Beyond Mumford's theorem on normal surfaces

by
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To Mihnea Colţoiu, on the occasion of a great anniversary

Abstract

Beyond normal surfaces there are several open questions concerning 2-dimensional spaces. We present some results and conjectures along this line.

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1 Questions

Let (X, x_0) be the germ of a surface. Let us assume that (X, x_0) has an isolated singularity at the origin and moreover that (X, x_0) is *normal*. Mumford has proved that the "triviality" of the singular germ may be translated into the "simplicity" of its link:

Theorem 1.1. (Mumford, 1961) *The link of a normal surface is simply connected if and only if the surface is non-singular.*

The "simplicity" of a surface is understood here as the simplicity of its link. We recall that the *link* (more precisely, *real link*) of an analytic set Y at some point $y \in Y$ is by definition $\mathrm{lk}_\varepsilon(Y, y) := Y \cap \rho^{-1}(\varepsilon)$, where $\rho : Y \to \mathbb{R}_{\geq 0}$ is a non-trivial analytic function such that $\rho^{-1}(0) = \{y\}$. By the *local conic structure* of analytic sets, result due to Burghelea and Verona [BV], the link $\mathrm{lk}_\varepsilon(Y, y)$ does not depend, up to homeomorphisms, neither on the choice of ρ, nor on that of $\varepsilon > 0$ provided that it is small enough. For instance ρ may be the distance function to the point y.
We address here two situations beyond Mumford's setting:

  (a) (X, x_0) normal with nontrivial $\pi_1(\mathrm{lk}(X, x_0))$.

  (b) (X, x_0) not normal but $\pi_1(\mathrm{lk}(X, x_0))$ trivial.

The issue (a) is discussed in §2. By Prill's result and by the more recent one by Mihnea Colţoiu and the author, normal surfaces fall into precisely two disjoint classes: the universal covering of the punctured germ $X \setminus \{x_0\}$ is a Stein manifold or not. The former corresponds to infinite $\pi_1(\mathrm{lk}(X, x_0))$ and the later to finite $\pi_1(\mathrm{lk}(X, x_0))$.

In sections §3-§6 we discuss several aspects of the issue (b). It turns out that Mumford's theorem does not extend to surfaces with isolated singularities, as we show by an example in §5. In case of non-isolated singularities, for surfaces in $\mathbb{C}^3$ we prove a criterion of simplicity in terms of the triviality of the complex link. This also yields an approach to the following "rank conjecture": *an injective map germ* $g : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ *has* $\mathrm{rank}\,(\mathrm{grad}\,g)(0) \geq 1$.

2   Steinness of the universal coverings

We assume in this section that (X, x_0) is a normal surface. One has the following classical result by D. Prill.

Theorem 2.1. [Pr]

If the fundamental group of the link is finite then the surface (X, x_0) *is isomorphic to the quotient of* $(\mathbb{C}^2, 0)$ *by a finite linear group.*

In particular this shows that the universal covering of the complement $X \setminus \{x_0\}$ is not Stein. One may say that X is of "concave" type, whereas the "convex" type surfaces would be those for which the universal covering of $X \setminus \{x_0\}$ is a Stein manifold. It turns out that all non-quotient surface singularities are in fact of convex type:

Theorem 2.2. [CT]
If $\pi_1(\mathrm{lk}(X, x_0))$ *is infinite, then the universal covering of* $X \setminus \{x_0\}$ *is a Stein manifold.*

The proof uses the structure of a resolution of the normal surface singularity $p : Y \to X$ in the neighbourhood of its exceptional divisor $A := p^{-1}(x_0)$ with normal crossings.

There are two cases to treat: $H_1(\mathrm{lk}(X, x_0))$ is infinite or finite. Starting from Mumford's result that the non-finiteness of $H_1(\mathrm{lk}(X, x_0))$ is equivalent to that of $\pi_1(A)$ one constructs special coverings of A and of its neighbourhood such that the former are Stein. There are mainly 3 cases: the dual graph of A is a cycle, contains a cycle, or does not contain. In order to settle the problem, one uses several results on Stein spaces, Runge pairs and other related properties, due to Napier [Nap], Nori, Gurjar, Coltoiu [Co], Simha [Sim] and others.

The case of finite homology group $H_1(\mathrm{lk}(X, x_0))$ is reduced to the previous case by topological methods involving the graph manifold structure of the link (or Waldhausen structure [Wa]) and its Jaco-Shalen-Johannson decomposition [JS], [Joh]. Namely one shows that there is a finite covering of the link with infinite H_1 and moreover one may attach to it a normal surface germ Z and an analytic covering $Z \to X$.

Finally, if one constructs a covering of the complement which is Stein, then all the coverings of it are Stein (by K. Stein's theorem [St]) and in particular the universal one is Stein.

The above result [CT] has been extended recently by Colţoiu, Joiţa and the author to higher dimensions [CJT].

3   Simplicity of surfaces via the simplicity of their complex links

One may ask if there is any "simplicity" criterion beyond the class of normal surfaces. We consider here this question for both types of non-normal surface germs: surfaces with isolated singularity and surfaces with non-isolated singularities.

We first observe that for normal surfaces Mumford's criterion is also equivalent to the "simplicity" of the *complex link* $\mathbb{C}\mathrm{lk}(X, 0)$, more precisely, that *a normal surface germ is non-singular if and only if its complex link is contractible* (Theorem 4.1). Using this new criterion (more precisely, Lemma 4.2 below) we show by an example in §5 that Mumford's criterion does not extend to isolated singularities.

Following the same idea of replacing the real link condition by a complex link condition, we show the following extension of Mumford's criterion to non-isolated singularities. This can be be compared to an observation by Lê D.T in [Lê2, Rem. 2, pag. 285].

Theorem 3.1. *Let $(X, 0) \subset (\mathbb{C}^3, 0)$ be a surface germ with 1-dimensional singularity. The complex link $\mathbb{C}\mathrm{lk}(X, 0)$ is contractible if and only if X is an equisingular deformation of a curve.*

We mean by "X is an equisingular deformation of a curve" that X is the total space of a one-parameter family of curves X_t with a single singularity and having the same Milnor number. It does not follow that the curves are irreducible, as shown by the following example: $X := \{xy = 0\} \times \mathbb{C} \subset \mathbb{C}^3$ is a surface with contractible complex link (see §4.1 for the definition) and a trivial family of reducible curves. However, if we impose in addition that the real link is homeomorphic to S^3 then the irreducibility follows too. It turns out that this criterion has an extension to higher dimensions (Corollary 6.1).

Under the same exchange between the real link condition and the complex link condition, our Theorem 3.1 corresponds to the following conjecture due to Lê Dũng Tráng which holds since about 30 years.

Conjecture 3.2. *Let $(X, 0) \subset (\mathbb{C}^3, 0)$ be a surface with 1-dimensional singular locus. Then the real link of $(X, 0)$ is simply connected if and only if $(X, 0)$ is an equisingular deformation of an irreducible curve.*

Several particular classes of surfaces have been checked out confirming the conjecture, see e.g. [Fe]. Lê D.T. observed that this conjecture can be equivalently formulated as follows.

Conjecture 3.3. *An injective map germ $g : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ has $\mathrm{rank}\,(\mathrm{grad}\,g)(0) \geq 1$.*

The latter looks like a more elementary statement but appears to be very delicate to prove even in particular cases, see for instance [KM] which provides a counterexample to such a proof in [Ne]. Our Theorem 3.1 yields in particular the following equivalent formulation of Conjecture 3.2:

Conjecture 3.4. *Let $(X, 0) \subset (\mathbb{C}^3, 0)$ be a surface with 1-dimensional singular locus. If the real link $\mathrm{lk}(X, 0)$ is simply connected then the complex link $\mathbb{C}\mathrm{lk}(X, 0)$ is contractible.*

4   Simplicity of surfaces via the simplicity of their complex links

4.1   A supplement to Mumford's criterion

We recall that the *link* (more precisely, *real link*) of an analytic set Y at some point $y \in Y$ is by definition $\mathrm{lk}_\varepsilon(Y, y) := \rho^{-1}(\varepsilon)$, where $\rho : (Y, y) \to (\mathbb{R}_{\geq 0}, 0)$ is a proper analytic function such that $\rho^{-1}(0) = \{y\}$. By the *local conic structure* of analytic sets, result due to Burghelea and Verona [BV], the link $\mathrm{lk}_\varepsilon(Y, y)$ does not depend, up to homeomorphisms, neither on the choice of ρ, nor on that of $\varepsilon > 0$ provided that it is small enough. For instance ρ may be the distance function to the point y. Given an analytic space germ $(Y, 0)$, any holomorphic function $g : (Y, 0) \to (\mathbb{C}, 0)$ defines a local *Milnor-Lê fibration* [Lê4], which means that :

$$g_| : B_\varepsilon \cap g^{-1}(D_\delta^*) \to D_\delta^* \tag{1}$$

is a locally trivial fibration for small enough $\varepsilon \gg \delta > 0$.

Moreover, the restriction:

$$g_| : \partial B_\varepsilon \cap g^{-1}(D_\delta) \to D_\delta \tag{2}$$

is a trivial fibration. Then $g^{-1}(\lambda) \cap B_\varepsilon$, for $0 < |\lambda| \ll \varepsilon$, is called *Milnor fibre.*

The *complex link* of $(Y, 0)$, denoted by $\mathbb{C}\mathrm{lk}(Y, 0)$, is defined as the Milnor fibre of a holomorphic *general* function. This notion has been introduced in the 1980's by Goresky and MacPherson [GM] who proved that its homotopy type does not depend on the choices of λ and ε provided they are small enough.

The complex link of a non-singular space germ $(Y, 0)$ is clearly contractible, whereas the converse turns out to be not true in general (see e.g. the example of §5). The following result is due to Looijenga [Lo, pag. 68]: *If $(Y, 0)$ is a complete intersection with isolated singularity then the complex link of $(Y, 0)$ is contractible if and only if $(Y, 0)$ nonsingular.*

We add up one more equivalence to Mumford's result, which shows that, in the quest for "simplicity", the complex link plays a similar role as the real link.

Theorem 4.1. (A supplement to Mumford's theorem)
Let $(X, 0)$ be an irreducible normal surface germ. The following are equivalent:

(a) *X is non-singular at 0.*

(b) *$\mathrm{lk}(X, 0)$ is simply connected.*

(c) *$\mathbb{C}\mathrm{lk}(X, 0)$ is contractible.*

4.2   Proof of Theorem 4.1

A smooth surface has contractible complex link, and its link is simply connected, thus (a) $\Leftrightarrow$ (c) and (a) $\Leftrightarrow$ (b) are obvious. The implication (b) $\Leftrightarrow$ (a) is of course Mumford's theorem. The implication (c) $\Rightarrow$ (b) follows from the next lemma, which assumes "isolated singularity" but not necessarily "normal".[1]

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[1]We do not know whether the converse of Lemma 4.2 is true or not.

Lemma 4.2. *Let* $(X, 0)$ *be a surface germ with isolated singularity. If* $\mathbb{C}\mathrm{lk}(X, 0)$ *is contractible then* $\mathrm{lk}(X, 0)$ *is homeomorphic to* S^3.

Proof: Let g be a generic function on the surface $(X, 0)$. We refer to the Milnor fibration (1) of the function g and we use the notation $X_M := g^{-1}(M) \cap B_\varepsilon$ for some subset $M \subset D_\delta$. By hypothesis, the fibre X_λ is contractible, $\lambda \in D_\delta \setminus \{0\}$. Then its boundary $\partial X_\lambda := \partial B_\varepsilon \cap X_\lambda$ is a circle S^1. Indeed, by (2) we have that ∂X_λ is diffeomorphic to the link of the function g, hence it is a disjoint union of circles. It is therefore enough to show that there is a single circle. From the homology exact sequence of the pair $(\bar{X}_\lambda, \partial X_\lambda)$ and since $H_1(\bar{X}_\lambda, \partial X_\lambda)$ is dual to $H^1(X_\lambda)$ thus isomorphic to $H_1(X_\lambda)$, it follows that $H_0(\partial X_\lambda) = \mathbb{Z}$. This shows our claim.

For the link $\mathrm{lk}(X, 0) := \partial B_\varepsilon \cap X$ we have the homotopy equivalence:

$$\mathrm{lk}(X, 0) \overset{\mathrm{ht}}{\simeq} (B_\varepsilon \cap X_{\partial D_\delta}) \cup_{\partial B_\varepsilon \cap X_{\partial D_\delta}} \partial B_\varepsilon \cap X_{D_\delta}.$$

By (1), the space $B_\varepsilon \cap X_{\partial D_\delta}$ is a locally trivial fibration over ∂D_δ with contractible fibre (the complex link) and it is therefore homeomorphic to a full torus $S^1 \times D_\delta$.

Due to (2), the term $\partial B_\varepsilon \cap X_{D_\delta}$ is a trivial fibration over D_δ with fibre $\partial X_0 \simeq S^1$, hence homeomorphic to a solid torus $D_\delta \times S^1$. The common boundary $\partial B_\varepsilon \cap X_{\partial D_\delta}$ is a trivial fibration over ∂D_δ with fibre S^1 as shown above, hence homeomorphic to a torus $S^1 \times S^1$. The glueing of the two tori produces a lens space and since this glueing is trivial by the fact that $S^1 \times S^1$ has trivial monodromy, it follows that this lens space is a 3-sphere.

This shows the homotopy equivalence $\mathrm{lk}(X, 0) \overset{\mathrm{ht}}{\simeq} S^3$, hence that $\mathrm{lk}(X, 0)$ is simply connected. $\qquad\square$

5 Mumford's criterion does not extend to surfaces with isolated singularity

Let us consider the following low degree example $F : \mathbb{C}^2 \to \mathbb{C}^4$, $F(x, y) = (x, y^2, xy^3, xy + y^3)$. Then F is injective and proper, with $Y = \mathrm{Im}F$ a surface which is singular with an isolated singularity at the origin and it is not a complete intersection.

A function $g : (Y, 0) \to \mathbb{C}$ has an isolated singularity if and only if g has no singularities on $Y \setminus \{0\}$. By the general bouquet theorem for functions $(X, 0) \to \mathbb{C}$ with isolated singularities with respect to some Whitney stratification of X [Ti1], the Milnor fibre of such a function has the homotopy type of a bouquet of spaces out of which one is $\mathbb{C}\mathrm{lk}(X, 0)$. This yields a "minimality" property of complex links, i.e. [Ti2, Corollary 2.6]: *The complex link* $\mathbb{C}\mathrm{lk}(X, 0)$ *is minimal among the Milnor fibres of functions* $(X, 0) \to \mathbb{C}$ *with isolated singularity.*

Consequently, in order to show that the complex link of our space $(Y, 0)$ is contractible it is enough to exhibit a function on Y with isolated singularity and with contractible Milnor fibre. If z_1 denotes the first coordinate of $\mathbb{C}^4$ then consider the restriction $z_{1|Y}$ of the linear function z_1. This has isolated singularity on Y. Its pull-back by F is the linear function x on $\mathbb{C}^2$, which has a trivial complex link. Since the two Milnor fibres, of $z_{1|Y}$ and of x, are homeomorphic by F, we have proved via [Ti2, Theorem 1.1] that the complex link of $(Y, 0)$ is contractible. By Lemma 4.2, the link $\mathrm{lk}(Y, 0)$ is simply connected. This example shows in particular that Mumford's criterion cannot be extended to surfaces with isolated singularity. One can directly

prove that the link of Y at 0 is simply connected by using the fact that F is the normalisation map of Y.

6 Surfaces with non-isolated singularity

Proof of Theorem 3.1. Let $(X, 0) \subset (\mathbb{C}^3, 0)$ be a hypersurface germ defined by $f = 0$.

"⇐". If X is an equisingular family of curves, then let $l : X \cap B_\varepsilon \cap l^{-1}(D_\delta) \to D_\delta$ be the projection of the family, for a small enough ball B_ε centered at 0, and let $X_t := l^{-1}(t)$. Consider the germ of the *polar curve*. $\Gamma(l, f) := \overline{\mathrm{Sing}\,(l, f) \setminus \mathrm{Sing}\,f}$ of the map $(l, f) : \mathbb{C}^3 \to \mathbb{C}^2$. The polar curve of a function f with respect to a linear function l is a old geometric notion which was brought into light in the 1970's by Lê D.T., see e.g. [Lê3], [LR], and used ever since by many authors.

By the equisingularity assumption, the curve X_t has a single singularity of Milnor number $\mu(X_t)$ independent on t. This implies that $\mathrm{Sing}\,X$ has multiplicity equal to 1 hence it is a non-singular irreducible curve. By [LR] (see also [Lê3]) the invariance of $\mu(X_t)$ implies that $\Gamma(l, f) = \emptyset$. Further on, these imply that X_t is contractible. Then by [Ti2, Corollary 2.6] the complex link of $(X, 0)$ is also contractible.

"⇒". If the complex link $X_t := g^{-1}(t) \cap B_\varepsilon$ is contractible, where g denotes a general linear function, then it follows that $\Gamma(g, f) = \emptyset$. This implies that the total Milnor number $\mu(X_t)$ is equal to the Milnor number $\mu(X_0)$ of the germ of X_0 at 0. By the non-splitting principle (cf [Lê1], [AC]), this means that X_t has a single singular point and that $\mathrm{Sing}\,X$ is a non-singular irreducible curve. Therefore X is an equisingular deformation of a plane curve (which may be reducible). □

Since the above proof is the same for a higher dimensional hypersurface $(X, 0)$, we actually get the following statement, which may be compared to an observation by Lê D.T. [Lê2, pag. 285, Rem. 2]:

Corollary 6.1. *Let $(X, 0) \subset (\mathbb{C}^n, 0)$, $n \geq 3$, be a hypersurface germ with 1-dimensional singularity. The complex link $\mathbb{C}\mathrm{lk}(X, 0)$ is contractible if and only if X is an equisingular deformation of a $(n-2)$-dimensional isolated hypersurface germ.*

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