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Rational toral rank of a map

by
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Abstract

Let X and Y be simply connected CW complexes with finite rational cohomologies. The rational toral rank $r_0(X)$ of a space X is the largest integer r such that the torus T^r can act continuously on a CW-complex in the rational homotopy type of X with all its isotropy subgroups finite [8]. As a rational homotopical condition to be a toral map preserving almost free toral actions for a map $f : X \to Y$, we define the rational toral rank $r_0(f)$ of f, which is a natural invariant with $r_0(id_X) = r_0(X)$ for the identity map id_X of X. We will see some properties of it by Sullivan models, which is a free commutative differential graded algebra over $\mathbb{Q}$ [4].

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1 Introduction

We assume that spaces X and Y are simply connected CW complexes with finite rational cohomologies. Let T^r be an r-torus $S^1 \times \cdots \times S^1$($r$-factors) and let $r_0(X)$ be the *rational toral rank*, which is the largest integer r such that a T^r can act continuously on a CW-complex in the rational homotopy type of X with all its isotropy subgroups finite [8]. Such an action is called *almost free*. Our motivation is in the following problem for an equivariant property of a map $f : X \to Y$.

Problem 1.1. *For an almost free T^r-action μ on X, when can one put an almost free T^r-action on Y so that f becomes T^r-equivariant ? Conversely, given an almost free T^r-action τ*

on Y, when does X admit an almost free T^r-action making f an T^r-equivariant map ?

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
ET^r \times^{\mu}_{T^r} X & \overset{F}{\dashrightarrow} & ? \\
\downarrow & & \downarrow \\
BT^r & = & BT^r
\end{array}
\qquad
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
? & \overset{F}{\dashrightarrow} & ET^r \times^{\tau}_{T^r} Y \\
\downarrow & & \downarrow \\
BT^r & = & BT^r
\end{array}
$$

Here $X \to ET^r \times^{\mu}_{T^r} X \to BT^r$ means the Borel fibration of a T^r-action μ on X. The integer r of Problem 1.1 is bounded from above by the following numerical invariant, obtained from a diagram which is a rational homotopy version of a T^r-equivariant map for almost free T^r-actions. In this paper, we propose

Definition 1.2. For a map $f : X \to Y$, we say that the *rational toral rank of* f, denoted as $r_0(f)$, is r when it is the largest integer such that there is a map F between fibrations over $BT^r_{\mathbb{Q}}$:

$$
\begin{array}{ccc}
X_{\mathbb{Q}} & \xrightarrow{f_{\mathbb{Q}}} & Y_{\mathbb{Q}} \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\
E_1 & \overset{F}{\dashrightarrow} & E_2 \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\
BT^r_{\mathbb{Q}} & = & BT^r_{\mathbb{Q}}
\end{array}
\qquad (*)
$$

with $\dim H^*(E_i; \mathbb{Q}) < \infty$ for $i = 1, 2$.

Here $X_{\mathbb{Q}}$ and $f_{\mathbb{Q}}$ are the rationalizations [10] of a simply connected CW complex X of finite type and a map f, respectively. Let the Sullivan minimal model of X be $M(X) = (\Lambda V, d)$. It is a free $\mathbb{Q}$-commutative differential graded algebra (DGA) with a $\mathbb{Q}$-graded vector space $V = \bigoplus_{i \geq 2} V^i$ where $\dim V^i < \infty$ and a decomposable differential; i.e., $d(V^i) \subset (\Lambda^+ V \cdot \Lambda^+ V)^{i+1}$ and $d \circ d = 0$. Here $\Lambda^+ V$ is the ideal of ΛV generated by elements of positive degree. Denote the degree of a homogeneous element x of a graded algebra as $|x|$. Then $xy = (-1)^{|x||y|} yx$ and $d(xy) = d(x)y + (-1)^{|x|} x d(y)$. Note that $M(X)$ determines the rational homotopy type of X. In particular, $H^*(\Lambda V, d) \cong H^*(X; \mathbb{Q})$ and $V^i \cong Hom(\pi_i(X), \mathbb{Q})$. Refer to [4] for details. If an r-torus T^r acts on a simply connected space X by $\mu : T^r \times X \to X$, there is the Borel fibration

$$X \to ET^r \times^{\mu}_{T^r} X \to BT^r,$$

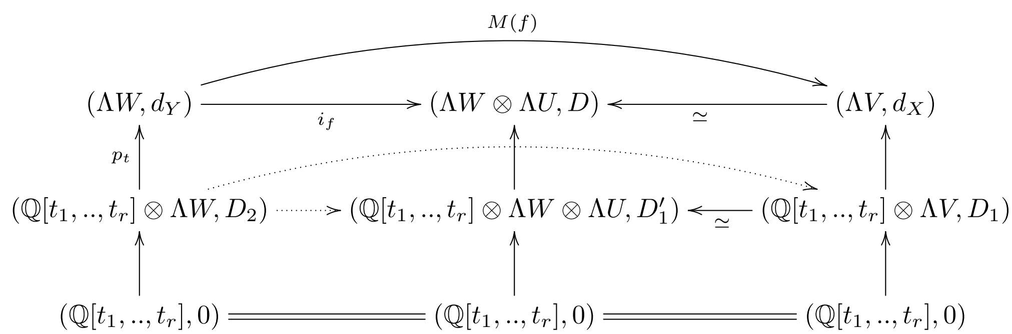
where $ET^r \times^{\mu}_{T^r} X$ is the orbit space of the action $g(e, x) = (e \cdot g^{-1}, g \cdot x)$ on the product $ET^r \times X$ for any $e \in ET^r$, $x \in X$ and $g \in T^r$. Note that $ET^r \times^{\mu}_{T^r} X$ is rational homotopy equivalent to the T^r-orbit space of X when μ is an almost free toral action [5]. The above Borel fibration is rationally given by the relative model (Koszul-Sullivan (KS) model)

$$(\mathbb{Q}[t_1, \ldots, t_r], 0) \to (\mathbb{Q}[t_1, \ldots, t_r] \otimes \Lambda V, D) \to (\Lambda V, d) \qquad (**)$$

where with $|t_i| = 2$ for $i = 1, \ldots, r$, $Dt_i = 0$ and $Dv \equiv dv$ modulo the ideal $(t_1, \ldots, t_r)$ for $v \in V$. The following criterion of Halperin is used in this paper.

Proposition 1.3. *[8, Proposition 4.2] Suppose that X is a simply connected CW-complex with* $\dim H^*(X;\mathbb{Q}) < \infty$. *Put $M(X) = (\Lambda V, d)$. Then $r_0(X) \geq r$ if and only if there is a relative model $(**)$ satisfying* $\dim H^*(\mathbb{Q}[t_1, \ldots, t_r] \otimes \Lambda V, D) < \infty$. *Moreover, if $r_0(X) \geq r$, then T^r acts freely on a finite complex X' in the rational homotopy type of X and $M(ET^r \times_{T^r} X') \cong (\mathbb{Q}[t_1, \ldots, t_r] \otimes \Lambda V, D)$.*

The diagram $(*)$ in Definition 1.2 is equivalent to a DGA homotopy commutative diagram: $(***)$



with $\dim H^*(\mathbb{Q}[t_1, .., t_r] \otimes \Lambda W, D_2) < \infty$ and $\dim H^*(\mathbb{Q}[t_1, .., t_r] \otimes \Lambda V, D_1) < \infty$.

For example, for the fibre inclusion of the Hopf fibration $f : S^3 \to S^7$, $r_0(S^3) = r_0(S^7) = r_0(f) = 1$ since it induces the natural inclusion $E_1 = \mathbb{C}P^1 \to \mathbb{C}P^3 = E_2$ satisfying $(*)$ (without rationalization). On the other hand, for a rationally non-trivial fibration $S^5 \to X \overset{f}{\to} Y = S^3 \times S^3$, $r_0(X) = 1$, $r_0(Y) = 2$ and $r_0(f) = 0$ from $(***)$. If $r_0(f) = 0$, the map f can not (rationally) be an S^1-equivariant map preserving almost free actions.

From the definition,

$$r_0(f) \leq \min\{r_0(X), r_0(Y)\}$$

for any map $f : X \to Y$. In particular,

$$r_0(i_X) = r_0(X) \quad \text{and} \quad r_0(p_Y) = r_0(Y)$$

for the inclusion $i_X : X \to X \times Y$, for the projection $p_Y : X \times Y \to Y$.

Recall the LS category $\mathrm{cat}(f) := \min \sharp\{U_i \subset X \mid X = \cup_i U_i$ is an open covering with $f|_{U_i} \simeq *\} - 1$ for a map $f : X \to Y$, where $\mathrm{cat}(id_X) = \mathrm{cat}(X)$, the LS category of a space X. Here $\sharp$ denotes the cardinality of a set. It satisfies $\mathrm{cat}(f) \leq \min\{\mathrm{cat}(X), \mathrm{cat}(Y)\}$ for any map $f : X \to Y$.

Theorem 1.4. *For maps $f : X \to Y$ and $g : Y \to Z$, $r_0(g \circ f)$ can be arbitrarily large compared with $r_0(f)$ and $r_0(g)$.*

This theorem follows from the second example in

Example 1.5. (1) For any m, n and $s \leq \min\{m, n\}$, there are maps $f : X \to Y$ and $g : Y \to Z$ with $r_0(f) = m$, $r_0(g) = n$ and $r_0(g \circ f) = s$. For example, put

$$f : S^3_1 \times \cdots \times S^3_m \to S^3_1 \times \cdots \times S^3_m \times S^5_1 \times \cdots \times S^5_n \quad \text{and}$$

$$g : S^3_1 \times \cdots \times S^3_m \times S^5_1 \times \cdots \times S^5_n \to S^3_1 \times \cdots \times S^3_s \times S^5_1 \times \cdots \times S^5_{n-s}$$

where $f|_{S^3_i} = id_{S^3_i}$ for all i, $g|_{S^3_i} = id_{S^3_i}$ for $i = 1, .., s$, $g|_{S^5_i} = id_{S^5_i}$ for $i = 1, .., n-s$ and $g|_{S^n_i} = *$ for other i. Then we have an example of it.

(2) Consider the maps $f : X \to Y$ and $g : Y \to Z = S^3 \times \cdots \times S^3$ (2n-factors) with the following models. For $n > 1$, put

$$M(g) : M(Z) = (\Lambda(w_1, .., w_{2n}), 0) \to (\Lambda(w_1, .., w_{2n}, w), d_Y) = M(Y)$$

with $|w_i| = 3$ for all i, $|w| = 6n - 1$, $d_Y(w) = w_1 \cdots w_{2n}$ and

$$M(f) : M(Y) = (\Lambda(w_1, .., w_{2n}, w), d_Y) \to (\Lambda(w_1, .., w_{2n}, w, y), d_X) = M(X)$$

with $|y| = 5$, $d_X(w) = w_1 \cdots w_{2n}$ and $d_X(y) = w_1 w_2$. Then we have

$$r_0(f) = 1, \ r_0(g) = 0 \text{ and } r_0(g \circ f) = 2n - 2.$$

In particular we can verify the third since

$$M(X) = (\Lambda(w_1, .., w_{2n}, w, y), d_X) \cong (\Lambda(w_3, .., w_{2n}, w), 0) \otimes A$$

with $A := (\Lambda(w_1, w_2, y), d_X)$ induces the $\mathbb{Q}[t_1, .., t_{2n-2}]$-map

$$F : (\mathbb{Q}[t_1, .., t_{2n-2}] \otimes \Lambda(w_1, .., w_{2n}), D) \to (\mathbb{Q}[t_1, .., t_{2n-2}] \otimes \Lambda(w_3, .., w_{2n}, w), D) \otimes A$$

with $Dw_i = t^2_{i-2}$ for $i = 3, .., 2n$ and $F(w_i) = w_i$ for all i.

On the other hand, $\mathrm{cat}(g \circ f) \leq \min\{\mathrm{cat}(f), \mathrm{cat}(g)\}$ [3, Exercise 1.16]. Futhermore, we know $\mathrm{cat}(c) = 0$ for the constant map $c : X \to Y$ for any space Y. But we can often rationally construct a suitable model $M(Y)$ such that $r_0(X) = r_0(c) = r_0(Y)$. For example, for $M(X) = (\Lambda(x, y, z), d)$ with $|x| = 3$, $|y| = 5$, $|z| = 7$, $dx = dy = 0$ and $dz = xy$, put $M(Y) = (\Lambda(x', y', z'), d)$ with $|x'| = 5$, $|y'| = 7$, $|z'| = 11$, $dx' = dy' = 0$ and $dz' = x'y'$. Then we can construct commutative diagram

$$\begin{array}{ccc} M(Y) = (\Lambda(x', y', z'), d) & \xrightarrow{\ 0\ } & (\Lambda(x, y, z), d) = M(X) \\ \uparrow{\scriptstyle t=0} & & \uparrow{\scriptstyle t=0} \\ M(E_2) = (\mathbb{Q}[t] \otimes \Lambda(x', y', z'), D_2) & \xrightarrow{\ F\ } & (\mathbb{Q}[t] \otimes \Lambda(x, y, z), D_1) = M(E_1) \end{array}$$

where $F(t) = t$, $F(x') = xt$, $F(y') = yt$, $F(z') = zt^2$, $D_1 z = xy + t^4$ and $D_2 z' = x'y' + t^6$. Since $\dim H^*(E_i; \mathbb{Q}) < \infty$, we see $r_0(X) = r_0(c) = r_0(Y) = 1$. Thus the two numerical invariants of a map, $r_0(f)$ and $\mathrm{cat}(f)$, have very different properties.

2 Examples

Suppose that G and K are compact connected Lie groups and K is a compact connected subgroup of G. Recall the result of Allday-Halperin [1, Remark(1)]:

Theorem 2.1. ([2, Corollary 4.3.8],[5, Corollaries 7.14 and 7.15])

$$r_0(G) = \mathrm{rank}G \ \ \textit{and} \ \ r_0(G/K) = \mathrm{rank}G - \mathrm{rank}K.$$

Theorem 2.1 says that there is a pure (two stage) Borel fibration $G/K \to ET^r \times_{T^r} G/K \to BT^r$ ([9]) with $\dim H^*(ET^r \times_{T^r} G/K;\mathbb{Q}) < \infty$ for $r = \mathrm{rank}G - \mathrm{rank}K$; i.e., the differential D in the relative model of $(***)$ in §1 satisfies $D_1 v \in \mathbb{Q}[t_1,..,t_r] \otimes \Lambda V^{even}$ for $v \in V^{odd}$ and $D_1 v = 0$ for $v \in V^{even}$ when $M(G/K) = (\Lambda V, d)$.

Theorem 2.2. *Let G and K be simply connected Lie groups and K a compact connected subgroup of G. For a principal K-bundle $K \overset{g}{\to} G \overset{f}{\to} G/K$, $r_0(g) = \mathrm{rank}K$ and $r_0(f) = \mathrm{rank}G - \mathrm{rank}K$.*

Proof: Put the relative model of $\xi : G \overset{f}{\to} G/K \overset{k}{\to} BK$ as

$$M(BK) = (\mathbb{Q}[x_1,..,x_n],0) \to (\mathbb{Q}[x_1,..,x_n] \otimes \Lambda V, d) \overset{p}{\to} (\Lambda V, 0) = M(G)$$

with $dv_i \in \mathbb{Q}[x_1,..,x_n]$ for $v_i \in V$ [4, Proposition 15.16]. Put $r = \mathrm{rank}G - \mathrm{rank}K$. From Theorem 2.1 and Proposition 1.3, there is a DGA $A := (\mathbb{Q}[t_1,..,t_r] \otimes \mathbb{Q}[x_1,..,x_n] \otimes \Lambda V, D)$ where $Dv_i = dv_i + g_i$ with $g_i \in (t_1,..,t_r)$, $Dx_i = 0$ and $\dim H^*(A) < \infty$. The DGA-projection p is extended to the $\mathbb{Q}[t_1,..,t_r]$-projection

$$F : A = (\mathbb{Q}[t_1,..,t_r] \otimes \mathbb{Q}[x_1,..,x_n] \otimes \Lambda V, D) \to (\mathbb{Q}[t_1,..,t_r] \otimes \Lambda V, \overline{D}) =: B,$$

which induces $\dim H^*(B) < \infty$. Thus $r_0(f) \geq \mathrm{rank}G - \mathrm{rank}K$. □

Example 2.3. Let $SU(n)$ be the n-th special unitary group. Then $M(SU(6))$ is given as $(\Lambda(v_1,v_2,v_3,v_4,v_5),0)$ with $|v_1| = 3$, $|v_2| = 5$, $|v_3| = 7$, $|v_4| = 9$ and $|v_5| = 11$.

(1) For the principal bundle $SU(3) \overset{g}{\to} SU(6) \overset{f}{\to} SU(6)/SU(3)$, the relative model is extended to

$$\begin{array}{ccccc}
\Lambda(v_3,v_4,v_5),0 & \longrightarrow & \Lambda(v_1,v_2,v_3,v_4,v_5),0 & \xrightarrow{M(g)} & \Lambda(v_1,v_2),0 \\
\| & \searrow & \uparrow & & \uparrow \\
M(SU(6)/SU(3)) & & \mathbb{Q}[t_1,t_2]\otimes\Lambda(v_1,v_2,v_3,v_4,v_5),D_2 & \xrightarrow{F} & \mathbb{Q}[t_1,t_2]\otimes\Lambda(v_1,v_2),D_1
\end{array}$$

with $D_1 v_1 = D_2 v_1 = t_1^2$, $D_1 v_2 = D_2 v_2 = t_2^3$, $D_2 v_3 = D_2 v_4 = D_2 v_5 = 0$. Thus $r_0(g) = \mathrm{rank}SU(3) = 2$.

(2) For the principal bundle $SU(3)\times SU(3) \stackrel{g}{\to} SU(6) \stackrel{f}{\to} SU(6)/SU(3)\times SU(3)$, the relative model is extended to

$$\begin{array}{ccccc}
\mathbb{Q}[x_1,x_2,x_1',x_2']\otimes \Lambda V, d & \longrightarrow & \Lambda V\otimes C, D & \longrightarrow & \Lambda(u_1,u_2,u_1',u_2'),0\\
\simeq\uparrow & \searrow & \uparrow & & \uparrow\\
M(SU(6)/SU(3)\times SU(3)) & & \mathbb{Q}[t_1,..,t_4]\otimes \Lambda V\otimes C, D_t & \stackrel{F}{\longrightarrow} & \mathbb{Q}[t_1,..,t_4]\otimes\Lambda(u_1,u_2,u_1',u_2'),\overline{D}_t
\end{array}$$

where $|x_1|=|x_1'|=4$, $|x_2|=|x_2'|=6$, $V=\mathbb{Q}(v_1,v_2,v_3,v_4,v_5)$, $dx_i=dx_i'=0$, $dv_1=x_1+x_1'$, $dv_2=x_2+x_2'$, $dv_3=x_1^2+{x_1'}^2$, $dv_4=x_1x_2+x_1'x_2'$, $dv_5=x_2^2+{x_2'}^2$ [6, p.486] and

$$C=\mathbb{Q}[x_1,x_2,x_1',x_2']\otimes\Lambda(u_1,u_2,u_1',u_2')$$

with $|u_i|=|u_i'|=2i+1$, $Du_i=x_i$ and $Du_i'=x_i'$; i.e., $H^*(C)=\mathbb{Q}$. Here $D_tu_1=x_1+t_1^2$, $D_tu_2=x_2+t_2^3$, $D_tu_1'=x_1'+t_3^2$, $D_tu_2'=x_2'+t_4^3$. Thus $r_0(g)=\mathrm{rank}SU(3)\times SU(3)=4$.

Also $r_0(f)=\mathrm{rank}SU(6)-\mathrm{rank}SU(3)\times SU(3)=1$. Indeed, for the minimal model $M(SU(6)/SU(3)\times SU(3))=(\mathbb{Q}[x_1,x_2]\otimes\Lambda(v_3,v_4,v_5),d)$ with $dx_1=dx_2=0$, $dv_3=x_1^2$, $dv_4=x_1x_2$ and $dv_5=x_2^2$, we have a commutative diagram

$$\begin{array}{ccc}
(\mathbb{Q}[x_1,x_2]\otimes\Lambda(v_3,v_4,v_5),d) & \stackrel{x_i=0}{\longrightarrow} & (\Lambda(v_1,v_2,v_3,v_4,v_5),0)\\
{\scriptstyle t=0}\uparrow & & \uparrow{\scriptstyle t=0}\\
(\mathbb{Q}[t,x_1,x_2]\otimes\Lambda(v_3,v_4,v_5),D) & \stackrel{F}{\longrightarrow} & (\mathbb{Q}[t]\otimes\Lambda(v_1,v_2,v_3,v_4,v_5),\overline{D})
\end{array}$$

where $Dv_3=x_1^2$, $Dv_4=x_1x_2+t^5$ and $Dv_5=x_2^2$.

Theorem 2.4. *If $G/K \stackrel{g}{\to} X \stackrel{f}{\to} Y$ is a fibration associated with a principal G-bundle, then $r_0(g)=\mathrm{rank}G-\mathrm{rank}K$.*

Proof: Put the model of the fibration $f: X\to Y$ as $i: M(Y)=(\Lambda W,d_Y)\to(\Lambda W\otimes\Lambda V,D)$. Then $M(G/K)=(\Lambda V,d)$ with $d=\overline{D}$. Note that $Dv\in\Lambda W\otimes\Lambda V^{even}$ for $v\in V^{odd}$ and $Dv=0$ for $v\in V^{even}$ [9, (3.4)] from the assumption. Put $r=\mathrm{rank}G-\mathrm{rank}K$. Fix a differential d_t on $\mathbb{Q}[t_1,..,t_r]\otimes\Lambda V$ with $\overline{d}_t=d$ and $\dim H^*(\mathbb{Q}[t_1,..,t_r]\otimes\Lambda V,d_t)<\infty$, which exists from Theorem 2.1. Note $d_t|_{V^{even}}=0$ and $d_t(V^{odd})\subset\mathbb{Q}[t_1,..,t_r]\otimes\Lambda V^{even}$. Then we have the relative model

$$(\Lambda W,d_Y)\to(\mathbb{Q}[t_1,..,t_r]\otimes\Lambda W\otimes\Lambda V,D_t)\to(\mathbb{Q}[t_1,..,t_r]\otimes\Lambda V,d_t)$$

with $D_t(v):=Dv+(d_t-d)(v)$ and $D_t(w):=d_Yw$. It is embedded into a commutative diagram

$$\begin{array}{ccccc}
(\Lambda W,d_Y) & \stackrel{i}{\longrightarrow} & (\Lambda W\otimes\Lambda V,D) & \longrightarrow & (\Lambda V,d)=M(G/K)\\
& \searrow & \uparrow{\scriptstyle t_i=0} & & \uparrow{\scriptstyle t_i=0}\\
& & (\mathbb{Q}[t_1,..,t_r]\otimes\Lambda W\otimes\Lambda V,D_t) & \stackrel{F}{\dashrightarrow} & (\mathbb{Q}[t_1,..,t_r]\otimes\Lambda V,d_t).
\end{array}$$

Since $\dim H^*(\mathbb{Q}[t_1,..,t_r]\otimes\Lambda V,d_t)<\infty$, we have $\dim H^*(\mathbb{Q}[t_1,..,t_r]\otimes\Lambda W\otimes\Lambda V,D_t)<\infty$ from the Serre spectral sequence. Thus we have $r_0(g)\geq r$. From Theorem 2.1, we have $r_0(g)=r$. □

Remark 2.5. *If a fibration $C \xrightarrow{g} X \xrightarrow{f} Y$ is not (associated with) a principal bundle, it does not hold that $r_0(g) = r_0(C)$. For example, for the rational fibration $SU(3) \to X \to S^3$ given by*

$$(\Lambda w, 0) \to (\Lambda(w, u_1, u_2), D) \to (\Lambda(u_1, u_2), 0)$$

where $|w| = |u_1| = 3$ and $|u_2| = 5$ and $Du_2 = wu_1$, we have $r_0(g) = 1 < 2 = r_0(SU(3))$. Also for a fibration over S^3 of the relative model

$$(\Lambda w, 0) \to (\Lambda(w, v_2, v_3, v_4, v_5, v_7), D) \to (\Lambda(v_2, v_3, v_4, v_5, v_7), \overline{D}) = M(C)$$

where $|w| = 3$, $|v_i| = i$, $Dv_2 = 0$, $Dv_3 = v_2^2$, $Dv_4 = wv_2$, $Dv_5 = v_2v_4 - wv_3$ and $Dv_7 = v_4^2 + 2wv_5$, we can check $r_0(g) = r_0(X) = 0$ from [12]. On the other hand, $r_0(C) = 1$ since $\dim H^*(\mathbb{Q}[t] \otimes \Lambda(v_2, v_3, v_4, v_5, v_7), \overline{D}_t) < \infty$ by $\overline{D}_t(v_3) = v_2^2$, $\overline{D}_t(v_5) = v_2v_4 + t^3$ and $\overline{D}_t(v_7) = v_4^2$. Note their fibres are pure but the above two fibrations are not pure [9]. Compare with Theorem 2.4.

Theorem 2.6. *For an odd-spherical fibration $\xi : S^{2n-1} \to X \xrightarrow{f} Y$, suppose $\pi_{>2n}(Y) \otimes \mathbb{Q} = 0$. Then, for any free T^r-action μ on X' with $X'_{\mathbb{Q}} \simeq X_{\mathbb{Q}}$ such that $r_0(ET^r \times^{\mu}_{T^r} X') = 0$, there is no map F between fibrations*

$$\begin{array}{ccccc} X_{\mathbb{Q}} & \longrightarrow & (ET^r \times^{\mu}_{T^r} X')_{\mathbb{Q}} & \longrightarrow & BT^r_{\mathbb{Q}} \\ \downarrow{\scriptstyle f_{\mathbb{Q}}} & & \downarrow{\scriptstyle F} & & \| \\ Y_{\mathbb{Q}} & \longrightarrow & (ET^r \times^{\tau}_{T^r} Y')_{\mathbb{Q}} & \longrightarrow & BT^r_{\mathbb{Q}} \end{array}$$

such that τ is a free T^r-actionon Y' with $Y'_{\mathbb{Q}} \simeq Y_{\mathbb{Q}}$. In particular, $r_0(f) < r_0(X)$.

Proof: Put $M(S^{2n-1}) = (\Lambda y, 0)$ and $M(Y) = (\Lambda V, d_Y)$. Supppose that there is a map $(\mathbb{Q}[t_1, .., t_j] \otimes \Lambda V, D_2) \to (\mathbb{Q}[t_1, .., t_j] \otimes \Lambda V \otimes \Lambda y, D_1)$ with $\dim H^*(D_i) < \infty$. Then, from degree reasons, there is a KS-extension of $(\mathbb{Q}[t_1, .., t_j] \otimes \Lambda V \otimes \Lambda y, D_1)$ by t_{j+1}, the DGA $A := (\mathbb{Q}[t_1, .., t_{j+1}] \otimes \Lambda V \otimes \Lambda y, D')$ with

$$D'(y) = D_1(y) + t_{j+1}^n \quad \text{and} \quad D'(v) = D_1(v) \ \text{ for } v \in V$$

satisies $\dim H^*(A) < \infty$. □

Question 2.7. *For a fibration $\xi : C \xrightarrow{g} X \xrightarrow{f} Y$ with fibre C simply connected of finite rational cohomology, does it hold that $r_0(g) + r_0(f) \leq r_0(X)$?*

Remark 2.8. *The above question is true for many cases. For example, it is true for the fibrations of Theorem 2.6 or when $r_0(C) = 0$. Of course, it is true when ξ is rationally trivial. But it may not be equal. Recall Halperin's inequality $r_0(X) = r_0(X) + r_0(S^{2n}) < r_0(X \times S^{2n})$ for a formal space X and an integer $n > 1$ [11]. For any even integer n, there is a space X_n such that $r_0(X_n) = 0$ and $r_0(X_n \times S^{6n+1}) \geq n$. In the following, we give an example of the model. Put*

$$M(X_n) = (\Lambda V, d) = (\Lambda(u_1, u_2, .., u_n, v_1, v_2, ..., v_{2n}, v, w), d)$$

with

$$dv_i = du_i = dw = 0 \quad \textit{for all } i \textit{ and}$$

$$dv = u_1u_2u_3\cdots u_n(v_1v_2 + v_3v_4 + v_5v_6 + \cdots + v_{2n-1}v_{2n}) + w^2$$

and $|v_i| = |u_i| = 3$ *for all* i, $|w| = (3n+6)/2$, $|v| = 3n+5$. *Then we can check that* $r_0(X_n) = 0$ *by Proposition 1.3 since* $\dim H^*(\mathbb{Q}[t]\otimes\Lambda V, D) = \infty$ *for any differential* D *by direct calculations. Put* $M(S^{6n+1}) = (\Lambda y, 0)$ *with* $|y| = 6n+1$ *and*

$$M(ET^n \times_{T^n} (X_n \times S^{6n+1})) = (\mathbb{Q}[t_1, .., t_n] \otimes \Lambda V \otimes \Lambda y, D)$$

by

$$Dv = dv + \sum_{i=1}^{n} u_i y t_i, \quad Dv_{2i} = t_i^2, \quad Dv_{2i-1} = 0 \ (i = 1, .., n)$$

$$\textit{and} \quad Dy = \sum_{i=1}^{n} (-1)^{i+1} v_{2i-1} u_1 \cdots \hat{u}_i \cdots u_n t_i.$$

Then $D \circ D = 0$ *and* $\dim H^*(\mathbb{Q}[t_1, .., t_n] \otimes \Lambda V \otimes \Lambda y) < \infty$. *Thus* $r_0(X \times Y)$ *can be arbitrarily large compared to* $r_0(X) + r_0(Y)$.

Remark 2.9. *Is there a good cohomological upper bound for* $r_0(f)$ *? Recall that S.Halperin proposes the toral rank conjecture (TRC) that the inequality*

$$\dim H^*(X; \mathbb{Q}) \geq 2^{r_0(X)}$$

holds [8] ([4, **39***], [5, Conjecture 7.20]). For example, a homogeneous space satisfies it [5, (7.23)]. It is natural to ask whether the inequality* $\dim Im(H^*(f;\mathbb{Q})) \geq 2^{r_0(f)}$ *holds. But that is not the case in general. For example, put* $M(X) = (\Lambda(v_1, v_2, v_3), 0)$ *with* $|v_i| = 3$ *and* $M(Y) = (\Lambda(x, y, v_1, v_2, v_3), d)$ *with* $dv_1 = x^2$, $dv_2 = xy$, $dv_3 = y^2$, $dx = dy = 0$, $|x| = |y| = 2$, *and* $M(f)(v_i) = v_i$ *and* $M(f)(x) = M(f)(y) = 0$. *Then* $H^*(f;\mathbb{Q})$ *is trivial; i.e.,* $\dim Im(H^*(f;\mathbb{Q})) = 1$. *On the other hand,* $r_0(f) = 1$. *Indeed,* $(\mathbb{Q}[t] \otimes \Lambda(x, y, v_1, v_2, v_3), D_2)$ *is given by* $D_2v_1 = x^2$, $D_2v_2 = xy + t^2$, $D_2v_3 = y^2$ *and* $(\mathbb{Q}[t] \otimes \Lambda(v_1, v_2, v_3), D_1)$ *is given by* $D_1v_1 = D_1v_3 = 0$, $D_1v_2 = t^2$. *Then* $\dim H^*(D_1) < \infty$, $\dim H^*(D_2) < \infty$ *and* $M(f)$ *is extended to a* $\mathbb{Q}[t]$*-morphism* $F : (\mathbb{Q}[t] \otimes \Lambda(x, y, v_1, v_2, v_3), D_2) \to (\mathbb{Q}[t] \otimes \Lambda(v_1, v_2, v_3), D_1)$ *with* $F(x) = F(y) = 0$.

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