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LCK metrics on Oeljeklaus-Toma manifolds versus Kronecker's theorem

by
Victor Vuletescu *

Abstract

A locally conformally Kähler (LCK) manifold is a manifold which is covered by a Kähler manifold, with the deck transform group acting by homotheties. We show that the search for LCK metrics on Oeljeklaus-Toma manifolds leads to a (yet another) variation on Kronecker's theorem on units. In turn, this implies that on any Oeljeklaus-Toma manifold associated to a number field with $2t$ complex embeddings and s real embeddings with $1 < s \leqslant t$ there is no LCK metric.

Key Words: Locally conformally Kähler manifold, number field, units.
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1 Introduction

1.1 Locally conformally Kähler structures

A **locally conformally Kähler** (LCK) manifold is a complex manifold X, $\dim_{\mathbb{C}} X > 1$, admitting a Kähler covering $(\tilde{X}, \tilde{\omega})$, with the deck transform group acting on $(\tilde{X}, \tilde{\omega})$ by holomorphic homotheties. In other words, for all $\gamma \in \pi_1(X) \subset Aut(\tilde{X})$ there exists some $\chi(\gamma) \in \mathbb{R}_{>0}$ such that

$$\gamma^*(\tilde{\omega}) = \chi(\gamma)\tilde{\omega}.$$

The positive numbers $\chi(\gamma)$ are called *the automorphy factors* of X.

LCK manifolds were introduced in the late 70's by I Vaisman, in an attempt to exhibit interesting metrics on non-Kähler manifolds. Basically, Vaisman noticed that the fundamental group of a standard Hopf manifold X (for simplicity, generated by $(z \mapsto 2z)$) acts on the standard flat metric ω_0 on $\mathbb{C}^n \setminus \{0\}$ by homotheties; consequently, the metric $\frac{1}{|z|}\omega_0$ descends to the quotient. Thus, even if X has no Kähler metric (for instance, since it has first Betti number equal to 1), it still carries a interesting metric, as $\frac{1}{|z|}\omega_0$ is locally conformal to a Kähler one.

Deciding whether a given (compact) complex manifold belongs or not to the class of manifolds carrying an LCK metric is a rather tricky problem. No general procedures can apply; this

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class is known not be closed under (even small!) deformations, and is still an open problem whether is closed or not under taking products or finite quotients. On the other hand, no general (e.g. topological) restrictions are known; except for the non-simply-connectedness, only some mild restrictions are known on the fundamental group of a compact compact manifold that prevent it from having LCK metrics with additional properties (see e.g. [OV3]).

Despite these difficulties, along the years, a rather suprising result emerged: almost all compact complex non-Kähler surfaces have LCK metrics! A rough chronological list would include (apart from standard Hopf surfaces from Vaisman's original paper): one class of Inoue surfaces (Tricerri, 1982, [Tri]), general Hopf surfaces (Gauduchon-Ornea, 1998, [GaOr]), elliptic surfaces and another class of Inoue surfaces (Belgun, 2000, [Bel]) and eventually the only known examples of surfaces in Kodaira's class VII_b with $b > 0$, namely Kato surfaces (Brunella, 2010-2011, [Bru1], [Bru2]). Let us mention, that the only class of non-Kähler surfaces known so far not to admit LCK structures is a third class of Inoue surfaces (Belgun, [Bel]) and, possibly, some hypothetical non-Kato surfaces in class $VII_b, b > 0$ - which are also supposed, by the *global spherical shells conjecture*, not to exist!

In higher dimensions, the only know examples to-day are complex structures on products of spheres of the form $S^1 \times S^{2n-1}$ (and their complex submanifolds; see e.g. [OV1], [OV2]) and some Oeljeklaus-Toma manifolds, which will be described below.

1.2 Oeljeklaus-Toma manifolds

We follow the original paper [OeTo]. Fix a number field K having $s > 0$ real embeddings and $2t > 0$ complex embeddings. Let $\mathbb{H}$ be the complex upper half plane; then the ring of integers $\mathcal{O}_K$ of K acts on $\mathbb{H}^s \times \mathbb{C}^t$ by

$$a \cdot (z_1, \ldots, z_{s+t}) = (z_1 + \sigma_1(a), \ldots, z_{s+t} + \sigma_{s+t}(a)) .$$

Next, the group of *totally positive units* $\mathcal{O}_K^{*,+}$ (i.e. units $u \in \mathcal{O}_K^*$ with positive value in all real embedings of K) also acts on $\mathbb{H}^s \times \mathbb{C}^t$ in a similar way by

$$u \cdot (z_1, \ldots, z_{s+t}) = (\sigma_1(u) z_1, \ldots, \sigma_{s+t}(u) z_{s+t}) .$$

If a subgroup $U \subset \mathcal{O}_K^{*,+}$ with $rank(U) = s$ is such that its projection onto its first s factors of its logarithmic embedding is a full lattice in $\mathbb{R}^s$ (such subgroups are called *admissible subgroups*) then combining the above action of $\mathcal{O}_K$ with the action of units in U gives a co-compact, properly discontinous action of $U \ltimes \mathcal{O}_K$ on $\mathbb{H}^s \times \mathbb{C}^t$; the resulting quotient will be denoted $X(K, U)$ and called an *Oeljeklaus-Toma manifold*.

We recollect some facts about the manifolds $X(K, U)$; once again, we refer the original paper [OeTo] for details and proofs.

Theorem 1. *a) For any choice of the number field K and of the admissible subgroup U, the manifold $X(K, U)$ is non-Kähler;*

b) for $t = 1, s > 0$ and any choice of admissible U, the manifold $X(K, U)$ has an LCK structure;

c) for $s = 1, t > 1$ and any choice of admissible U, the manifold $X(K, U)$ has no LCK structure.

2 The results

2.1 The geometrical issues

A classical theorem due to L. Kronecker (in 1857) asserts that if a unit of some number field K has the same absolute value in *all* the embeddings of K it must be a root of unity.

Since then, many variations of this theme (*algebraic integres with specified restrictions on the absolute values of it Galois conjugates*) appeared. As we shall see below, the search for LCK metrics on Oeljeklaus-Toma manifolds leads naturally to a (yet another) problem in this theme, namely the search for *units of number fields with the same absolute value in all complex embeddings.*

This is due to the following:

Lemma 1. If an Oeljeklaus-Toma manifold $X = X(K, U)$ has an LCK metric, then its automorphy factors $\chi(u)$ for $u \in U$ are given by:

$$\chi(u) = |\sigma_{s+1}(u)|^2 = \cdots = |\sigma_{s+t}(u)|^2 \tag{2.1}$$

for all $u \in U$. In particular, for all $u \in U$ one has

$$|\sigma_{s+1}(u)| = \cdots = |\sigma_{s+t}(u)|.$$

Remark. Notice that for at least one unit $u \in U$ we must have $\chi(u) \neq 1$, otherwise $X(K, U)$ would be Kähler.

Proof. Assume that ω is Kähler metric on $\mathbb{H}^s \times \mathbb{C}^t$ upon which $U \ltimes \mathcal{O}_K$ acts by homotheties. Then ω can be written as

$$\omega = \sum_{i,j=\overline{1,s+t}} h_{i\bar{j}} dz_i \wedge d\bar{z}_j.$$

By an average argument, as in [OeTo], we can assume that all the coefficients of ω depend only on $z_1, \ldots, z_s$. Next, we infer that all $h_{i\bar{i}}, i = s+1, \ldots, s+t$ are constant. Indeed, if this would not be the case, then

$$\frac{\partial h_{i\bar{i}}}{\partial z_k} \neq 0$$

for some $1 \leqslant k \leqslant s$. But from the Kählerianity assumption we get that also

$$\frac{\partial h_{k\bar{i}}}{\partial z_i} \neq 0,$$

a contradiction with our assumption on $h_{k\bar{i}}$. Now the conclusion on the automorphy factors follows at once.

We are now in position to state the main result of the paper.

Theorem 2. Let $X = X(K, U)$ be an Oeljeklaus-Toma manifold associated to a number field K with s real embeddings and $2t$ complex embeddings and to an admissible subgroup $U \subset \mathcal{O}_K^{*,+}$. If $1 < s \leqslant t$ then X has no lck metric.

The proof follows immediately from Theorem 3 in the next section, which is basically just a reformulation of it.

2.2 The number-theoretical issues

In this section, we state and prove the main number-theoretical ingredients needed for the proof of the main result. Since this section may be of interest for number-theorists (who may wish to skip the other sections), we recall the setup.

Fix K a number field with s real mebedings and $2t$ complex embeddings; we label $\sigma_i, i = 1, \dots, s+2t$ its embeddings, with the convention that the first s ones are real, and for any $i = 1, \dots, t$ one has $\sigma_{s+t+i} = \overline{\sigma}_{s+i}$.

We will introduce the following *ad-hoc* terminology, inspired by the equalities (2.1), which we consider suggestive for the geometrical context we are working in.

Definition. A unit $u \in \mathcal{O}_K^*$ will be called *homothetical* if

$$|\sigma_{s+1}(u)| = \cdots = |\sigma_{s+t}(u)| \neq 1$$

and respectively *isometrical* if

$$|\sigma_{s+1}(u)| = \cdots = |\sigma_{s+t}(u)| = 1.$$

To give examples of isometrical or homothetical units, we proceed as in [PaVu].

For isometrical units, fix a totally real number field L of arbitrary degree m, fix $\theta \in \mathcal{O}_L$ a primitive element of it. We let $K = L(u)$ where u has minimal polynomial over L of the form

$$X^2 - \theta X + 1.$$

Then u is an isometrical unit. Notice that the procedure used to obtain isometrical units is rather general, since the minimal polynomial over $\mathbb{Q}$ of any such unit is reciprocal.

Producing homothetical units for general t is less immediate. Of course, for $t = 1$ every unit is a homothetical unit. This already gives a way of producing homothetical units for larger t; indeed, pick a number field L with 2 complex embeddings, and consider any finite extension $L \subset K$ unramified at infinity (that is, no real embedding of L extends to a complex embedding of K). Then any unit $u \in \mathcal{O}_L$ will give a homothetical unit of K.

Another way of getting homothetical units is as follows. Pick any monic irreducible polynomial $f \in \mathbb{Z}[X]$ with exactely two non-real roots and with $f(0) = 1$. Let $g(X) = f(X^2)$; g is also an irreducible polynomial for general f. Also, for general f, its complex roots will have absolute value $\neq 1$. Then the class of X in $\mathbb{Q}[X]/(g)$ will be a homothetical unit.

The search for such units, or, more generally, for algebraic integers with many galois conjugates of equal absolute value has been the scope of many papers; we cite here only [Boyd], later extended by [Fe], or [Dix]. Actually, the key step in the proof main result of the present paper is based on the result in [Fe].

We recall also the notion of *admissible subgroup* of units, needed for the construction of Oeljeklaus-Toma manifolds; it is a subgroup $U \subset \mathcal{O}_K^{*,+}$ whose projection onto its first s factors of its logarithmic embedding form a full lattice in $\mathbb{R}^s$.

Recall that for the construction of Oeljeklaus-Toma manifolds, we need admissible subgroups $U \subset \mathcal{O}_K^{*,+}$ with $rank(U) = s$ containing only homothetical or isometrical units; recall also that any admissible subgroup we consider must contain at least one homothetical unit, (see the remark after 1). The main scope of this section is to prove that for $1 < s \leqslant t$ such subgroups do not exists.

Before stating and proving the theorem, let us give a rough idea why this should be true. Indeed, inspecting the above ways of constructing homothetical units, we see the first procedure can not yield admissible subgoups, as units coming form proper subfields lead to violation of the admissibility condition, while units obtained by taking radicals of units coming from proper subfields will usually be not totally positive.

Theorem 3. If K is a number field with s real embeddings and $2t$ complex embeddings with $1 < s \leqslant t$ then there are no admissible subgroups $U \subset \mathcal{O}_K^{*,+}$ of rank s containing only homothetical or isometrical units, and at least one homothetical unit.

Proof. The plan of proof is as follows. We prove first that all homothetical units in U have degree $< [K : \mathbb{Q}]$. This will imply that there is some proper subfield $L \subset K$ such that $U \subset \mathcal{O}_L^*$. But this will force some of the complex embeddings of these units to have different absolute value, contradiction.

To prove the first assertion, let $u \in U$ be a homothetical unit of maximal degree $deg(u) = [K : \mathbb{Q}]$, and let us label by $r_1, \dots, r_s$ its images under the real embeddings and respectively $z_1, \dots, z_{2t}$ its images under the complex embeddings (with the convention that $z_{t+i} = \overline{z}_i$ for all $i = 1, \dots t$). Let us also denote by R the common value of $|z_k|, k = 1, \dots, t$.
Recall that we have:

$$z_1\overline{z}_1 = \cdots = z_t\overline{z}_t. \tag{2.2}$$

Let K^{ncl} be the normal closure of $\mathbb{Q} \subset K$; then for any $i = 1, \dots, s$ there exists some $\sigma \in Gal(K^{ncl}|\mathbb{Q})$ such that $\sigma(z_1) = r_i$. Applying σ to (2.2) we get

$$r_i\sigma(\overline{z}_1) = \sigma(z_2)\sigma(\overline{z}_2) \cdots = \sigma(z_t)\sigma(\overline{z}_t). \tag{2.3}$$

Recalling $s \leqslant t$, we see that in the above equations (2.3) either

- all factors $\sigma(z_k)\sigma(\overline{z}_k)$ with $k \geqslant 2$ are of the form $r_{i(j)}z_{\alpha(j)}$ for some $i(j) \in \{1, \dots, s\}$ and some $\alpha(j) \in \{1, \dots, 2t\}$, or
- we have at least one factor of the form $\sigma(z_k)\sigma(\overline{z}_k)$ equal to some $z_\alpha z_\beta$ for some $k = 2, \dots t$ and some $\alpha, \beta \in 1, \dots, 2t$.

In the first case, one has $r_i = r_j$ for some different $i, j = 1, \dots, s$, a contradiction with the assumption that $deg(u)$ is maximal. Hence, we are left with the remaining case, so we have

$$r_i \sigma(\overline{z}_1) = z_\alpha z_\beta. \tag{2.4}$$

We consider the occuring possibilities.
Case 1. There is some $\gamma(i) \in 1, \dots, 2t$ such that $\sigma(\overline{z}_1) = z_{\gamma(i)}$. Taking absolute values, we get $r_i = R$. But then, since u was assumed to be of maximal degree $deg(u) = [K : \mathbb{Q}]$ we have that all the $z_1, \dots, z_{2t}$ are distinct, so by [Fe], the minimal polynomial $f \in \mathbb{Q}[X]$ of u is of the form $f(X^{2t+1})$. We get $2t + 1 | s + 2t$; but this is absurd, as $1 < s \leqslant t$.

So we are left with:
Case 2. For all $i = 1, \dots, s$, there exists $\varphi(i) \in \{1, \dots, s\}$ and some $\alpha(i), \beta(i) \in \{1, \dots 2t\}$ such that $r_i r_{\varphi(i)} = z_{\alpha(i)} z_{\beta(i)}$. Again, taking absolute values we get

$$r_i r_{\varphi(i)} = R^2.$$

Noticing that φ is a bijection, (since by assumption u was of maximal degree, so all the $r_i's$ are distinct), we this implies

$$\prod_{i=1}^{s} r_i = R^s.$$

But as u is a totally positive unit, we have

$$\left(\prod_{i=1}^{s} r_i \right) R^{2t} = 1,$$

hence we get $R = 1$, again a contradiction, as u was assumed to be a homothetical unit. We conclude that every homothetical unit has degree $< [K : \mathbb{Q}]$.

Next, let $L_1, \dots, L_M$ be the set of proper subfields of K generated by the homothetical units in U (i.e. for each i there is some homothetical unit $u_i \in U$ such that $L_i = \mathbb{Q}(u_i)$) and for each $i \in \{1, \dots, M\}$ let

$$C_i = \{u \in U | u\text{=homothetical unit}, u \in L_i\}.$$

Let us also $Isom(U)$ for the subset of U formed by the isometrical units; it is a proper subgroup of U. As

$$\bigcup_{i=1}^{M} C_i = U \setminus Isom(U)$$

we see $U =< C_{i_0} >$ for some i_0 (where $< C_{i_0} >$ is the subgroup generated by C_{i_0}). Hence,

$$U \subset \mathcal{O}^*_{L_{i_0}}.$$

But then, at least two complex embeddings $\sigma_k, \sigma_l, (k, l \geqslant s + 1)$ of K lie over different real embeddings of L_{i_0}. To see this, let s' (respt $2t'$) be the number of real (respectively complex) embeddings of L_{i_0} and let $l = [K : L_{i_0}]$. As U is admissible, we must have $s' = s$ (cf [OeTo], Lemma 1.6) and the restriction of any two different real embeddings of K to L_{i_0} cannot coincide. But this means that over each real embedding of L_{i_0} there is at least one complex embedding of K, so there exists $k, l \geqslant s+1$ such that $|\sigma_k(u_{i_0})| \neq |\sigma_l(u_{i_0})|$, contradiction with the assumption on u_{i_0}. Q.E.D.

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University of Bucharest, Faculty of Mathematics,
14 Academiei str., 70109 Bucharest, Romania
E-mail: `vuli@fmi.unibuc.ro`