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On a generalization of the Landesman-Lazer condition and Neumann problem for nonuniformly semilinear elliptic equations in an unbounded domain with nonlinear boundary condition

by

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Abstract

This paper deals with the existence of weak solutions of Neumann problem for a nonuniformly semilinear elliptic equation :

$$\begin{cases} \text{-div}(h(x)\nabla u) + a(x)u = \lambda\theta(x)u + f(x,u) - k(x) & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = g(x,u) & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset R^N$, $N \geq 3$ is an unbounded domain with smooth and bounded boundary $\partial\Omega$, $\bar{\Omega} = \Omega \cup \partial\Omega$, $h(x) \in L^1_{loc}(\bar{\Omega})$, $a(x) \in C(\bar{\Omega})$, $a(x) \to +\infty$ as $|x| \to +\infty$, $f(x,s)$, $x \in \Omega$, $g(x,s)$, $x \in \partial\Omega$ are Carathéodory, $k(x) \in L^2(\Omega)$, $\theta(x) \in L^\infty(\bar{\Omega})$, $\theta(x) \geq 0$.
Our arguments is based on the minimum principle and rely essentially on a generalization of the Landesman-Lazer type condition.

Key Words: Semilinear elliptic equation, Non-uniform, Landesman-Lazer condition, Minimum principle.
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1 Introduction and Preliminaries

Let Ω be an unbounded domain in R^N , $N \geq 3$ with smooth and bounded boundary $\partial\Omega$, $\bar{\Omega} = \Omega \cup \partial\Omega$. We are concerned with the study of the existence of weak solutions of Neumann problem for a nonuniformly semilinear elliptic equation:

$$\begin{cases} \text{-div}(h(x)\nabla u) + a(x)u = \lambda_1\theta(x)u + f(x,u) - k(x) & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = g(x,u) & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where $\frac{\partial u}{\partial n}$ denotes the derivative of u with respect to the outward unit normal to $\partial\Omega$ and $f : \Omega \times R \to R$, $g : \partial\Omega \times R \to R$ are Carathéodory functions which will be specified later.

$$h(x) \in L^1_{loc}(\bar{\Omega}), \quad h(x) \geq 1, \quad \text{for a.e } x \in \bar{\Omega}. \tag{1.2}$$

$$a(x) \in C(\bar{\Omega}), a(x) \geq 1, \forall x \in \bar{\Omega}, \quad a(x) \to +\infty \text{ as } |x| \to +\infty. \tag{1.3}$$

λ_1 denotes the first eigenvalue of the problem:

$$\begin{cases} \text{-div}(h(x)\nabla u) + a(x)u = \lambda_1 \theta(x) u & \text{in } \bar{\Omega}, \\ u(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

We firstly make some comments on the problem (1.1).When Ω is bounded domain in R^N, and $h(x) = 1$ there were extensive studies in the last decades dealing with the Neumann problem for nonlinear elliptic equations involving the p-Laplacian, where different techniques of finding solutions are illutrated. When Ω is unbounded domain and $h(x) \in L^1_{loc}(\Omega)$, we refer the reader to [13] where the authors have considered the existence of weak solutions of Neumann problem for nonuniformly quasilinear elliptic equations involving p-Laplacian type in an unbounded domain $\Omega \subset R^N$ with smooth and bounded boundary $\partial\Omega$ by using variational techniques via the Mountain Pass Theorem.

The goal of this work we consider the existence of weak solutions of nonuniformly semilinear elliptic equations in an unbounded domain $\Omega \subset R^N$ with smooth and bounded boundary $\partial\Omega$ with nonlinear Neumann boundary condition under potential Landesman-Lazer type condition which is more general than classical Landesman-Lazer condition.

On the Landesman-Lazer condition we refer the reader to [1,2,3,7,8]. In [1,2,3] the authors have considered a resonant problem involving p-Laplacian

$$\text{-div}(|\nabla|^{p-2}\nabla u) = \lambda_1 |u|^{p-2} + f(x,u) - h(x) \qquad \text{in } \Omega,$$

where Ω is bounded domain in R^N and the existence of weak solutions $u \in W^{1,p}_0(\Omega)$ is shown provided that one of the following two conditions are satisfies:

$$\int_\Omega f^{+\infty}(x)\varphi_1(x)dx < \int_\Omega h(x)\varphi_1(x)dx < \int_\Omega f^{-\infty}(x)\varphi_1(x)dx$$

or

$$\int_\Omega f^{-\infty}(x)\varphi_1(x)dx < \int_\Omega h(x)\varphi_1(x)dx < \int_\Omega f^{+\infty}(x)\varphi_1(x)dx,$$

where (λ_1, φ_1) is eigenvalue and eigenfunction associated with λ_1 of the operator $(-\Delta_p, W^{1,p}_0(\Omega))$, $f^{\pm\infty}(x) = \lim_{s\to\pm\infty} f(x,s)$, $x \in \Omega$.

In [7,8] the authors have extended some results in [1,2,3] to Dirichlet problem for nonuniformly nonlinear general elliptic equations in divergence form in bounded domain.

In this paper, by assuming an extended type of Landesman-Lazer condition, we consider Neumann problem for nonuniformly semilinear elliptic equations with nonlinear boundary condition in an unbounded domain $\Omega \subset R^N$.

Recall that due to $h(x) \in L^1_{loc}(\Omega)$ the problem (1.1) now is nonuniformly in sense that the Euler-Lagrange functional associated to the problem may be infinity at some u in $H^1(\Omega)$. Hence we must consider problem (1.1) in some suitable subspace of $H^1(\Omega)$.

Denotes by
$$C_0^\infty(\bar{\Omega}) = \{\varphi \in C^\infty(\bar{\Omega}) : \mathrm{Supp}\varphi \text{ Compact} \subset \bar{\Omega}\},$$
where $\bar{\Omega} = \Omega \cup \partial\Omega$.

Then $H^1(\Omega)$ is usual Sobolev space which can be defined as the completion of $C_0^\infty(\bar{\Omega})$ under the norm:
$$||\varphi|| = (\int_\Omega (|\nabla\varphi|^2 + |\varphi|^2)dx)^{\frac{1}{2}}.$$
We now define following subspace E of $H^1(\Omega)$ as:
$$E = \{u \in H^1(\Omega) : \int_\Omega (h(x)|\nabla u|^2 + a(x)|u|^2)dx < +\infty\}.$$
By similar arguments as those used in proof of Proposition 1.2 in [11], we deduce that E is a Hilbert space with the norm:
$$||u||_E = (\int_{R^N} (h(x)|\nabla u|^2 + a(x)|u|^2)dx)^{\frac{1}{2}}, \quad u \in E$$
and the continuous embeddings $E \hookrightarrow H^1(\Omega) \hookrightarrow L^q(\Omega)$, $2 \leq q \leq 2^*$ hold true.

Moreover by condition (1.3) embedding E into $L^2(\Omega)$ is compact.

Besides since $\partial\Omega$ is bounded and smooth boundary, hence with $R > 0$ large enough $\partial\Omega \subset B_R(0)$, where $B_R(0)$ is ball of radius R.

Denote $\Omega_R = \bar{\Omega} \cap B_R(0)$, the map $E \hookrightarrow H^1(\Omega_R)$ by $u \to u|_{\Omega_R}$ is continuous.
Therefore from Theorem A_8 in [10] we deduce that $E \hookrightarrow L^2(\partial\Omega)$ compactly.

Remark 1.1. *With similar arguments as those used in the proof of the Lemma 2.3 in [4], we infer that the functional $J_0 : E \to R$ given by*
$$J_0(u) = ||u||_E^2 = \int_\Omega (h(x)|\nabla u|^2 + a(x)|u|^2)dx, \quad u \in E$$
is weakly lower semicontinuous on E.

Next, we will prove a proposition which concerns the existence of the first eigenvalue and eigenfunction of the problem (1.4) and the proof is made by adapting some arguments used in proof of the Proposition 2.2 in [6].

Proposition 1.1. *Let $\theta(x) \in L^\infty(\bar{\Omega})$, $\theta(x) > 0$, $x \in \bar{\Omega}$.*

Denotes by
$$\lambda_1 = inf\{\int_\Omega (h(x)|\nabla u|^2 + a(x)|u|^2)dx : u \in E, \int_\Omega \theta(x)|u|^2 dx = 1\}. \tag{1.5}$$
Then:

(i) $S = \{u \in E : \int_\Omega \theta(x)|u|^2(x)dx = 1\} \neq \phi$.

(ii) There exists $\varphi_1 \in S$, $\varphi_1 > 0$ *in* $\bar{\Omega}$ *such that :*

$$\int_\Omega (h(x)|\nabla\varphi_1|^2 + a(x)|\varphi_1|^2)dx = \lambda_1.$$

Proof:

(i) Let $u(x) \in C_0^\infty(\bar{\Omega})$, $u \neq 0$ then $u \in E$ and $\int_\Omega \theta(x)|u|^2(x)dx > 0$.
Choose $\bar{u} \in E$ as:

$$\bar{u}(x) = \frac{u(x)}{(\int_\Omega \theta(x)|u|^2(x)dx)^{\frac{1}{2}}} \quad \text{for } x \in \bar{\Omega}.$$

Then $\int_\Omega \theta(x)|\bar{u}(x)|^2dx = 1$. So $\bar{u} \in S$ and $S \neq \phi$.

(ii) Let $u_m \subset E$ be a minimizing sequence , i.e

$$\int_\Omega \theta(x)|u_m(x)|^2dx = 1, m = 1, 2, ..$$

and $\lim\limits_{m\to+\infty} \int_\Omega (h(x)|\nabla u_m|^2 + a(x)|u_m|^2)dx = \lambda_1$. So $\{u_m\}$ is bounded in E. Then there exists a subsequence $\{u_{m_k}\}_k$ such that $\{u_{m_k}\}_k$ converges weakly to $\widehat{u}$ in E. Since the embedding E into $L^2(\Omega)$ is compact, subsequence $\{u_{m_k}\}$ converges strongly to $\widehat{u}$ in $L^2(\Omega)$.

Moreover since $\theta(x) \in L^\infty(\Omega)$ we infer that:

$$\lim_{k\to+\infty} \int_\Omega \theta(x)(u_{m_k}^2 - \widehat{u}^2)dx = 0,$$

hence

$$1 = \lim_{k\to+\infty} \int_\Omega \theta(x)|u_{m_k}|^2dx = \int_\Omega \theta(x)|\widehat{u}|^2dx.$$

So $\widehat{u} \in S$.

By the minimizing properties and the weakly lower semicontinuity of the functional $J_0(u) = \int_\Omega (h(x)|\nabla u|^2 + a(x)|u|^2)dx$ on E (see Remark 1.1), we have:

$$\lambda_1 = \liminf_{k\to+\infty} \int_\Omega (h(x)|\nabla u_{m_k}|^2 + a(x)|u_{m_k}|^2)dx \geq \int_\Omega (h(x)|\nabla\widehat{u}|^2 + a(x)|\widehat{u}|^2)dx \geq \lambda_1$$

So we obtain

$$\lambda_1 = \int_\Omega (h(x)|\nabla\widehat{u}|^2 + a(x)|\widehat{u}|^2)dx.$$

Thus $\widehat{u}$ is a minimizer of (1.5).

Observe further that since $\widehat{u} \in E \subset H^1(\Omega)$ then $|\widehat{u}| \in H^1(\Omega)$ (see Lema 7.6 p.152 [5]). Moreover,

$$\int_\Omega (h(x)|\nabla\widehat{|u|}|^2 + a(x)|\widehat{u}|^2)dx = \int_\Omega (h(x)|\nabla\widehat{u}|^2 + a(x)|\widehat{u}|^2)dx < +\infty$$

and
$$\int_\Omega (\theta(x)|\widehat{u}|^2)dx = \int_\Omega (\theta(x)|\widehat{|u|}|^2)dx = 1,$$
hence $|\widehat{u}| \in S$ and we have
$$\lambda_1 = \int_\Omega (h(x)|\nabla \widehat{|u|}|^2 + a(x)|\widehat{u}|^2)dx.$$
Thus $|\widehat{u}|$ is a minimizer too.

Applying the Lagrange multiplier theorem, we deduce that
$$\text{-div}(h(x)\nabla|\widehat{u}|) + a(x)|\widehat{u}| = \lambda_1\theta(x)|\widehat{u}|, \text{ in } \bar{\Omega}.$$
Then for any Ω' compact $\subset \bar{\Omega}$, we have $h(x) \in L^1(\Omega')$, $a(x) \in L^\infty(\Omega')$, $|\widehat{u}| \geq 0$ in Ω' and
$$\text{-div}(h(x)\nabla|\widehat{u}|) + a(x)|\widehat{u}| = \lambda_1\theta(x)|\widehat{u}|, \text{ in } \Omega'.$$
So by the Harnack inequality (see [5], Theorem 8.19 or Theorem 8.20 and Corollary 8.21), it follows that $|\widehat{u}| > 0$ in Ω'. This implies that $|\widehat{u}| > 0$ in $\bar{\Omega}$.

Denotes $\varphi_1(x) = |\widehat{u}|$, then $\varphi_1(x) > 0$ and φ_1 is λ_1 eigenfunction of the problem (1.4). The proof of Proposition 1.1 is complete. □

On the other hand by similar argument we also show that the eigenfunctions of λ_1 are either positive or negative in $\bar{\Omega}$. Hence by the compact embedding E into $L^2(\Omega)$ and the standard spectral theory for compact, self-adjoint operators we can infer that the λ_1-eigenfunction φ_1 is unique (up to a multiplicitive constant) and
$$\lambda_1 = \inf_{0\neq u\in E} \frac{\int_\Omega (h(x)|\nabla u|^2 + a(x)|u|^2)dx}{\int_\Omega (\theta(x)|u|^2)dx}.$$

In order to state our main results, let us introduce following some hypotheses:

(H1) (i) $k(x) \in L^2(\Omega)$, $\theta(x) \in L^\infty(\Omega)$, $\theta(x) > 0$, for all $x \in R^N$.

(ii) $f : \Omega \times R \to R$, $g : \partial\Omega \times R \to R$ are Carathéodory functions, $f(x, 0) = 0$ and there exist positive functions $\tau_1(x) \in L^2(\Omega)$, $\tau_2(x) \in L^2(\partial\Omega)$ such that:
$$\begin{aligned} |f(x, s)| &\leq \tau_1(x), \quad \text{for a.e } x \in \Omega,\ s \in R, \\ |g(x, s)| &\leq \tau_2(x), \quad \text{for a.e } x \in \partial\Omega,\ s \in R. \end{aligned}$$

(H2) Denotes $F(x, s) = \int_0^s f(x, t)dt$, $x \in \Omega$ and $G(x, s) = \int_0^s g(x, t)dt$, $x \in \partial\Omega$, we define:
$$F^{+\infty}(x) = \lim_{s\to+\infty}\inf \frac{F(x, s)}{s}, \qquad F_{-\infty}(x) = \lim_{s\to-\infty}\sup \frac{F(x, s)}{s}, \qquad x \in \Omega,$$
$$G^{+\infty}(x) = \lim_{s\to+\infty}\inf \frac{G(x, s)}{s}, \qquad G_{-\infty}(x) = \lim_{s\to-\infty}\sup \frac{G(x, s)}{s}, \qquad x \in \partial\Omega.$$

Assume that the following potential Landesman-Lazer type condition holds

$$\int_\Omega F^{+\infty}(x)\varphi_1(x)dx + \int_{\partial\Omega} h(x)G^{+\infty}(x)\varphi_1(x)ds < \int_\Omega k(x)\varphi_1(x)dx < \\ < \int_\Omega F_{-\infty}(x)\varphi_1(x)dx + \int_{\partial\Omega} h(x)G_{-\infty}(x)\varphi_1(x)ds \tag{1.6}$$

We remark that as those proved in [12] the condition (1.6) is more general the classical Landesman-Lazer condition.

Definition 1.1. *Function $u \in E$ is called a weak solution of the problem (1.1) if and only if*

$$\int_\Omega (h(x)\nabla u\nabla\varphi(x) + a(x)u\varphi(x))dx - \lambda_1 \int_\Omega \theta(x)u(x)\varphi(x)dx - \int_\Omega f(x,u)\varphi(x)dx + \\ + \int_\Omega k(x)\varphi(x)dx - \int_{\partial\Omega} h(x)g(x,u)\varphi(x)ds = 0, \text{for all } \varphi(x) \in C_0^\infty(\bar\Omega). \tag{1.7}$$

Proposition 1.2. *If function $u_0 \in C^2(\bar\Omega)$ satisfies the condition (1.7), hence u_0 is a classical solution of the problem (1.1).*

Proof: Indeed, let $R > 0$ be large enough such that: $\partial\Omega \subset B(0,R)$, where $B(0,R)$ is open ball of radius R.
Denotes $\Omega_R = \Omega \cap B(0,R)$, $\bar\Omega_R = \bar\Omega \cap B(0,R)$ and $\partial\Omega_R = \partial\Omega \cup \partial B(0,R)$. Since u_0 satisfies the condition (1.7) we have:

$$\int_\Omega (h(x)\nabla u_0\nabla\varphi(x) + a(x)u_0\varphi(x))dx - \lambda_1 \int_\Omega \theta(x)u_0\varphi(x)dx - \int_\Omega f(x,u_0)\varphi(x)dx + \\ + \int_\Omega k(x)\varphi(x)dx - \int_{\partial\Omega} h(x)g(x,u_0)\varphi(x)ds = 0, \text{ for all } \varphi(x) \in C_0^\infty(\bar\Omega).$$

Applying Green's formula and remark that for $\varphi(x) \in C_0^\infty(\bar\Omega_R)$, $\varphi(x) = 0$ for $x \in \partial B(0,R)$ we get

$$\int_{\Omega_R} (\text{-div}(h(x)\nabla u_0) + a(x)u_0)\varphi dx + \int_{\partial\Omega} h(x)\frac{\partial u_0}{\partial n}\varphi ds - \\ - \lambda_1 \int_{\Omega_R} \theta(x)u_0\varphi dx - \int_{\Omega_R} f(x,u_0)\varphi dx + \int_{\Omega_R} k(x)\varphi dx - \\ - \int_{\partial\Omega} h(x)g(x,u_0)\varphi ds = 0, \text{ for all } \varphi(x) \in C_0^\infty(\bar\Omega_R). \tag{1.8}$$

Since $C_0^\infty(\Omega_R) \subset C_0^\infty(\bar\Omega_R)$, from (1.8) we have

$$\int_{\Omega_R} (\text{-div}(h(x)\nabla u_0) + a(x)u_0)\varphi dx - \lambda_1 \int_{\Omega_R} \theta(x)u_0\varphi dx -$$
$$- \int_{\Omega_R} f(x,u_0)\varphi dx + \int_{\Omega} k(x)\varphi dx = 0, \text{ for all } \varphi(x) \in C_0^\infty(\Omega_R).$$

This implies that

$$\text{-div}(h(x)\nabla u_0) + a(x)u_0 = \lambda_1\theta(x)u_0(x) + f(x,u_0) - k(x), \text{ in } \Omega_R. \tag{1.9}$$

Combining (1.8), (1.9) we obtain

$$\int_{\partial\Omega} h(x)\frac{\partial u_0}{\partial n}\varphi ds - \int_{\partial\Omega} h(x)g(x,u_0)\varphi ds = 0, \text{ for all } \varphi(x) \in C_0^\infty(\bar{\Omega}_R).$$

Hence

$$h(x)\frac{\partial u_0}{\partial n} = h(x)g(x,u_0), \quad x \in \partial\Omega.$$

Since $h(x) \geq 1$ for $x \in \partial\Omega$ it follows

$$\frac{\partial u_0}{\partial n} = g(x,u_0), \qquad x \in \partial\Omega. \tag{1.10}$$

Letting $R \to +\infty$ from (1.9),(1.10) we infer that

$$\begin{cases} \text{-div}(h(x)\nabla u_0) + a(x)u_0 = \lambda_1\theta(x)u_0 + f(x,u_0) - k(x) & \text{in } \Omega, \\ \dfrac{\partial u_0}{\partial n} = g(x,u_0) & \text{on } \partial\Omega. \end{cases}$$

That is u_0 is a classical solution of (1.1). □

Our main result is given by the following theorem

Theorem 1.1. *Assuming conditions (H1), (H2) are fulfilled. Then the problem (1.1) has at least nontrivial weak solution in E.*

Proof of Theorem 1.1 is based on variational techniques and the minimum principle.

Theorem 1.2. *(Minimum principle) (see [10],[7],[8])*
Let X be a Banach space and $I \in C^1(X)$. Assume that:
(i) I is bounded from below on X and $c = \inf_X I$.
(ii) I satisfies the Palais-Smale condition on X.
Then c is a critical value of I (i.e there exists a critical point $u_0 \in X$ such that $I(u_0) = c$).

2 Proof of existence of a weak solution

The Euler-Lagrange functional associated to the problem (1.1) $I : E \to R$ is given by

$$I(u) = \frac{1}{2}\int_\Omega (h(x)|\nabla u|^2 + a(x)|u|^2)dx - \frac{\lambda_1}{2}\int_\Omega \theta(x)|u|^2 dx - \int_\Omega F(x,u)dx + \\ + \int_\Omega k(x)udx - \int_{\partial\Omega} h(x)G(x,u)ds, \quad \text{for all } u \in E. \tag{2.1}$$

Denote

$$J(u) = \frac{1}{2}\int_\Omega (h(x)|\nabla u|^2 + a(x)|u|^2)dx, \tag{2.2}$$

$$T(u) = -\frac{\lambda_1}{2}\int_\Omega \theta(x)|u|^2 dx - \int_\Omega F(x,u)dx + \int_\Omega k(x)udx - \int_{\partial\Omega} h(x)G(x,u)ds. \tag{2.3}$$

Then $I(u) = J(u) + T(u)$, $u \in E$.

By hypotheses (H1) on functions $f(x,s)$, $g(x,s)$, $k(x)$, $\theta(x)$ the functionals T and $I = J+T$ are well-defined on E. The following proposition which concerns the smoothness of functional $I = J+T$ on E.

Proposition 2.1. *The Euler-Lagrange functional I given by (2.1) is Fréchet differentiable on E and we have:*

$$(I'(u),v) = \int_\Omega (h(x)\nabla u\nabla v + a(x)uv)dx - \lambda_1\int_\Omega \theta(x)u(x)v(x)dx - \int_\Omega f(x,u)v(x)dx + \\ + \int_\Omega k(x)v(x)dx - \int_{\partial\Omega} h(x)g(x,u)vds, \quad \text{for all } u,v \in E. \tag{2.4}$$

Proof: With similar arguments as those used in the proof of the Proposition 2.2 (iii) in [11] we deduce that the functional J given by (2.2) is Gateaux differentiable on E and whose the Gateaux derivative is given by:

$$(J'(u),v) = \int_\Omega (h(x)\nabla u\nabla v + a(x)uv)dx, \quad \text{for all } u,v \in E. \tag{2.5}$$

Now let $\{u_m\}$ be a sequence converging to u in E, i.e

$$\lim_{m\to+\infty}||u_m - u||_E = \lim_{m\to+\infty}(\int_\Omega (h(x)|\nabla(u_m-u)|^2 + a(x)|u_m-u|^2)dx)^{\frac{1}{2}} = 0.$$

Then for any $v \in E$ we have

$$
\begin{aligned}
|(J'(u_m) - J'(u)), v| &= |\int_\Omega (h(x)\nabla(u_m - u)\nabla v + a(x)(u_m - u)v)dx| \\
&\leq \int_\Omega h(x)|\nabla(u_m - u)||\nabla v|dx + \int_{R^N} a(x)|u_m - u||v|dx \\
&\leq (\int_\Omega h(x)|\nabla(u_m - u)|^2 dx)^{\frac{1}{2}}.(\int_\Omega h(x)|\nabla v|^2 dx)^{\frac{1}{2}} + \\
&\quad + (\int_\Omega a(x)|u_m - u|^2 dx)^{\frac{1}{2}}.(\int_\Omega a(x)|v|^2 dx)^{\frac{1}{2}}.
\end{aligned}
$$

This implies that

$$|(J'(u_m) - J'(u), v)| \leq 2||v||_E||u_m - u||_E$$

and so

$$||J'(u_m) - J'(u)||_{E^*} \leq 2||u_m - u||_E.$$

Let $m \to +\infty$ we obtain: $\lim\limits_{m\to+\infty} J'(u_m) = J'(u)$ in E^*.

Hence J' is continuous on E. Thus $J \in C^1(E, R)$.

Besides, from hypotheses (H1) on the functions $f(x, s)$, $g(x, s)$, $\theta(x)$ and $k(x)$ for some standard computations we infer that the functional T given by (2.3) is Fréchet differentiable on E and we have

$$
\begin{aligned}
(T'(u), v) = -\lambda_1 \int_\Omega \theta(x)uvdx - \int_\Omega f(x, u)vdx + \int_\Omega k(x)vdx- \\
- \int_\Omega h(x)g(x, u)vds, \quad \text{for all } u, v \in E. \qquad (2.6)
\end{aligned}
$$

Finally the functional $I = J + T \in C^1(E, R)$ and by combining (2.5) and (2.6) we obtain (2.4). The Proposition 2.1 is proved. □

Remark 2.1. *By Proposition 2.1 the critical points of the functional I are precisely the weak solutions of the problem (1.1).*

Proposition 2.2. *The functional I satisfies the Palais-Smale condition on E provided that condition (H2) holds.*

Proof: Let $\{u_m\}$ be a sequence in E and β be a positive number such that

$$|I(u_m)| \leq \beta \qquad \text{for } m = 1, 2, .. \qquad (2.7)$$

$$I'(u_m) \to 0 \qquad \text{in } E^* \text{ as } m \to +\infty. \qquad (2.8)$$

Firstly we prove that $\{u_m\}$ is a bounded sequence in E. Indeed, by contradiction we assume that $||u_m||_E \to +\infty$ as $m \to +\infty$.

Let $v_m = \frac{u_m}{||u_m||_E}$ for every m. Thus $\{v_m\}$ is bounded in E.
Then there exists a subsequence $\{v_{m_k}\}$ which converges weakly to some v in E.
Since the embeddings E into $L^2(\Omega)$ and $L^2(\partial\Omega)$ are compact, the subsequence $\{v_{m_k}\}$ converges strongly to v in $L^2(\Omega)$ and $L^2(\partial\Omega)$, i.e

$$\lim_{k\to+\infty} ||v_{m_k} - v||_{L^2(\Omega)} = 0, \qquad \lim_{k\to+\infty} ||v_{m_k} - v||_{L^2(\partial\Omega)} = 0.$$

From (2.7) dividing by $||u_{m_k}||_E^2$ we deduce that

$$\lim_{k\to+\infty} \sup\{\frac{1}{2}\int_\Omega (h(x)\left|\nabla v_{m_k}\right|^2 + a(x)\left|v_{m_k}\right|^2)dx - \frac{\lambda_1}{2}\int_\Omega \theta(x)\left|v_{m_k}\right|^2 dx$$
$$- \int_\Omega \frac{F(x,u_{m_k})}{||u_{m_k}||_E^2}dx + \int_\Omega k(x)\frac{u_{m_k}}{||u_{m_k}||_E^2}dx - \int_{\partial\Omega} h(x)\frac{G(x,u_{m_k})}{||u_{m_k}||_E^2}dx\} \le 0. \tag{2.9}$$

Observe that by condition (H1) we have

$$|F(x,u_{m_k})| \le \tau_1(x)\left|u_{m_k}\right| \text{ where } \tau_1(x) \in L^2(\Omega),$$

it follows that

$$\left|\int_\Omega \frac{F(x,u_{m_k})}{||u_{m_k}||_E^2}dx\right| \le \int_\Omega \frac{\tau_1(x)\left|u_{m_k}\right|}{||u_{m_k}||_E^2}dx \le \frac{1}{||u_{m_k}||_E}\,||\tau_1||_{L^2(\Omega)}\,||v_{m_k}||_{L^2(\Omega)}.$$

Since v_{m_k} converges strongly in $L^2(\Omega)$, $\{||v_{m_k}||_E\}$ is bounded, letting $k\to+\infty$, since $||u_{m_k}||_E \to +\infty$, we obtain:

$$\lim_{k\to+\infty} \sup \int_\Omega \frac{F(x,u_{m_k})}{||u_{m_k}||_E^2}dx = 0.$$

Similarly we also obtain

$$\lim_{k\to+\infty} \sup \int_\Omega k(x)\frac{u_{m_k}}{||u_{m_k}||_E^2}dx = 0,$$

$$\lim_{k\to+\infty} \sup \int_{\partial\Omega} h(x)\frac{G(x,u_{m_k})}{||u_{m_k}||_E^2}ds = 0.$$

Moreover

$$\lim_{k\to+\infty} \sup \int_\Omega \theta(x)|v_{m_k}|^2dx = \int_\Omega \theta(x)|v|^2dx.$$

From (2.9) we deduce that

$$\lim_{k\to+\infty} \sup \int_\Omega (h(x)\left|\nabla v_{m_k}\right|^2 + a(x)\left|v_{m_k}\right|^2)dx \le \lambda_1 \int_\Omega \theta(x)|v|^2dx.$$

By Remark 1.1 and the variational characterization of λ_1, we get:

$$\begin{aligned}\lambda_1 \int_\Omega \theta(x)|v|^2 dx &\leq \int_\Omega (h(x)\left|\nabla v\right|^2 + a(x)\left|v\right|^2)dx \\ &\leq \liminf_{k\to+\infty} \int_\Omega (h(x)\left|\nabla v_{m_k}\right|^2 + a(x)\left|v_{m_k}\right|^2)dx \\ &\leq \limsup_{k\to+\infty} \int_\Omega (h(x)\left|\nabla v_{m_k}\right|^2 + a(x)\left|v_{m_k}\right|^2)dx \\ &\leq \lambda_1 \int_\Omega \theta(x)|v|^2 dx.\end{aligned}$$

Thus, these inequalities are indeed equalities and

$$\int_\Omega (h(x)\left|\nabla v\right|^2 + a(x)\left|v\right|^2)dx = \lambda_1 \int_\Omega \theta(x)|v|^2 dx.$$

This implies, by definition of φ_1, that $v = \pm\varphi_1$.
On the other hand, from (2.7) we deduce that

$$\begin{aligned}-2\beta \leq &- \int_\Omega (h(x)\left|\nabla u_{m_k}\right|^2 + a(x)\left|u_{m_k}\right|^2)dx + \lambda_1 \int_\Omega \theta(x)|u_{m_k}|^2 dx \\ &+ 2\int_\Omega F(x,u_{m_k})dx - 2\int_\Omega k(x)u_{m_k}dx + 2\int_{\partial\Omega} h(x)G(x,u_{m_k})ds \leq 2\beta\end{aligned} \tag{2.10}$$

and in view of (2.8) there exists a sequence of positive numbers $\{\epsilon_k\}_k$, $\epsilon_k \to 0$ as $k \to +\infty$ such that :

$$\begin{aligned}-\epsilon_k \left|\left|u_{m_k}\right|\right|_E \leq &\int_\Omega (h(x)\left|\nabla u_{m_k}\right|^2 + a(x)\left|u_{m_k}\right|^2)dx - \lambda_1 \int_\Omega \theta(x)|u_{m_k}|^2 dx \\ &- \int_\Omega f(x,u_{m_k})u_{m_k}dx + \int_\Omega k(x)u_{m_k}dx - \int_{\partial\Omega} h(x)g(x,u_{m_k})u_{m_k}ds \leq \\ &\leq \epsilon_k \left|\left|u_{m_k}\right|\right|_E .\end{aligned} \tag{2.11}$$

We consider the following two cases:

<u>Case 1:</u> Suppose $v_{m_k} \to -\varphi_1$. Then $u_{m_k}(x) \to -\infty$ a.e $x \in \Omega$.
Firstly from (2.8) we get

$$|< I'(u_{m_k}), \varphi_1 >| \to 0, \quad \text{as } k \to +\infty.$$

i.e

$$\begin{aligned}\lim_{k\to+\infty} \{&\int_\Omega (h(x)\nabla u_{m_k}\nabla\varphi_1 + a(x)u_{m_k}\varphi_1)dx - \lambda_1 \int_\Omega \theta(x)u_{m_k}\varphi_1 dx - \\ &- \int_\Omega f(x,u_{m_k})\varphi_1 dx + \int_\Omega k(x)\varphi_1 dx - \int_{\partial\Omega} h(x)g(x,u_{m_k})\varphi_1 ds\} = 0.\end{aligned} \tag{2.12}$$

Observe that
$$\int_\Omega (h(x)\nabla u_{m_k}\nabla\varphi_1 + a(x)u_{m_k}\varphi_1)dx = \lambda_1\int_\Omega \theta(x)u_{m_k}\varphi_1 dx.$$
Hence we infer from (2.12) that
$$\lim_{k\to+\infty}\{-\int_\Omega f(x,u_{m_k})\varphi_1 dx + \int_\Omega k(x)\varphi_1 dx - \int_{\partial\Omega} h(x)g(x,u_{m_k})\varphi_1 ds\} = 0. \tag{2.13}$$
By summing up (2.10) and (2.11) we get
$$\begin{aligned}-2\beta - \epsilon_k||u_{m_k}||_E \le \int_\Omega [2F(x,u_{m_k}) - f(x,u_{m_k})u_{m_k}]dx - \int_\Omega k(x)u_{m_k}dx+\\ +\int_{\partial\Omega}[2G(x,u_{m_k}) - g(x,u_{m_k})u_{m_k}]ds \le 2\beta + \epsilon_k||u_{m_k}||_E\end{aligned}$$
and after dividing by $||u_{m_k}||_E$ we obtain
$$\begin{aligned}-\frac{2\beta}{||u_{m_k}||_E} - \epsilon_k \le \int_\Omega [2\frac{F(x,u_{m_k})}{u_{m_k}}v_{m_k} - f(x,u_{m_k})v_{m_k}]dx - \int_\Omega k(x)v_{m_k}dx+\\ +\int_{\partial\Omega} h(x)[2\frac{G(x,u_{m_k})}{u_{m_k}}v_{m_k} - g(x,u_{m_k})v_{m_k}]ds \le \frac{2\beta}{||u_{m_k}||_E} + \epsilon_k.\end{aligned} \tag{2.14}$$
By hypotheses (1-2) on $h(x)$, (H1) on $f(x,s)$, $g(x,s)$, $k(x)$, since $v_{m_k} \to (-\varphi_1)$ in $L^2(\Omega)$ and $L^2(\partial\Omega)$ we have
$$\lim_{k\to+\infty}\int_\Omega f(x,u_{m_k})v_{m_k}dx = \lim_{k\to+\infty} -\int_\Omega f(x,u_{m_k})\varphi_1 dx, \tag{2.15}$$
$$\lim_{k\to+\infty}\int_\Omega k(x)v_{m_k}dx = -\int_\Omega k(x)\varphi_1(x)dx \tag{2.16}$$
and
$$\lim_{k\to+\infty}\int_{\partial\Omega} h(x)g(x,u_{m_k})v_{m_k}ds = \lim_{k\to+\infty} -\int_{\partial\Omega} h(x)g(x,u_{m_k})\varphi_1 ds. \tag{2.17}$$
Then from (2.14) taking $\lim\limits_{k\to+\infty}$ sup to both sides with together (2.15), (2.16), (2.17) and using (2.13) we deduce that
$$0 \le -2\int_\Omega F_{-\infty}(x)\varphi_1(x)dx - 2\int_{\partial\Omega} h(x)G_{-\infty}(x)\varphi_1(x)ds + 2\int_\Omega k(x)\varphi_1(x)dx \le 0,$$
which gives
$$\int_\Omega k(x)\varphi_1(x)dx = \int_\Omega F_{-\infty}(x)\varphi_1(x)dx + \int_{\partial\Omega} h(x)G_{-\infty}(x)\varphi_1(x)ds,$$
which contradicts (H2).

<u>Case 2:</u> Suppose $v_{m_k} \to \varphi_1$. Then $u_{m_k}(x) \to +\infty$ a.e $x \in \Omega$. Since $v_{m_k} \to \varphi_1$ in $L^2(\Omega)$ we have

$$\lim_{k\to+\infty} \int_\Omega f(x, u_{m_k}) v_{m_k} dx = \lim_{k\to+\infty} \int_\Omega f(x, u_{m_k}) \varphi_1 dx,$$
$$\lim_{k\to+\infty} \int_\Omega k(x) v_{m_k} dx = \int_\Omega k(x)\varphi_1(x) dx$$

and

$$\lim_{k\to+\infty} \int_{\partial\Omega} h(x) g(x, u_{m_k}) v_{m_k} ds = \lim_{k\to+\infty} \int_{\partial\Omega} h(x) g(x, u_{m_k}) \varphi_1 ds$$

Using (2.13) from (2.14) by taking $\lim\limits_{k\to+\infty}$ inf to both sides we obtain:

$$0 \le 2\int_\Omega F^{+\infty}(x)\varphi_1(x)dx + 2\int_{\partial\Omega} h(x)G^{+\infty}(x)\varphi_1(x)ds - 2\int_\Omega k(x)\varphi_1(x)dx \le 0,$$

which gives

$$\int_\Omega k(x)\varphi_1(x)dx = \int_\Omega F^{+\infty}(x)\varphi_1(x)dx + \int_{\partial\Omega} h(x)G^{+\infty}(x)\varphi_1(x)ds,$$

which contradicts (H2).

Thus the sequence $\{u_m\}$ is bounded in E.

Next, we prove that the sequence $\{u_m\}$ has a subsequence which converges strongly in E.

Since the sequence $\{u_m\}$ is bounded in E, then there exists a subsequence $\{u_{m_k}\}$ which converges weakly to u in E. By the weak lower semicontinuity of the functional J (see remark 1.1) we have

$$J(u) \le \lim_{k\to+\infty} \inf J(u_{m_k}). \tag{2.18}$$

Moreover by the compact embeddings E into $L^2(\Omega)$ and $L^2(\partial\Omega)$, $\{u_{m_k}\}$ converges strongly in $L^2(\Omega)$ and $L^2(\partial\Omega)$, i.e

$$\lim_{k\to+\infty} ||u_{m_k} - u||_{L^2(\Omega)} = 0 \qquad \text{and} \qquad \lim_{k\to+\infty} ||u_{m_k} - u||_{L^2(\partial\Omega)} = 0. \tag{2.19}$$

On the other hand, from condition (H1), we deduce that

$$\begin{aligned} |< T'(u_{m_k}), u_{m_k} - u >| &\le \lambda_1 \int_\Omega \theta(x) |u_{m_k}| \, |u_{m_k} - u| \, dx + \int_\Omega |f(x, u_{m_k})| \, |u_{m_k} - u| \, dx + \\ &+ \int_{\partial\Omega} h(x) |g(x, u_{m_k})| \, |u_{m_k} - u| \, ds + \int_\Omega |k(x)| \, |u_{m_k} - u| \, dx + \\ &\le C_1 \{\lambda_1 ||u_{m_k}||_{L^2(\Omega)} + ||\tau_1(x)||_{L^2(\Omega)} + ||k(x)||_{L^2(\Omega)}\} \, ||u_{m_k} - u||_{L^2(\Omega)} \\ &+ C_2 \, ||\tau_2(x)||_{L^2(\partial\Omega)} \, ||u_{m_k} - u||_{L^2(\partial\Omega)} \, . \end{aligned}$$

From this, letting $k \to +\infty$, by (2.19) and remark that $\{||u_{m_k}||_{L^2(\Omega)}\}$ is bounded, we obtain

$$\lim_{k\to+\infty} < T'(u_{m_k}), u_{m_k} - u >= 0. \tag{2.20}$$

Moreover from (2.8) it implies

$$\lim_{k\to+\infty} < I'(u_{m_k}), u_{m_k} - u >= 0. \tag{2.21}$$

Combining (2.20), (2.21) it follows that

$$\lim_{k\to+\infty} < J'(u_{m_k}), u_{m_k} - u >= \lim_{k\to+\infty}\{< I'(u_{m_k}), u_{m_k} - u > +T'(u_{m_k}), u_{m_k} - u >\} = 0.$$

Observe further that since J is convex, the following inequality holds true

$$J(u) - J(u_{m_k}) \geq (J'(u_{m_k}), u_{m_k} - u).$$

Letting $k \to +\infty$ we get

$$J(u) - \lim_{k\to+\infty} J(u_{m_k}) \geq \lim_{k\to+\infty} (J'(u_{m_k}), u_{m_k} - u) = 0.$$

Thus

$$J(u) \geq \lim_{k\to+\infty} J(u_{m_k}). \tag{2.22}$$

Relation (2.18) and (2.22) implies that

$$J(u) = \lim_{k\to+\infty} J(u_{m_k}).$$

Now we prove that the subsequence $\{u_{m_k}\}$ converges strongly to u in E, i.e:

$$\lim_{k\to+\infty} ||u_{m_k} - u||_E = 0.$$

Indeed, we suppose by contradiction that the subsequence $\{u_{m_k}\}$ is not converging strongly to u in E.
Then there exist a constant $\epsilon_0 > 0$ and subsequence $\{u_{m_{k_j}}\}_j$ such that:

$$\left|\left|u_{m_{k_j}} - u\right|\right|_E \geq \epsilon_0 > 0, \qquad (j = 1, 2, ...).$$

Recalling the equality

$$\left|\frac{\alpha+\beta}{2}\right|^2 + \left|\frac{\alpha-\beta}{2}\right|^2 = \frac{1}{2}(\alpha^2 + \beta^2), \qquad \alpha, \beta \in R.$$

We deduce that for any $j = 1, 2, ...$

$$\frac{1}{2}J(u_{m_{k_j}}) + \frac{1}{2}J(u) - J(\frac{u_{m_{k_j}} + u}{2}) = \frac{1}{4}\left|\left|u_{m_{k_j}} - u\right|\right|_E^2 \geq \frac{1}{4}\epsilon_0^2. \tag{2.23}$$

Again instead of the remark that since the sequence $\{\frac{u_{m_{k_j}}+u}{2}\}$ converges weakly to u in E, we have

$$J(u) \leq \lim_{j\to+\infty} J(\frac{u_{m_{k_j}}+u}{2}).$$

Moreover from (2.20) letting $j \to +\infty$ we get

$$J(u) - \lim_{j\to+\infty} J(\frac{u_{m_{k_j}}+u}{2}) \geq \frac{1}{4}\epsilon_0^2.$$

Hence $\frac{1}{4}\epsilon_0^2 \leq 0$ which is a contradiction.

Thus, the subsequence $\{u_{m_k}\}$ converges strongly to u in E. This implies that the functional I satisfies the Palais-Smale condition in E.

The proof of Proposition 2.2 is complete. □

Proposition 2.3. *The functional I is coercive on E provided that condition (H2) holds true.*

Proof: We firstly note that, in the proof of the Proposition 2.2, we have proved that if the sequence $\{I(u_m)\}_m$ is bounded from above with $||u_m||_E \to +\infty$, then (up to a subsequence), $v_m = \frac{u_m}{||u_m||_E} \to \pm\varphi_1$ in E, using this fact we will prove that the functional I is coercive on E provided that condition (H2) holds .

Indeed, if I is not coercive on E, then there exists a sequence $\{u_m\} \subset E$ such that $||u_m||_E \to +\infty$ and $I(u_m) \leq c$, c is positive constant. Observe that by the variational characterization of λ_1 we have

$$\int_\Omega (h(x)\left|\nabla u_m\right|^2 + a(x)\left|u_m\right|^2)dx \geq \lambda_1 \int_\Omega \theta(x)\left|u_m\right|^2 dx, \qquad m = 1, 2, ...$$

Hence

$$-\int_\Omega F(x, u_m)dx - \int_{\partial\Omega} h(x)G(x, u_m)ds + \int_\Omega k(x)u_m(x)dx \leq I(u_m). \tag{2.24}$$

Assume that $v_m \to \varphi_1$, letting $m \to +\infty$ after dividing by $||u_m||_E$ we get

$$\begin{aligned} -\int_\Omega F^{+\infty}(x)\varphi_1(x)dx - \int_{\partial\Omega} h(x)G^{+\infty}(x)\varphi_1(x)ds + \\ + \int_\Omega k(x)\varphi_1(x)dx \leq \lim_{m\to+\infty}\sup\frac{I(u_m)}{||u_m||_E} = 0, \end{aligned}$$

which gives

$$\int_\Omega k(x)\varphi_1(x)dx \leq \int_\Omega F^{+\infty}(x)\varphi_1(x)dx + \int_{\partial\Omega} h(x)G^{+\infty}(x)\varphi_1(x)ds.$$

This contradicts condition (H2).

If $v_m \to -\varphi_1$. By similar aguments above, we deduce that

$$\int_\Omega k(x)\varphi_1(x)dx \geq \int_\Omega F_{-\infty}(x)\varphi_1(x)dx + \int_{\partial\Omega} h(x)G_{-\infty}(x)\varphi_1(x)ds.$$

Hence we get a contradiction. Thus I is coercive on E and Proposition 2.3 is proved. $\square$

Proof of theorem 1.1:

By the minimum principle (see Theorem 1.2), the coerciveness of the functional I and the Palais-Smale condition are enough to prove that I attains its proper infimum at some $u_0 \in E$, so that the problem (1.1) has at least a weak solution $u_0 \in E$. Moreover by hypotheses (H1) on functions $f(x, s)$, $k(x)$, it is clear that u_0 is nontrivial and the Theorem 1.1 is proved.

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