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Cofiniteness of weakly Laskerian local cohomology modules

by
[1]Moharram Aghapournahr and [2]Kamal Bahmanpour

Dedicated to Professor Leif Melkersson

Abstract

Let I be an ideal of a Noetherian ring R and M be a finitely generated R-module. We introduce the class of extension modules of finitely generated modules by the class of all modules T with $\dim T \leq n$ and we denote it by $\mathrm{FD}_{\leq n}$ where $n \geq -1$ is an integer. We prove that for any $\mathrm{FD}_{\leq 0}$(or minimax) submodule N of $H^t_I(M)$ the R-modules $\mathrm{Hom}_R(R/I, H^t_I(M)/N)$ and $\mathrm{Ext}^1_R(R/I, H^t_I(M)/N)$ are finitely generated, whenever the modules $H^0_I(M)$, $H^1_I(M)$, ..., $H^{t-1}_I(M)$ are $\mathrm{FD}_{\leq 1}$ (or weakly Laskerian). As a consequence, it follows that the set of associated primes of $H^t_I(M)/N$ is finite. This generalizes the main results of Bahmanpour and Naghipour [4] and [5], Brodmann and Lashgari [6], Khashyarmanesh and Salarian [21] and Hong Quy [18]. We also show that the category $FD^1(R,I)_{cof}$ of I-cofinite $\mathrm{FD}_{\leq 1}$ R-modules forms an Abelian subcategory of the category of all R-modules.

Key Words: Local cohomology module, cofinite module, Weakly Laskerian modules.
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1 Introduction

The following conjecture was made by Grothendieck in [15]:

Conjecture: *For any ideal I of a Noetherian ring R and any finite R–module M, the module $\mathrm{Hom}_R(R/I, H^j_I(M))$ is finitely generated for all $j \geq 0$.*

Here, $H^j_I(M)$ denotes the j^{th} local cohomology module of M with support in I. Although the conjecture is not true in general as was shown by Hartshorne in [16], there are some attempts to show that under some conditions, for some number t, the module $\mathrm{Hom}_R(R/I, H^t_I(M))$ is finite, see [2, Theorem 3.3], [11, Theorem 6.3.9], [13, Theorem 2.1], [4, Theorem 2.6] and [5, Theorem 2.3]. In [16], Hartshorne defined an R-module L to be *I-cofinite*, if $\mathrm{Supp}(L) \subseteq V(I)$ and $\mathrm{Ext}^i_R(R/I, L)$ is finitely generated module for all i. He asked:

If I is an ideal of R and M is a finitely generated R-module, when is $H^i_I(M)$ I–cofinite for all i ?

In this direction in section 3 we generalize [2, Theorem 3.3], [4, Theorem 2.6] and [5, Theorem 2.3] to the class of extension modules of finitely generated modules by the class of all modules T with $\dim T \leq 1$ ($\mathrm{FD}_{\leq 1}$). Note that the class of weakly Laskerian modules is contained in the class of $\mathrm{FD}_{\leq 1}$ modules. More precisely, we shall show that:

Theorem 1.1. *Let R be a Noetherian ring and I an ideal of R. Let M be a finitely generated R-module and $t \geq 1$ be a positive integer such that the R-modules $H^i_I(M)$ are $\mathrm{FD}_{\leq 1}$ R-modules (or weakly Laskerian) for all $i < t$. Then, the following conditions hold:*
(i) *The R-modules $H^i_I(M)$ are I-cofinite for all $i < t$.*
(ii) *For all $\mathrm{FD}_{\leq 0}$ (or minimax) submodule N of $H^t_I(M)$, the R-modules*

$$\mathrm{Hom}_R(R/I, H^t_I(M)/N) \text{ and } \mathrm{Ext}^1_R(R/I, H^t_I(M)/N)$$

are finitely generated.

As an immediate consequence we prove the following corollary that is a generalization of Bahmanpour-Naghipour's results in [4] and also the Delfino-Marley's result in [10] and Yoshida's result in [28] for an arbitrary Noetherian ring.

Corollary 1.2. *Let R be a Noetherian ring and I an ideal of R. Let M be a finitely generated R-module such that the R-modules $H^i_I(M)$ are $\mathrm{FD}_{\leq 1}$ (or weakly Laskerian) R-modules for all i. Then,*
(i) *the R-modules $H^i_I(M)$ are I-cofinite for all i.*
(ii) *For any $i \geq 0$ and for any $\mathrm{FD}_{\leq 0}$ (or minimax) submodule N of $H^i_I(M)$, the R-module $H^i_I(M)/N$ is I-cofinite.*

Abazari and Bahmanpour in [1] studied cofiniteness of extension functors of cofinite modules as a generalization of Huneke-Koh's results in [17]. In Corollary 3.8 we generalize the results of Abazari and Bahmanpour.

Hartshorne also posed the following question:

Whether the category $M(R,I)_{cof}$ of I-cofinite modules forms an Abelian subcategory of the category of all R-modules? That is, if $f : M \longrightarrow N$ is an R-module homomorphism of I-cofinite modules, are $\ker f$ and $\mathrm{coker} f$ I-cofinite?

Hartshorne proved that if I is a prime ideal of dimension one in a complete regular local ring R, then the answer to his question is yes. On the other hand, in [10], Delfino and Marley extended this result to arbitrary complete local rings. Recently, Kawasaki [20] generalized the Delfino and Marley's result for an arbitrary ideal I of dimension one in a local ring R. Finally, more recently, Melkersson in [24] completely have removed local assumption on R. One of the main results of this section is to prove that the class of *I-cofinite* $\mathrm{FD}_{\leq 1}$ modules compose an Abelian category (see Theorem 3.7).

Let R denote a commutative Noetherian ring, and let I be an ideal of R. Throughout this paper, R will always be a commutative Noetherian ring with non-zero identity and I will be an ideal of R. We denote $\{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{p} \supseteq \mathfrak{a}\}$ by $V(\mathfrak{a})$. If we say a $\mathrm{D}_{\leq n}$($\mathrm{FD}_{\leq n}$ or FSF) module, we will understand that an R-module belongs to the class $\mathrm{D}_{\leq n}$($\mathrm{FD}_{\leq n}$ or FSF). For any unexplained notation and terminology we refer the reader to [7], [8] and [23].

2 Preliminaries

Yoshizawa in [29, Definition 2.1] defined classes of extension modules of Serre subcategory by another one as below.

Definition 2.1. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be Serre subcategories of the category of all R-modules. We denote by $(\mathcal{S}_1, \mathcal{S}_2)$ the class of all R-modules M with some R-modules $S_1 \in \mathcal{S}_1$ and $S_2 \in \mathcal{S}_2$ such that a sequence $0 \longrightarrow S_1 \longrightarrow M \longrightarrow S_2 \longrightarrow 0$ is exact.*

We will denote the class of all modules M with $\dim M \leq n$ by $\mathrm{D}_{\leq n}$ and the class of extension modules of finitely generated modules by the class of $\mathrm{D}_{\leq n}$ modules by $\mathrm{FD}_{\leq n}$ where $n \geq -1$ is an integer. Note that the class of $\mathrm{FD}_{\leq -1}$ is the same as finitely generated R-modules. Recall that a module M is a *minimax* module if there is a finitely generated submodule N of M such that the quotient module M/N is Artinian. Thus the class of minimax modules is the class of extension modules of finitely generated modules by the class of Artinian modules. Minimax modules have been studied by Zink in [30] and Zöschinger in [31, 32]. See also [27]. Recall too that an R-module M is called *weakly Laskerian* if $\operatorname{Ass}(M/N)$ is a finite set for each submodule N of M. The class of weakly Laskerian modules introduced in [14], by Divaani-Aazar and Mafi. Recently, Hung Quy [18], introduced the class of extension modules of finitely generated modules by the class of all modules of finite support and named it FSF modules. By the following theorem over a Noetherian ring R an R-module M is weakly Laskerian if and only if is FSF.

Theorem 2.2. *Let R be a Noetherian ring and M a nonzero R-module. The following statements are equivalent:*

1. *M is a weakly Laskerian module;*
2. *M is an* FSF *module.*

Proof: See [3, Theorem 3.3]. □

Lemma 2.3. *Let R be a Noetherian ring. Then the following conditions hold:*

(i) *Any finitely generated R-module and any* $\mathrm{D}_{\leq n}$ *R-module are* $\mathrm{FD}_{\leq n}$.

(ii) *The class of* $\mathrm{FD}_{\leq n-1}$ *modules is contained in the class of* $\mathrm{FD}_{\leq n}$ *modules for all* $n \geq 0$.

(iii) *The class of minimax modules is contained in the class of* $\mathrm{FD}_{\leq 0}$ *that is the class of extension modules of finitely generated modules by semiartinian modules.*

(iv) *The class of weakly Laskerian modules is contained in the class of* $\mathrm{FD}_{\leq 1}$.

(v) *The class of* $\mathrm{FD}_{\leq n}$ *R-modules forms a Serre subcategory of the category of all R-modules.*

Proof: (i), (ii), (iii) are trivial.
(iv) Use Theorem 2.2.
(v) See [29, Corollary 4.3 or 4.5]. □

Example 2.4. (i) *Let R be a Notherian ring with* $\dim R \geq 2$ *and let* $\mathfrak{p} \in \operatorname{Spec}(R)$ *such that* $\dim R/\mathfrak{p} = 1$. *Let* $M = R \oplus E(R/\mathfrak{p})$. *It is easy to see that M is an* $\mathrm{FD}_{\leq 1}$ *R-module that is neither finitely generated nor* $\mathrm{D}_{\leq 1}$.
(ii) *Suppose the set Ω of maximal ideals of R is infinite. Then the module* $\oplus_{\mathfrak{m} \in \Omega} R/\mathfrak{m}$ *is* $\mathrm{FD}_{\leq 0}$ *module and thus* $\mathrm{FD}_{\leq 1}$ *but it is not a weakly Laskerian module.*

The following lemma represents an adaption of [24, Lemma 2.2].

Lemma 2.5. *Let I be an ideal of the Noetherian ring R and let M be an R-module such that* $\dim M = 1$ *and* $\operatorname{Supp}_R(M) \subseteq V(I)$. *If* $\operatorname{Hom}_R(R/I, M)$ *is a finite R-module, then there is a finite submodule N of M and an element $x \in I$ such that* $\operatorname{Supp}_R(M/(xM + N)) \subseteq \operatorname{Max}(R)$.

Proof: Since $\operatorname{Hom}_R(R/I, M)$ is finite R-module we conclude that $\operatorname{Ass}_R(M)$ is finite and therefore $\operatorname{Assh}_R(M) = \{\mathfrak{p} \in \operatorname{Supp} M \mid \dim R/\mathfrak{p} = 1\}$ is finite. Consider $S = R \setminus \bigcup_{\mathfrak{p} \in \operatorname{Assh}_R(M)} \mathfrak{p}$. It is easy to see that $\operatorname{Supp}_{S^{-1}R}(S^{-1}M) \subseteq V(S^{-1}I) \cap \operatorname{Max}(S^{-1}R)$ and $\operatorname{Hom}_{S^{-1}R}(S^{-1}R/S^{-1}I, S^{-1}M)$ is a finite $S^{-1}R$-module. From [24, Lemma 2.1] we conclude that $S^{-1}M$ is an Artinian $S^{-1}R$-module and $S^{-1}I$-cofinite. By [26, Corollary 1.2] the $S^{-1}R$-module $S^{-1}M/IS^{-1}M$ is finite. Hence there is a finite submodule N of M, such that $S^{-1}(M/(IM + N)) = 0$. Put $\overline{M} = M/N$. Then $S^{-1}\overline{M}$ (as a homomorphic image of $S^{-1}M$) is an Artinian $S^{-1}R$-module. Furthermore $S^{-1}\overline{M} = IS^{-1}\overline{M}$. Then by [22, 2.8], there is $x \in I$, such that $S^{-1}\overline{M} = xS^{-1}\overline{M}$. Therefore $S^{-1}(\overline{M}/x\overline{M}) = 0$ and hence $\operatorname{Supp}_R(\overline{M}/x\overline{M}) \subseteq \operatorname{Supp}_R(M) \setminus \operatorname{Assh}_R(M) \subseteq \operatorname{Max}(R)$. Together with the isomorphism $\overline{M}/x\overline{M} \cong M/(xM + N)$, this proves our assertion. □

Proposition 2.6. *Let I be an ideal of a Noetherian ring R and M be a* $\mathrm{D}_{\leq 1}$ *module such that* $\operatorname{Supp} M \subseteq V(I)$. *Then the following statements are equivalent:*
(i) *M is I-cofinite,*
(ii) *The R-modules* $\operatorname{Hom}_R(R/I, M)$ *and* $\operatorname{Ext}^1_R(R/I, M)$ *are finitely generated.*

Proof: The conclusion (i)⇒(ii) is obvious. In order to prove (ii)⇒(i) using [24, Lemma 2.1], we may assume $\dim M = 1$. Now use Lemma 2.5 and the method of the proof of [24, Theorem 2.3]. □

3 Cofiniteness of local cohomology

In what follows the next theorem plays an important role.

Theorem 3.1. *Let I be an ideal of a Noetherian ring R and M be an* $\mathrm{FD}_{\leq 1}$ *R-module such that* $\operatorname{Supp} M \subseteq V(I)$. *Then the following statements are equivalent:*
(i) *M is I-cofinite,*
(ii) *The R-modules* $\operatorname{Hom}_R(R/I, M)$ *and* $\operatorname{Ext}^1_R(R/I, M)$ *are finitely generated.*

Proof: $(i) \Rightarrow (ii)$ is clear. In order to prove $(ii) \Rightarrow (i)$, by definition there is a finitely generated submodule N of M such that the R-module $\dim(M/N) \leq 1$ and $\operatorname{Supp} M/N \subseteq V(I)$. Also, the exact sequence

$$0 \to N \to M \to M/N \to 0, \quad (*)$$

induces the following exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(R/I, N) \longrightarrow \operatorname{Hom}_R(R/I, M) \longrightarrow \operatorname{Hom}_R(R/I, M/N)$$

$$\longrightarrow \operatorname{Ext}^1_R(R/I, N) \longrightarrow \operatorname{Ext}^1_R(R/I, M) \longrightarrow \operatorname{Ext}^1_R(R/I, M/N) \longrightarrow \operatorname{Ext}^2_R(R/I, N).$$

Whence, it follows that the R-modules $\operatorname{Hom}_R(R/I, M/N)$ and $\operatorname{Ext}^1_R(R/I, M/N)$ are finitely generated. Therefore, in view of Proposition 2.6, the R-module M/N is I-cofinite. Now it follows from the exact sequence $(*)$ that M is I-cofinite. □

Lemma 3.2. *Let I be an ideal of a Noetherian ring R, M a non-zero R-module and $t \in \mathbb{N}_0$. Suppose that the R-module $H^i_I(M)$ is I-cofinite for all $i = 0, ..., t-1$, and the R-modules $\operatorname{Ext}^t_R(R/I, M)$ and $\operatorname{Ext}^{t+1}_R(R/I, M)$ are finitely generated. Then the R-modules $\operatorname{Hom}_R(R/I, H^t_I(M))$ and $\operatorname{Ext}^1_R(R/I, H^t_I(M))$ are finitely generated.*

Proof: See [13, Theorem 2.1] and [12, Theorem A]. □

Lemma 3.3. *Let I be an ideal of a Noetherian ring R and M be an $\mathrm{FD}_{\leq 0}$ R-module such that $\operatorname{Supp} M \subseteq V(I)$. Then the following statements are equivalent:*
(i) *M is I-cofinite,*
(ii) *The R-module $\operatorname{Hom}_R(R/I, M)$ is finitely generated.*

Proof: The proof is similar to the proof of [25, Proposition 4.3]. □

We are now ready to state and prove the following main results (Theorem 3.4 and the Corollaries 3.5 and 3.6) which are extension of Bahmanpour-Naghipour's results in [4] and [5], Brodmann-Lashgari's result in [6], Khashyarmanesh-Salarian's result in [21], Hong Quy's result in [18], and also the Delfino-Marley's result in [10] and Yoshida's result in [28] for an arbitrary Noetherian ring.

Theorem 3.4. *Let R be a Noetherian ring and I an ideal of R. Let M be a finitely generated R-module and $t \geq 1$ be a positive integer such that the R-modules $H^i_I(M)$ are $\mathrm{FD}_{\leq 1}$ R-modules for all $i < t$. Then, the following conditions hold:*
(i) *The R-modules $H^i_I(M)$ are I-cofinite for all $i < t$.*
(ii) *For all $\mathrm{FD}_{\leq 0}$ (or minimax) submodule N of $H^t_I(M)$, the R-modules*

$$\operatorname{Hom}_R(R/I, H^t_I(M)/N) \text{ and } \operatorname{Ext}^1_R(R/I, H^t_I(M)/N)$$

are finitely generated. In particular the set $\operatorname{Ass}_R(H^t_I(M)/N)$ is finite.

Proof: (i) We proceed by induction on t. By Lemma 3.2 the case $t = 1$ is obvious since $H^0_I(M)$ is finitely generated. So, let $t > 1$ and the result has been proved for smaller values of t. By the inductive assumption, $H^i_I(M)$ is I-cofinite for $i = 0, 1, ..., t-2$. Hence by Lemma 3.2 and assumption, $\mathrm{Hom}_R(R/I, H^{t-1}_I(M))$ and $\mathrm{Ext}^1_R(R/I, H^{t-1}_I(M))$ are finitely generated. Therefore by Theorem 3.1, $H^i_I(M)$ is I-cofinite for all $i < t$. This completes the inductive step.
(ii) In view of (i) and lemma 3.2, $\mathrm{Hom}_R(R/I, H^t_I(M))$ and $\mathrm{Ext}^1_R(R/I, H^t_I(M))$ are finitely generated. On the other hand, according to Lemma 3.3 or Melkersson's result [25, Proposition 4.3], N is I-cofinite. Now, the exact sequence

$$0 \longrightarrow N \longrightarrow H^t_I(M) \longrightarrow H^t_I(M)/N \longrightarrow 0$$

induces the following exact sequence,

$$\mathrm{Hom}_R(R/I, H^t_I(M)) \longrightarrow \mathrm{Hom}_R(R/I, H^t_I(M)/N) \longrightarrow \mathrm{Ext}^1_R(R/I, N) \longrightarrow$$
$$\mathrm{Ext}^1_R(R/I, H^t_I(M)) \longrightarrow \mathrm{Ext}^1_R(R/I, H^t_I(M)/N) \longrightarrow \mathrm{Ext}^2_R(R/I, N).$$

Consequently

$$\mathrm{Hom}_R(R/I, H^t_I(M)/N) \text{ and } \mathrm{Ext}^1_R(R/I, H^t_I(M)/N)$$

are finitely generated, as required. □

Corollary 3.5. *Let R be a Noetherian ring and I an ideal of R. Let M be a finitely generated R-module such that the R-modules $H^i_I(M)$ are $\mathrm{FD}_{\leq 1}$ (or weakly Laskerian) R-modules for all i. Then,*

(i) *The R-modules $H^i_I(M)$ are I-cofinite for all i.*

(ii) *For any $i \geq 0$ and for any $\mathrm{FD}_{\leq 0}$ (or minimax) submodule N of $H^i_I(M)$, the R-module $H^i_I(M)/N$ is I-cofinite.*

Proof: (i) Clear.
(ii) In view of (i) the R-module $H^i_I(M)$ is I-cofinite for all i. Hence the R-module $\mathrm{Hom}_R(R/I, N)$ is finitely generated, and so it follows from Lemma 3.3 or [25, Proposition 4.3] that N is I-cofinite. Now, the exact sequence

$$0 \longrightarrow N \longrightarrow H^i_I(M) \longrightarrow H^i_I(M)/N \longrightarrow 0,$$

implies that the R-module $H^i_I(M)/N$ is I-cofinite. □

Corollary 3.6. *Let R be a Noetherian ring and I an ideal of R. Let M be a finitely generated R-module and $t \geq 1$ be a positive integer such that the R-modules $H^i_I(M)$ are weakly Laskerian for all $i < t$. Then, the following conditions hold:*

(i) *The R-modules $H^i_I(M)$ are I-cofinite for all $i < t$.*

(ii) *For all $\mathrm{FD}_{\leq 0}$ (or minimax) submodule N of $H^t_I(M)$, the R-modules*

$$\mathrm{Hom}_R(R/I, H^t_I(M)/N) \text{ and } \mathrm{Ext}^1_R(R/I, H^t_I(M)/N)$$

are finitely generated. In particular the set $\mathrm{Ass}_R(H^t_I(M)/N)$ is finite.

Proof: Use Theorem 2.2 and note that the category of weakly Laskerian modules is contained in the category of $\mathrm{FD}_{\leq 1}$ modules. □

One of the main result of this section is to prove that for an arbitrary ideal I of a Noetherian ring R, the category of I-cofinite $\mathrm{FD}_{\leq 1}$ modules is an Abelian category.

Theorem 3.7. *Let I be an ideal of a Noetherian ring R. Let $FD^1(R,I)_{cof}$ denote the category of I-cofinite $\mathrm{FD}_{\leq 1}$ R-modules. Then $FD^1(R,I)_{cof}$ is an Abelian category.*

Proof: Let $M, N \in FD^1(R,I)_{cof}$ and let $f : M \longrightarrow N$ be an R-homomorphism. By Lemma 2.3 (v) $\ker f$ and $\mathrm{coker} f$ are $\mathrm{FD}_{\leq 1}$, so it is enough to show that the R-modules $\ker f$ and $\mathrm{coker} f$ are I-cofinite.

To this end, the exact sequence

$$0 \longrightarrow \ker f \longrightarrow M \longrightarrow \mathrm{im} f \longrightarrow 0,$$

induces an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(R/I, \ker f) \longrightarrow \mathrm{Hom}_R(R/I, M) \longrightarrow \mathrm{Hom}_R(R/I, \mathrm{im} f)$$
$$\longrightarrow \mathrm{Ext}^1_R(R/I, \ker f) \longrightarrow \mathrm{Ext}^1_R(R/I, M),$$

that implies the R-modules $\mathrm{Hom}_R(R/I, \ker f)$ and $\mathrm{Ext}^1_R(R/I, \ker f)$ are finitely generated. Therefore it follows from Theorem 3.1 that $\ker f$ is I-cofinite. Now, the assertion follows from the following exact sequences

$$0 \longrightarrow \ker f \longrightarrow M \longrightarrow \mathrm{im} f \longrightarrow 0,$$

and

$$0 \longrightarrow \mathrm{im} f \longrightarrow N \longrightarrow \mathrm{coker} f \longrightarrow 0.$$

□

The following corollary is a generalization of [1, Theorem 2.7].

Corollary 3.8. *Let I be an ideal of a Noetherian ring R. Let M be an $\mathrm{FD}_{\leq 1}$ I-cofinite R-module. Then, the R-modules $\mathrm{Ext}^i_R(N,M)$ and $\mathrm{Tor}^R_i(N,M)$ are I-cofinite and $\mathrm{FD}_{\leq 1}$ modules, for all finitely generated R-modules N and all integers $i \geq 0$.*

Proof: Since N is finitely generated it follows that N has a free resolution of finitely generated free modules. Now the assertion follows using Theorem 3.7 and computing the modules $\mathrm{Tor}^R_i(N,M)$ and $\mathrm{Ext}^i_R(N,M)$, by this free resolution. □

A particular case of following corollary was handled in [9]. In that paper, R is a Gorenstein local ring of dimension d and the module M coincides with R.

Corollary 3.9. *Let I be an ideal of a Noetherian ring R, M a non-zero finite R-module such that $\dim M/IM \leq 1$(e.g., $\dim R/I \leq 1$). Then for each finite R-module N, the R-modules $\mathrm{Ext}^j_R(N, H^i_I(M))$ and $\mathrm{Tor}^R_j(N, H^i_I(M))$ are I-cofinite and $\mathrm{FD}_{\leq 1}$ modules for all $i \geq 0$ and $j \geq 0$.*

Proof: Note that $\dim \operatorname{Supp} H_I^i(M) \leq \dim M/IM \leq 1$ thus $H_I^i(M)$ is an $\mathrm{FD}_{\leq 1}$ R-module and by Corollary 3.5 it is I-cofinite. □

Lemma 3.10. *Let R be a Noetherian ring, I a proper ideal of R and M be a non-zero $\mathrm{D}_{\leq 1}$ and I-cofinite R-module. Then for each non-zero finitely generated R-module N with support in $V(I)$, the R-modules $\operatorname{Ext}_R^i(M,N)$ are finitely generated, for all integers $i \geq 0$.*

Proof: See [19, Theorem 2.8]. □

Corollary 3.11. *Let R be a Noetherian ring and I be an ideal of R. Let M be an $\mathrm{FD}_{\leq 1}$ and I-cofinite R-module. Then, the R-modules $\operatorname{Ext}_R^i(M,N)$ are finitely generated, for all finitely generated R-modules N with $\operatorname{Supp}(N) \subseteq V(I)$ and all integers $i \geq 0$.*

Proof: The assertion follows from the definition using Lemma 3.10 and [25, Theorem 2.1 and Corollary 2.5]. □

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[1]Department of Mathematic,
Faculty of Science, Arak University,
Arak, 38156-8-8349, Iran.
E-mail: m-aghapour@araku.ac.ir

[2]Department of Mathematic,
Faculty of Mathematical Sciences,
University of Mohaghegh Ardabili,
Ardabil , Iran.
E-mail: bahmanpour.k@gmail.com