ v(y)\simeq |y|^{2m-n} \quad \text{near } \infty \end{cases} \tag{2.11}$$

admits a unique classical solution v satisfying

$$|y|^{m-n}(|y|-1)^m \preceq v(y) \preceq |y|^{m-n+1}(|y|-1)^{m-1}, \quad \text{on } D. \tag{2.12}$$

3 The Kato class $K_{m,n}^{\infty}(D)$ and Schrödinger polyharmonic equation

3.1 The Kato class $K_{m,n}^{\infty}(D)$

Example 3.1. *Given $s > \frac{n}{2m} > 1$. Then $L^s(D) \cap L^1(D) \subset K_{m,n}^{\infty}(D)$.*
Indeed, let $0 < r < 1$ and $q \in L^s(D) \cap L^1(D)$ with $s > \frac{n}{2m} > 1$.
Since for each $x, y \in D$, we have

$$G_{m,n}^{D}(x,y) = |x|^{2m-n}|y|^{2m-n} G_{m,n}^{B}(j(x), j(y)), \tag{3.1}$$

then by using (2.8), there exists a constant $C > 0$, such that for each $x, y \in D$

$$\left(\frac{\rho(y)}{\rho(x)}\right)^m G_{m,n}^{D}(x,y) \leq C\frac{1}{|x-y|^{n-2m}}. \tag{3.2}$$

This fact and the Hölder inequality imply that

$$\begin{aligned}
\int_{B(x,r)\cap D}\left(\tfrac{\rho(y)}{\rho(x)}\right)^m G_{m,n}^{D}(x,y)|q(y)|dy \quad &\leq C\int_{B(x,r)\cap D}\tfrac{|q(y)|}{|x-y|^{n-2m}}dy \\
&\leq C\left(\int_D |q(y)|^s dy\right)^{\frac{1}{s}} \\
&\quad \times \left(\int_{B(x,r)} |x-y|^{(2m-n)\frac{s}{s-1}}\, dy\right)^{\frac{s-1}{s}} \\
&\leq C\left(\int_0^r t^{(2m-n)\frac{s}{s-1}+n-1}dt\right)^{\frac{s-1}{s}} \to 0,
\end{aligned}$$

as $r \to 0$, since $(2m-n)\frac{s}{s-1} + n - 1 > -1$ when $s > \frac{n}{2m}$.

This shows that q satisfies (1.7).
We claim that q satisfies (1.8). Indeed, let $M > 0$, then for each $\varepsilon > 0$, there exists $r > 0$, such that

$$\begin{aligned}
&\int_{(|y|\geq M)}\left(\tfrac{\rho(y)}{\rho(x)}\right)^m G_{m,n}^{D}(x,y)|q(y)|dy \\
&\leq \tfrac{\varepsilon}{2} + \int_{(|x-y|\geq r)\cap(|y|\geq M)}\left(\tfrac{\rho(y)}{\rho(x)}\right)^m G_{m,n}^{D}(x,y)|q(y)|dy \\
&\leq \tfrac{\varepsilon}{2} + C\int_{(|x-y|\geq r)\cap(|y|\geq M)}\tfrac{|q(y)|}{|x-y|^{n-2m}}dy \\
&\leq \tfrac{\varepsilon}{2} + C\int_{(|y|\geq M)}|q(y)|dy \to 0, \ \text{as } M \to \infty.
\end{aligned}$$

□

Next we collect some properties of the Kato class $K_{m,n}^{\infty}(D)$, which are useful to establish of our main results. For the proofs we refer to [3].

Proposition 3.2. [3] *Let q be a nonnegative function in $K_{m,n}^{\infty}(D)$. Then we have*
(i) $\|q\| < \infty$.
(ii) $x \to q(x) \in L_{loc}^{1}(D)$.
(iii) For each bounded function h in $\mathfrak{J}$, the function
$x \to \int_D \frac{(|y|^2-1)^{m-1}}{(|x|^2-1)^{m-1}} h(y) G_{m,n}^D(x,y)\,|q(y)|\,dy$, *is continuous in $\overline{D}$, vanishes at the boundary ∂D.*
(iv) The family of functions $\left\{ \frac{1}{h_{m,n}(.)} \int_D G_{m,n}^D(.,y) h_{m,n}(y)\,\zeta(y)dy : \ |\zeta| \leq q \right\}$ *is relatively compact in $C(\overline{D} \cup \{\infty\})$.*
Furthermore,

$$\lim_{|x|\to\infty} \frac{1}{h_{m,n}(x)} \int_D G_{m,n}^D(x,y) h_{m,n}(y)\, q(y)dy = 0, \quad \text{for } n \geq 2m.$$

Lemma 3.3. *Let q be a nonnegative function in $K_{m,n}^{\infty}(D)$. Then we have*
(i)

$$\alpha_q := \sup_{x,y\in D} \int_D \frac{G_{m,n}^D(x,z) G_{m,n}^D(z,y)}{G_{m,n}^D(x,y)} \,|q(z)|\, dz < \infty. \tag{3.3}$$

(ii)

$$V(qG_{m,n}^D(.,y))(x) \leq \alpha_q G_{m,n}^D(x,y), \text{ for each } x,y \in D. \tag{3.4}$$

Proof: *(i)* It follows from the 3-G Theorem (1.5) and proposition 3.2, that $\alpha_q \leq 2C_{m,n}\,\|q\| < \infty$.

(ii) Inequality (3.4) follows immediately from the definitions of the potential function V and α_q. □

Proposition 3.4. *Let $q \in K_{m,n}^{\infty}(D)$. Then for each $x \in D$,*

$$\int_D G_{m,n}^D(x,y) h_{m,n}(y)\,|q(y)|\,dy \leq \alpha_q h_{m,n}(x) \tag{3.5}$$

Proof: It follows from (3.1), that

$$h_{m,n}(x) = \lim_{|y|\to\infty} |y|^{n-2m}\, G_{m,n}^D(x,y). \tag{3.6}$$

In particular

$$\lim_{|y|\to\infty} \frac{G_{m,n}^D(z,y)}{G_{m,n}^D(x,y)} = \frac{h_{m,n}(z)}{h_{m,n}(x)}. \tag{3.7}$$

Thus by Fatou's lemma and (3.7), we deduce that, for $x \in D$

$$\int_D G_{m,n}^D(x,y) \frac{h_{m,n}(y)}{h_{m,n}(x)}\,|q(y)|\,dy \leq \liminf_{|z|\to\infty} \int_D \frac{G_{m,n}^D(x,y) G_{m,n}^D(y,z)}{G_{m,n}^D(x,z)} |q(y)| dy \leq \alpha_q.$$

□

Proposition 3.5. *For all $q \in K_{m,n}^{\infty}(D)$ and any $h \in \mathfrak{J}$, we have for each $x \in D$,*

$$\int_D G_{m,n}^D(x,z)\left(|z|^2-1\right)^{m-1} h(z)\,|q(z)|\,dz \le \alpha_q \left(|x|^2-1\right)^{m-1} h(x). \tag{3.8}$$

Proof: Let $h \in \mathfrak{J}$. Then there exists a nonnegative measure ν on ∂D such that

$$h(x) = \int_{\partial D} P(x,\xi)\nu(d\xi).$$

So we need only to verify(3.8) for $h(y) = P(y,\xi)$ uniformly in $\xi \in \partial D$.

From (3.1) and [12, lemma 2.1], we deduce that the Green function $G_{m,n}^D$ satisfies

$$G_{m,n}^D(x,y) = k_{m,n}|\,x-y\,|^{2m-n}\int_1^{\frac{[x,y]}{|x-y|}} \frac{(v^2-1)^{m-1}}{v^{n-1}}dv. \tag{3.9}$$

Using the transformation $v^2 = 1+\frac{\theta(x,y)}{|x-y|^2}(1-t)$ in (3.9) , we obtain

$$G_{m,n}^D(x,y) = \frac{k_{m,n}}{2}\frac{(\theta(x,y))^m}{[x,y]^n}\int_0^1 \frac{(1-t)^{m-1}}{\left(1-t\,\frac{\theta(x,y)}{[x,y]^2}\right)^{\frac{n}{2}}}\,dt.$$

This implies that for each $x, z \in D$ and $\xi \in \partial D$,

$$\lim_{y\to\xi}\frac{G_{m,n}^D(z,y)}{G_{m,n}^D(x,y)} = \frac{(|z|^2-1)^{m-1}}{(|x|^2-1)^{m-1}}\frac{P(z,\xi)}{P(x,\xi)}, \tag{3.10}$$

Thus by Fatou's lemma and (3.10), we deduce that, for $x \in D$, and $\xi \in \partial D$,

$$\begin{aligned} &\int_D G_{m,n}^D(x,z)\frac{(|z|^2-1)^{m-1}}{(|x|^2-1)^{m-1}}\frac{P(z,\xi)}{P(x,\xi)}|q(z)|dz \\ \le\ & \liminf_{y\to\xi}\int_D \frac{G_{m,n}^D(x,z)G_{m,n}^D(z,y)}{G_{m,n}^D(x,y)}|q(z)|dz \le \alpha_q. \end{aligned}$$

□

3.2 The Schrödinger polyharmonic equation

For a nonnegative function q in $K_{m,n}^{\infty}(D)$ such that $\alpha_q \le \frac{1}{2}$, we put

$$\mathcal{G}_{m,n}(x,y) = \begin{cases} \sum\limits_{k\ge 0}(-1)^k\left(V(q.)\right)^k\left(G_{m,n}^D(.,y)\right)(x) \text{ if } x \ne y \\ +\infty \text{ if } x = y. \end{cases}$$

Then we have

Lemma 3.6. *Let q be a nonnegative function in $K_{m,n}^{\infty}(D)$ such that $\alpha_q \leq \frac{1}{2}$. Then for each x, y in D, we have*

$$\left(1-\alpha_q\right) G_{m,n}^{D}(x,y) \leq \mathcal{G}_{m,n}(x,y) \leq G_{m,n}^{D}(x,y). \tag{3.11}$$

Proof: Since $\alpha_q \leq \frac{1}{2}$, we deduce from (3.4), that for $x \neq y$

$$\left|\mathcal{G}_{m,n}(x,y)\right| \leq \sum_{k\geq 0} \left(\alpha_q\right)^k G_{m,n}^{D}(x,y) = \frac{1}{1-\alpha_q} G_{m,n}^{D}(x,y).$$

On the other hand, from the expression for $\mathcal{G}_{m,n}$, we deduce that for $x \neq y$

$$\mathcal{G}_{m,n}(x,y) = G_{m,n}^{D}(x,y) - V\left(q\mathcal{G}_{m,n}(.,y)\right)(x). \tag{3.12}$$

Using these facts and (3.4), we obtain that

$$\mathcal{G}_{m,n}(x,y) \geq G_{m,n}^{D}(x,y) - \frac{\alpha_q}{1-\alpha_q} G_{m,n}^{D}(x,y) = \frac{1-2\alpha_q}{1-\alpha_q} G_{m,n}^{D}(x,y) \geq 0.$$

Hence the result follows from (3.12) and (3.4). ∎

In the sequel, for a given nonnegative function $q \in K_{m,n}^{\infty}(D)$ such that $\alpha_q \leq \frac{1}{2}$, we define the operator V_q on $\mathcal{B}^{+}(D)$ by

$$V_q\psi(x) = \int_D \mathcal{G}_{m,n}(x,y)\,\psi(y)dy, \quad x \in D.$$

Then, we have the following Lemma.

Lemma 3.7. *Let q be a nonnegative function in $K_{m,n}^{\infty}(D)$ such that $\alpha_q \leq \frac{1}{2}$ and $\psi \in \mathcal{B}^{+}(D)$. Then $V_q\psi$ satisfies the following resolvent equation:*

$$V\psi = V_q\psi + V_q\left(qV\psi\right) = V_q\psi + V\left(qV_q\psi\right). \tag{3.13}$$

Proof: From the expression for $\mathcal{G}_{m,n}$, we deduce for $\psi \in \mathcal{B}^{+}(D)$ such that $V\psi < \infty$,

$$V_q\psi = \sum_{k\geq 0} (-1)^k \left(V(q.)\right)^k V\psi.$$

So we obtain that

$$\begin{aligned} V_q\left(qV\psi\right) &= \sum_{k\geq 0} (-1)^k \left(V(q.)\right)^k \left[V\left(qV\psi\right)\right] \\ &= -\sum_{k\geq 1} (-1)^k \left(V(q.)\right)^k V\psi \\ &= V\psi - V_q\psi. \end{aligned}$$

The second equality is proved by integrating (3.12). ∎

Proposition 3.8. *Let q be a nonnegative function in $K_{m,n}^{\infty}(D)$ such that $\alpha_q \leq \frac{1}{2}$ and let $\psi \in L_{loc}^{1}(D)$ be such that $V\psi \in L_{loc}^{1}(D)$. Then $V_q\psi$ is a solution of the time-independent Schrödinger polyharmonic equation* (1.13).

Proof: Using the resolvent equation (3.13), we have

$$V_q\psi = V\psi - V\left(qV_q\psi\right).$$

Applying the operator $(-\Delta)^m$ on both sides of the above equality, we obtain that

$$(-\Delta)^m\left(V_q\psi\right) = \psi - qV_q\psi \text{ (in the sense of distributions).}$$

This completes the proof. ∎

4 Proofs of Theorems 1.4 and 1.5

Proof of Theorem 1.4.

Let φ be a nonnegative continuous function on ∂D and $H_D\varphi$ the bounded continuous solution of the Dirichlet problem (1.11).We recall that

$$\omega\left(x\right) = h_{m,n}(x) + \left(|x|^2 - 1\right)^{m-1} H_D\varphi.$$

Since g satisfies (H_2), there exists a nonnegative function $q \in K_{m,n}^{\infty}(D)$ such that $\alpha_q \leq \frac{1}{2}$ and for each $x \in D$, the map $t \to t\left(q\left(x\right) - g\left(x, t\omega\left(x\right)\right)\right)$ is continuous and nondecreasing on $[0,1]$. We consider the closed convex set Λ given by

$$\Lambda := \left\{v \in \mathcal{B}^{+}(D) : (1-\alpha_q) \leq v \leq 1\right\}.$$

We define the operator T on Λ by

$$Tv\left(x\right) := \frac{1}{\omega\left(x\right)}\left[\omega\left(x\right) - V_q\left(q\omega\right)\left(x\right)\right] + \frac{1}{\omega\left(x\right)}V_q\left[\left(q - g\left(., \omega v\right)\right)\omega v\right]\left(x\right), \text{ for } x \in D. \tag{4.1}$$

By (H_2), we deduce that

$$0 \leq g(x, t\omega\left(x\right)) \leq q(x), \quad \text{for each } x \in D \text{ and } t \in [0,1].$$

Hence,

$$0 \leq g(., \omega v) \leq q, \quad \text{for all } v \in \Lambda. \tag{4.2}$$

So the operator T is well defined on Λ.
On the other hand, using (3.5), (3.8) and (3.3) we have

$$\frac{1}{\omega}V_q\left(q\omega\right) \leq \alpha_q < \infty. \tag{4.3}$$

We claim that Λ is invariant under T. Indeed, using (4.1) and (4.3) we have for $v \in \Lambda$,

$$Tv \leq \frac{1}{\omega}\left[\omega - V_q\left(q\omega\right)\right] + \frac{1}{\omega}V_q\left(q\omega v\right) \leq 1.$$

Furthermore, from (4.1), (4.2) and (4.3) , we obtain

$$Tv \geq \frac{1}{\omega}\left[\omega - V_q\left(q\omega\right)\right] \geq \left(1-\alpha_q\right).$$

Next, we will prove that the operator T is nondecreasing on Λ. Indeed, let $u, v \in \Lambda$ be such that $u \leq v$. Since the map $t \rightarrow t\left(q\left(x\right) - g\left(x, t\omega\left(x\right)\right)\right)$ is nondecreasing on $[0,1]$, for $x \in D$, we obtain

$$Tv - Tu = \frac{1}{\omega} V_q\left[\omega\left[v\left(q - g\left(.,\omega v\right)\right) - u\left(q - g\left(.,\omega u\right)\right)\right]\right] \geq 0.$$

Now, we consider the sequence (v_k) defined by $v_0 = (1-\alpha_q) \in \Lambda$ and $v_{k+1} = Tv_k$ for $k \in \mathbb{N}$. Since Λ is invariant under T , then $v_1 = Tv_0 \geq v_0$, and so from the monotonicity of T, we deduce that

$$(1-\alpha_q) = v_0 \leq v_1 \leq \ldots \leq v_k \leq v_{k+1} \leq 1.$$

Furthermore, by (H_2) it is clear for each $x \in D$ that the map $t \rightarrow tg\left(., t\omega\left(x\right)\right)$ is continuous on $[0,\infty)$. Which together with the dominated convergence theorem imply that the sequence (v_k) converges to a function $v \in \Lambda$ which is a fixed point of T. We let $u\left(x\right) = \omega\left(x\right) v\left(x\right)$, for each $x \in D$.
Then u satisfies $\left(1-\alpha_q\right)\omega \leq u \leq \omega$ and

$$u = \left(I - V_q\left(q.\right)\right)\omega + V_q\left[\left(q - g\left(.,u\right)\right)u\right].$$

That is

$$\left(I - V_q\left(q.\right)\right)u = \left(I - V_q\left(q.\right)\right)\omega - V_q\left(ug\left(.,u\right)\right).$$

Applying the operator $\left(I + V\left(q.\right)\right)$ on both sides of the above equality and using (3.13) we deduce that u satisfies

$$u = \omega - V\left(ug\left(.,u\right)\right). \tag{4.4}$$

Finally, we need to verify that u is a positive continuous solution for the problem (1.3). Indeed, from (4.2) we obtain

$$ug\left(.,u\right) \leq \omega q. \tag{4.5}$$

We deduce by Proposition 3.2(ii), that $ug\left(.,u\right) \in L^1_{loc}\left(D\right)$ and by (3.5) and (3.8) that $V\left(ug\left(.,u\right)\right) \leq V(\omega q) \leq \alpha_q \omega \in L^1_{loc}\left(D\right)$.
Hence we conclude by [7], that u satisfies (in the sense of distributions) the elliptic differential equation

$$\left(-\Delta\right)^m u + ug\left(.,u\right) = 0 \quad \text{in} \quad D.$$

Finally, since by Proposition 3.2, (3.5) and (3.8) the function $x \mapsto \dfrac{V(\omega q)(x)}{\omega(x)}$, is continuous and bounded, then by writing

$$\frac{1}{\omega} V(\omega q) = \frac{1}{\omega}\left[V\left(ug\left(.,u\right)\right) + V(\omega q - ug\left(.,u\right))\right],$$

we deduce that $u \in C(D)$. Using (4.4), (4.5), (1.12) and again Proposition 3.2, we obtain that $\lim\limits_{x \rightarrow \zeta \in \partial D} \frac{u(x)}{\left(|x|^2-1\right)^{m-1}} = \varphi(\zeta)$ and for $n \geq 2m$, $\lim\limits_{|x| \rightarrow \infty} \frac{u(x)}{h_{m,n}(x)} = 1$. This ends the proof. □

Example 4.1. *Let $\gamma,\sigma\in\mathbb{R}_+$ and $\lambda<2m\leq 2m+\max(0,2m-n)<\mu$.*
Let φ be a nonnegative continuous bounded function on ∂D. Put $\omega(x)=h_{m,n}(x)+\left(|x|^2-1\right)^{m-1}H_D\varphi$. Assume that p is a nonnegative Borel measurable function on D satisfying

$$p(x)\leq\frac{\nu}{|x|^{\mu-\lambda}\left(|x|-1\right)^{\lambda}\omega^{\gamma}(x)\left(1+\omega^{\sigma}(x)\right)},$$

where ν is a sufficiently small positive constant. Then the problem

$$\begin{cases}(-\Delta)^m u+p(x)u^{\gamma}Log(1+u^{\sigma})=0, & \text{in } D \text{ (in the sense of distributions)}\\ \lim\limits_{x\to\zeta\in\partial D}\frac{u(x)}{\left(|x|^2-1\right)^{m-1}}=\varphi(\zeta), & \\ u(x)\simeq\rho_0(x), \ \text{near } x=\infty, & \end{cases}$$

has a continuous positive solution u satisfying

$$u(x)\simeq h_{m,n}(x)+\left(|x|^2-1\right)^{m-1}H_D\varphi.$$

Moreover, for $n\geq 2m$ we have

$$\lim_{|x|\to\infty}\frac{u(x)}{h_{m,n}(x)}=1.$$

Proof of Theorem 1.5.

We recall that h_0 is a fixed positive harmonic function in D, which is continuous and bounded in $\overline{D}$. Let φ be a nonnegative nontrivial continuous bounded function on ∂D and let $H_D\varphi$ be the bounded continuous solution of the Dirichlet problem (1.11).
Let $q\in K^{\infty}_{m,n}(D)$ be given by (A_2) and put $c_1=\frac{1}{(1-\alpha_q)}>1$. Let $\widetilde{c}\geq c_1$ and assume that

$(\mathbf{A}_3)$ $\varphi(x)\geq c_1h_0(x),\ \forall x\in\partial D.$

Put $\omega_0(x)=h_{m,n}(x)+\left(|x|^2-1\right)^{m-1}h_0(x)$ and $\omega(x)=\widetilde{c}h_{m,n}(x)+\left(|x|^2-1\right)^{m-1}H_D\varphi(x)$.
We consider the closed convex set S given by

$$S:=\left\{u\in\mathcal{B}^+(D):\omega_0(x)\leq u(x)\leq\omega(x)\,,\ \text{for all } x\in D\right\}.$$

Since $H_D\varphi=\varphi$ on ∂D and h_0 is continuous and bounded in $\overline{D}$, we obtain by (A_3) that $H_D\varphi\geq c_1h_0$ on D. So S is a well defined nonempty set in $\mathcal{B}^+(D)$.
By (A_2), we deduce that

$$0\leq f(.,u)\leq qu,\quad \text{for any } u\in S. \tag{4.6}$$

So we define the operator L on S by

$$Lu:=\omega-V_q(q\omega)+V_q[qu-f(.,u)]. \tag{4.7}$$

It is easy to verify that S is invariant under L and that the operator L is nondecreasing on S. Now, we consider the sequence (u_k) defined by $u_0=\omega_0\in S$ and $u_{k+1}=Lu_k$ for $k\in\mathbb{N}$. Then we have

$$\omega_0\leq u_1\leq ...\leq u_k\leq u_{k+1}\leq\omega.$$

Using (A_2) and similar argument as in the proof of Theorem 1.4, we prove that the sequence (u_k) converges to a function $u \in S$, which satisfies

$$u = \omega - Vf\left(., u\right). \tag{4.8}$$

Finally, we verify that u is the required solution.□

Example 4.2. *Let $\gamma \in (0,1]$ $n > 2m$ and $\lambda < 2m < \mu$. Let φ be a nonnegative continuous bounded function on D and h_0 be a positive harmonic function in D, which is bounded and continuous in $\overline{D}$. Then from $[2,\ p.258]$, there exists a constant $C > 0$, such that for each $x \in D$,*

$$C\frac{(|x|-1)}{(|x|+1)^{n-1}} \leq h_0\left(x\right).$$

Suppose that p is a nonnegative Borel measurable function on D satisfying

$$p(x) \leq \frac{\nu}{|x|^{\mu-\lambda-(\gamma-1)(n-m)}\left(|x|-1\right)^{\lambda+(\gamma-1)m}},$$

where ν is a sufficiently small positive constant. Then there exists a constant $c_1 > 1$ such that if $\widetilde{c} \geq c_1$ and $\varphi \geq c_1 h_0$ on ∂D, the problem

$$\begin{cases} (-\Delta)^m u + p(x)u^\gamma(x) = 0, & \text{in } D \text{ (in the sense of distributions)} \\ \lim\limits_{x\to\zeta\in\partial D} \frac{u(x)}{\left(|x|^2-1\right)^{m-1}} = \varphi(\zeta), \\ \lim\limits_{|x|\to\infty} \frac{u(x)}{h_{m,n}(x)} = \widetilde{c}. \end{cases}$$

has a continuous positive solution u satisfying for each $x \in D$

$$\omega_0(x) \leq u(x) \leq \omega(x).$$

Acknowledgement 1. *The author thanks the referee for a careful reading of the paper and useful suggestions. This project was supported by King Saud University, Deanship of Scientific Research, College of Science Research Center.*

References

[1] M. Aizenman, B. Simon, Brownian motion and harnack inequality for Schrödinger operators. *Commun. Pure Appl. Math.*, 35:209-273, 1982.

[2] D.-H. Armitage, S.J. Gardiner, *Classical potential theory*. Springer-Verlag, 2001.

[3] I. Bachar, H. Mâagli, N. Zeddini, Estimates on the Green function and existence of positive solutions for some nonlinear polyharmonic problems outside the unit ball. *Analysis and Applications.*, 6(2):121-150, 2008.

[4] F. Chiarenza, E. Fabes, N. Garofalo, Harnack's inequality for Schrödinger operators and the continuity of solutions. *Proc. Amer. Math. Soc.*, 98:415-425, 1986.

[5] G. Caristi, E. Mitidieri, Harnack inequality and applications to solutions of biharmonic equations, *Operator Theory, Advances and Applications.*, 168:1-26, DOI: 10.1007/3-7643-7601-5_1 Birkhäuser Verlag Basel/Switzerland, 2006.

[6] F. Cîrstea, V. Rădulescu, Blow-up boundary solutions of semilinear elliptic problems. *Nonlinear Anal.*, 48:521-534, 2002.

[7] K.L. Chung, Z. Zhao, *From Brownian motion to Schrödinger's equation.* Springer-Verlag, 1995.

[8] R. Dautray, J.L. Lions, *Analyse mathématique et calcul numérique pour les sciences et les techniques, L'operateur de Laplace.* Masson, 1987.

[9] E.-B. Fabes, D.W. Stroock, The L^p-integrability of Green's function and fundamental solutions for elliptic and parabolic equations. *Duke Math. J.*, 51:997-1016, 1984.

[10] F. Gazzola, H.C. Grunau, G. Sweers, Polyharmonic boundary value problems positivity preserving and nonlinear higher order elliptic equations in bounded domains. *Lecture Notes in Mathematics.,* 1991, DOI: 10.1007/978-3-642-12245-3, 2010.

[11] A. Ghanmi, H. Mâagli, V. Rădulescu, N. Zeddini, Large and bounded solutions for a class of nonlinear Schrödinger stationary systems. *Analysis and Applications.,* 7:391-404, 2009.

[12] H.C. Grunau, G. Sweers, Positivity for equations involving polyharmonic operators with Dirichlet boundary conditions. *Math.Ann.*, 307:589-626, 1997.

[13] H.C. Grunau, G. Sweers, Positivity properties of elliptic boundary value problems of higher order. *Nonlinear Analysis.*, 30(8):5251-5258, 1997.

[14] A.M. Hinz, H. Kalf, Subsolution estimates and harnack inequality for Schrödinger operators. *J. Reine Angew. Math.*, 404:118–134, 1990.

[15] H. Mâagli, S. Masmoudi, Positive solutions of some nonlinear elliptic problems in unbounded domain. *Ann Acad Scien Fenn Math.*, 29:151-166, 2004.

[16] V. Rădulescu, *Singular phenomena in nonlinear elliptic problems. From blow-up boundary solutions to equations with singular nonlinearities*, *in handbook of differential equations*: Stationary partial differential equations, ed..M. Chipot. Vol. 4:483-591. Elsevier, 2007.

[17] Ch.G. Simader, An elementary proof of harnack's inequality for Schrödinger operators and related topics. *Math. Z.*, 203:129-152, 1990.

[18] F. Toum, N. Zeddini, Existence of positive solutions for nonlinear boundary-value problems in unbounded domains of R^n. *Electron. J. Differential Equations.*, 143:14 pp., 2005.

[19] J. Weiand, X. Xu, Classification of solutions of higher order conformally invariant equations. *Math. Ann.*, 313:207-228, 1999.

[20] Z. Zhao, Green function for Schrödinger operator and conditioned Feynman-Kac gauge. *J. Math. Anal. Appl.*, 116:309-334, 1986.

[21] Z. Zhao, Uniform boundedness of conditional gauge and Schrödinger equations. *Commun.Math. Phys.*, 93:19-31, 1984.

[22] Z. Zhao, Conditional gauge with unbounded potential. *Z. Wahrsch. Verw. Gebiete.*, 65:13-18, 1983.

Received: 24.06.2012

Accepted: 10.08.2012

King Saud University College of Sciences Mathematics Department
P.O. Box 2455 Riyadh 11451 Kingdom of Saudi Arabia
E-mail: abachar@ksu.edu.sa