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Module nuclearity and module injectivity of C^*-modules

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Abstract

We define module nuclearity and module injectivity for C^*-algebras which are C^*-module over another C^*-algebra with compatible actions and extend Connes-Haagerup result to this context by showing that module nuclearity is equivalent to module amenability. We also solve the module version of an open problem of Alan L.T. Paterson, by showing that the C^*-algebra of an inverse semigroup S is module nuclear over the C^*-algebra of its idempotents if and only if S is amenable.

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1 Introduction

One of the main objectives of the monograph [9] by Alan L.T. Paterson is to study the operator algebras on an inverse semigroup S by relating them to the corresponding algebras on its universal groupoid. Using this technique it is shown that the semigroup von Neumann algebra $VN(S)$ is injective if and only if all the maximal subgroups of S (indexed by the set E of idempotents of S) are amenable [9, Theorem 4.5.2]. It is asked if this is also equivalent to the nuclearity of the reduced and full C^*-algebras $C^*_{red}(S)$ and $C^*(S)$ [9, page 210]. It is not hard to see that the answer of this question is negative, as the free inverse semigroup on two generators has trivial maximal subgroups, but its (reduced) C^*-algebra is not nuclear (c.f. [4]). An affirmative solution is proposed in [4] by showing that $C^*_{red}(S)$ is nuclear if and only if a family of groups indexed by the unit space of the universal groupoid of S are amenable and S is hyperfinite [4, Corollary 3.14]. In the case of free inverse semigroup on two generators, this unit space is a one-point compactification of E and the group standing on the point at infinity is the free group $\mathbb{F}_2$. The Cuntz-Renault semigroup is an example of a hyperfinite semigroup and the bicyclic semigroup is not hyperfinite [4].

In this paper, we propose an alternative solution to the Paterson's problem by considering $C^*_{red}(S)$ as a module over $C^*_{red}(E)$ and show that there is a natural notion of nuclearity for

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C^*-modules and when E acts on S trivially from left and by multiplication from right, then $C^*_{red}(S)$ is module nuclear if and only if S is amenable. We show that module nuclearity is related to the notion of module amenability (of Banach modules) introduced by the first author in [1]. We extend Connes-Haagerup result to this context by showing that when we have a trivial left action, module nuclearity of a C^*-module A is equivalent to its module amenability and also to module Connes amenability (c.f.[2]) and module injectivity of A^{**}.

The paper is organized as follows: In section 2 we introduce the notion of module nuclearity for a C^*-algebra A which is C^*-module over another C^*-algebra $\mathfrak{A}$ with compatible actions, and show that in the case of a trivial left action, module nuclearity of A is equivalent to nuclearity of an appropriate quotient A/J of A. We show that in this case, module nuclearity is equivalent to module amenability [1]. As the main example, we show that for an inverse semigroup S with set of idempotents E with trivial left action, the reduced C^*-algebra $C^*_{red}(S)$ is $C^*_{red}(E)$-module nuclear if and only if S is amenable. In section three, we introduce the notion of module injectivity for von Neumann algebra modules and relate it to the notion of module Connes amenability [2]. Also we relate the latter module notions to their classical counterparts in the case of trivial left action.

2 Module nuclearity

Let $\mathfrak{A}, A$ be Banach algebras such that A is a Banach $\mathfrak{A}$-module with compatible actions,

$$\alpha.(ab) = (\alpha.a)b, \ (ab).\alpha = a(b.\alpha) \ \ (a, b \in A, \alpha \in \mathfrak{A}). \tag{2.1}$$

We know that $A\hat{\otimes}_{\mathfrak{A}}A = A\hat{\otimes}A/I$ which $A\hat{\otimes}A$ is the projective tensor product of A and A and I is the closed ideal generated by elements of the form $a.\alpha \otimes b - a \otimes \alpha.b$ for $\alpha \in \mathfrak{A}$, $a, b \in A$ [11]. Let $J = \overline{\langle \omega(I) \rangle}$ be the closed ideal of A generated by $\omega(I)$. We define $\omega : A\hat{\otimes}A \longrightarrow A$ by $\omega(a \otimes b) = ab$, and $\tilde{\omega} : A\hat{\otimes}_{\mathfrak{A}}A = A\hat{\otimes}A/I \longrightarrow A/J$ by

$$\tilde{\omega}(a \otimes b + I) = ab + J \quad (a, b \in A), \tag{2.2}$$

both extended by linearity and continuity. Then $\tilde{\omega}, \tilde{\omega}^{**}$ are A-$\mathfrak{A}$-module homomorphisms [1].

Let V be a Banach A-module and a Banach $\mathfrak{A}$-module with compatible actions,

$$\alpha.(a.x) = (\alpha.a).x, \ \ (a.\alpha).x = a.(\alpha.x)$$
$$(\alpha.x).a = \alpha.(x.a) \quad (a \in A, \alpha \in \mathfrak{A}, x \in V), \tag{2.3}$$

and the same for the right or two-sided actions. Then we say that V is a *Banach A-$\mathfrak{A}$-module*. If moreover

$$\alpha.x = x.\alpha \quad (\alpha \in \mathfrak{A}, x \in V), \tag{2.4}$$

then V is called a *commutative* A-$\mathfrak{A}$-module.

Given a Banach A-$\mathfrak{A}$-module V, a bounded map $D : A \longrightarrow V$ is called a *module derivation* if

$$D(a \pm b) = D(a) \pm D(b), \quad D(ab) = D(a).b + a.D(b) \quad (a, b \in A) \tag{2.5}$$

and

$$D(\alpha.a) = \alpha.D(a), \quad D(a.\alpha) = D(a).\alpha \quad (\alpha \in \mathfrak{A}, a \in A). \tag{2.6}$$

A module derivation D is called *inner* if there exists $v \in V$ such that

$$D(a) = a.v - v.a \quad (a \in A). \tag{2.7}$$

Definition 2.1. *Let A be a Banach $\mathfrak{A}$-module, A is called module amenable if for every commutative A-$\mathfrak{A}$-module V with compatible actions, each module derivation $D : A \longrightarrow V^*$ is inner.*

A left trivial action of $\mathfrak{A}$ on A is defined as $e.a = f(e)a$ for $e \in \mathfrak{A}, a \in A$,where f is a continuous character on $\mathfrak{A}$.

Let A be a Banach $\mathfrak{A}$-module with trivial left action, A is called module nuclear if for every C^*-algebra B which is a Banach $\mathfrak{A}$-module with trivial left and right actions, $A \otimes_{\mathfrak{A}} B$ has a unique C^* norm.

From now on, we suppose that A, $\mathfrak{A}$ are C^*-algebras. If J is the closed ideal of A described above, since C^*-algebra A/J has bounded approximate identity, it follows from [5, Lemma 2.7, Theorem 2.8] that if $\mathfrak{A}$ acts trivially from left on A then for each $\alpha \in \mathfrak{A}, a \in A$, we have $a.\alpha - f(\alpha)a \in J$ and A/J is amenable if and only if A is module amenable.

We know that if J is a closed ideal in C^*-algebra A and if J and A/J are nuclear, then A will be nuclear [14, Theorem 6.5.3]. The following theorem states that for the ideal J described above, nuclearity of A/J suffices for A to be module nuclear.

Theorem 2.2. *If $\mathfrak{A}$ acts trivially from left on A, A is module nuclear if and only if A/J is nuclear.*

Proof: Let A/J be nuclear, and B be a C^*-algebra. We know that $A \otimes_{\mathfrak{A}} B = A \otimes B/I_B$ that I_B is the ideal generated by the elements of the form $a.\alpha \otimes b - a \otimes \alpha.b$ in which $a \in A, b \in B, \alpha \in \mathfrak{A}$. We have

$$a.\alpha \otimes b - a \otimes \alpha.b = a.\alpha \otimes b - f(\alpha)a \otimes b = (a.\alpha - f(\alpha)a) \otimes b \in J \otimes B \tag{2.8}$$

Hence if γ_1, γ_2 are C^*- norms on $A \otimes B/J \otimes B$, and if

$$\phi : \frac{A}{J} \otimes B \longrightarrow \frac{A \otimes B}{J \otimes B} \tag{2.9}$$

is the canonical map between algebraic tensor products, then for each $x \in \frac{A}{J} \otimes B$ the equalities

$$\|x\|_1 = \|\phi(x)\|_{\gamma_1}, \quad \|x\|_2 = \|\phi(x)\|_{\gamma_2} \tag{2.10}$$

define C^*- norms on $\frac{A}{J} \otimes B$ and since A/J is nuclear $\|.\|_1 = \|.\|_2$ and so $\gamma_1 = \gamma_2$.

Now suppose that A is module nuclear, we know that

$$\frac{A}{J} \otimes_{\max} B \cong \frac{A \otimes_{\max} B}{J \otimes_{\max} B}. \tag{2.11}$$

We should show that

$$\frac{A}{J} \otimes_{\min} B \cong \frac{A \otimes_{\min} B}{J \otimes_{\min} B}. \tag{2.12}$$

For this, we need to show that the sequence

$$0 \longrightarrow J\otimes_{\min}B \longrightarrow A\otimes_{\min}B \longrightarrow \frac{A}{J}\otimes_{\min}B \longrightarrow 0 \tag{2.13}$$

is exact. As in the proof of [14, Theorem 6.5.2], there exists a unique surjective $*$-homomorphism

$$\bar{\pi} : \frac{A\otimes_{\min}B}{J\otimes_{\min}B} \longrightarrow \frac{A}{J}\otimes_{\min}B. \tag{2.14}$$

It suffices to find a left inverse for $\bar{\pi}$. Again let ϕ be the natural map described above and define the norm $\|.\|$ as

$$\|\phi(x)\| = \|x\|_{\min}, \quad (x \in \frac{A}{J}\otimes B, \phi(x) \in \frac{A\otimes B}{J\otimes B}). \tag{2.15}$$

This is a C^*- norm on $A\otimes B/J\otimes B$ and by module nuclearity of A, this norm is equal to the norm on $A\otimes_{\min}B/J\otimes_{\min}B$. Hence ϕ extends to a continuous $*$-homomorphism

$$\bar{\phi} : \frac{A}{J}\otimes_{\min}B \longrightarrow \frac{A\otimes_{\min}B}{J\otimes_{\min}B} \tag{2.16}$$

with $\bar{\phi}\bar{\pi} = id$. Thus we have

$$\frac{A}{J}\otimes_{\min}B \cong \frac{A}{J}\otimes_{\max}B \tag{2.17}$$

and so A/J is nuclear. □

Corollary 2.3. *If $\mathfrak{A}$ acts trivially from left on A, A is module nuclear if and only if A is module amenable.*

Proposition 2.4. *If $\mathfrak{A}$ is considered as a $\mathfrak{A}$-module with trivial left and right action by multiplication, then $\mathfrak{A}$ is module nuclear.*

Proof: We show that $\mathfrak{A}/J$ is amenable. We know that $a.\alpha = a\alpha, \alpha.a = f(\alpha)a$, for $a, \alpha \in \mathfrak{A}$. Since $a.\alpha - \alpha.a \in J$, $a\alpha - f(\alpha)a \in J$ for $a, \alpha \in \mathfrak{A}$. Now let $\bar{a} = a + J, \bar{b} = b + J$ be elements of $\mathfrak{A}/J$, then $ab - f(b)a \in J$ and so $\bar{a}\bar{b} = f(b)\bar{a}$.

Let $\bar{a}_\alpha$ be a bounded approximate identity for $\mathfrak{A}/J$. We have $\bar{a}_\alpha\bar{b} = f(b)\bar{a}_\alpha$. But $\bar{a}_\alpha\bar{b} \to \bar{b}$, thus $f(b)\bar{a}_\alpha \to \bar{b}$. When $f(b) \neq 0$, $\bar{a}_\alpha \to \bar{b}/f(b)$. Since $\bar{a}_\alpha$ is a norm-convergent bounded approximate identity, its limit has to be the identity and we have $\bar{b} = f(b)1$. This clearly also holds when $f(b) = 0$. Therefore $\mathfrak{A}/J$ is one dimensional and we have

$$\bar{a}\bar{b} = f(a)1f(b)1 = f(a)f(b)1 = f(b)f(a)1 = \bar{b}\bar{a}.$$

Therefore $\mathfrak{A}/J$ is a commutative unital C^*-algebra and so is amenable. □

In the next result, $A\otimes\mathfrak{A}$ is considered as an $\mathfrak{A}$-module with the following actions:

$$\alpha\cdot(a\otimes\beta) = a\otimes\beta, \ (a\otimes\beta)\cdot\alpha = a\otimes\beta\alpha \quad (a \in A, \alpha, \beta \in \mathfrak{A}).$$

Proposition 2.5. *If A is a nuclear C^*-algebra then $A\otimes\mathfrak{A}$ is $\mathfrak{A}$-module nuclear.*

Proof: We should show that $(A \otimes \mathfrak{A})/J$ is nuclear. But J is generated by elements of the form $(a \otimes e).\alpha - \alpha.(a \otimes e)$ in which $a \in A, e, \alpha \in \mathfrak{A}$. We have

$$(a \otimes e).\alpha - \alpha.(a \otimes e) = a \otimes e\alpha - f(\alpha)a \otimes e = a \otimes (e\alpha - f(\alpha)e) = a \otimes (e.\alpha - \alpha.e)$$

But if $\mathfrak{A}$ is considered as a $\mathfrak{A}$ module and $J_{\mathfrak{A}}$ is its associated ideal, then $J_{\mathfrak{A}}$ is generated by elements of the form $e.\alpha - \alpha.e$ and since A is nuclear we have

$$\frac{A \otimes \mathfrak{A}}{J} = \frac{A \otimes \mathfrak{A}}{A \otimes J_{\mathfrak{A}}} \cong A \otimes \frac{\mathfrak{A}}{J_{\mathfrak{A}}}$$

Now nuclearity of $(A \otimes \mathfrak{A})/J$ is followed from nuclearity of A and $\mathfrak{A}/J_{\mathfrak{A}}$. □

As an example, if M_n is the C^*-algebra of n by n matrices with entries in $\mathbb{C}$, then $M_n(\mathfrak{A})$ is $\mathfrak{A}$-module nuclear.

A discrete semigroup S is called an inverse semigroup if for each $x \in S$, there is a unique element $x^* \in S$ such that $xx^*x = x$ and $x^*xx^* = x$. An element $e \in S$ is called an idempotent if $e = e^* = e^2$. The set of idempotents of S is denoted by E [8].

We know that E is a commutative subsemigroup of S. As in [1], we let $l^1(E)$ act on $l^1(S)$ by multiplication from right and trivially from left, that is

$$\delta_e.\delta_s = \delta_s, \quad \delta_s.\delta_e = \delta_s * \delta_e = \delta_{se}. \tag{2.18}$$

Let us observe that these actions continuously extend to actions of $C^*(E)$ on $C^*(S)$. For the right action,

$$\begin{aligned} \|(\sum_{s\in S}\alpha_s\delta_s)\cdot(\sum_{e\in E}\beta_e\delta_e)\|_{C^*(S)} &= \|\sum_{s,e}\alpha_s\beta_e\delta_{se}\|_{C^*(S)} \\ &= \sup_{\pi}\|\sum_{s,e}\alpha_s\beta_e\pi(\delta_{se})\| \\ &= \sup_{\pi}\|\sum_{s,e}\alpha_s\beta_e\pi(\delta_s)\pi(\delta_e)\| \\ &\le \sup_{\pi}\|\sum_{s}\alpha_s\pi(\delta_s)\|\sup_{\tilde{\pi}}\|\sum_{e}\beta_e\tilde{\pi}(\delta_e)\| \\ &\le \|\sum_{s}\alpha_s\delta_s\|_{C^*(S)}\|\sum_{e}\beta_e\delta_e\|_{C^*(E)}, \end{aligned}$$

where π ranges over all non-degenerate representations of $\ell^1(S)$, and $\tilde{\pi}$ is the restriction of π to the subalgebra $\ell^1(E)$. Note that $\tilde{\pi}$ is not necessarily non-degenerate (unless to be taken on a smaller Hilbert space), yet the last inequality is clearly satisfied. For the left action, defined via the augmentation character ψ on $\ell^1(E)$

$$\psi(\sum_{e}\beta_e\delta_e) = \sum_{e}\beta_e$$

by

$$(\sum_{e\in E}\beta_e\delta_e)\cdot(\sum_{s\in S}\alpha_s\delta_s) = \sum_{s,e}\alpha_s\beta_e\delta_s = \psi(\sum_{e}\beta_e\delta_e)\sum_{s}\alpha_s\delta_s,$$

since the augmentation character could be considered as a one dimensional representation of $\ell^1(E)$, we have $|\psi(\sum_e \beta_e\delta_e)| \le \|\sum_e \beta_e\delta_e\|_{C^*(E)}$, hence

$$\|(\sum_{e\in E}\beta_e\delta_e)\cdot(\sum_{s\in S}\alpha_s\delta_s)\|_{C^*(S)} \le \|\sum_s \alpha_s\delta_s\|_{C^*(S)}\|\sum_e \beta_e\delta_e\|_{C^*(E)}.$$

Therefore, since $\ell^1(E)$ is dense in $C^*(E)$ and $\ell^1(S)$ is dense in $C^*(S)$, the actions of $\ell^1(E)$ on $\ell^1(S)$ extend continuously to actions of $C^*(E)$ on $C^*(S)$, where the left action is trivial and given by the unique extension of ψ on $C^*(E)$.

We don't know if the augmentation character is continuous in the reduced C^*-norm, thus it is not clear how to extend the left action to an action of $C^*_{red}(E)$ on $C^*_{red}(S)$ (the right action extends with an argument similar to the one above). However, since the left regular representation extends to a surjective $*$-homomorphism $\lambda : C^*(S) \to C^*_{red}(S)$ and $ker(\lambda)$ is clearly invariant under both left and right actions of $\ell^1(E)$ on $C^*(S)$, these actions lift continuously to actions of $C^*(E)$ on $C^*_{red}(S)$, where the left action remains trivial. Finally, since $VN(S) \simeq C^*_{red}(S)^{**}$, where the Arens product of the right hand side corresponds to the operator product of the left hand side, $VN(S)$ is a $C^*(E)$ module with compatible actions, where the left action is trivial (given by the extension of the augmentation character).

We consider an equivalence relation on S as follows:

$$s \approx t \Leftrightarrow \delta_s - \delta_t \in J \ \ (s,t \in S).$$

Let $e, f \in E$. Since E is a semilattice, $ef \le f, e$. Now, a slight modification of the discussion before Theorem 2.4 in [3] shows that the quotient $S/\!\approx$ is a discrete group (indeed it is not hard to see that it is the maximal group homomorphic image of S). As in [10, Theorem 3.3], we may observe that $\ell^1(S)/J \cong \ell^1(S/\approx)$.

It is well known that for a discrete group G, nuclearity of the reduced (or full) group C^*-algebra $C_{red}(G)$ (or $C^*(G)$) and injectivity of the group von Neumann algebra $VN(G)$ are both equivalent to amenability of G (see for instance [7]). This is far from being true for (inverse) semigroups (see the introduction). However we prove that a module version of this result holds for any inverse semigroup.

Theorem 2.6. *Let S be an inverse semigroup with set of idempotents E which acts on S trivially from left, and by multiplication from right. Then the following statements are equivalent.*
(i) S is amenable.
(ii) $S/\approx$ is amenable.
(iii) $C^*_{red}(S)$ is $C^*(E)$-module nuclear.
(iv) $C^*(S)$ is $C^*(E)$-module nuclear.

Proof: Since $S/\approx$ is the maximal group homomorphic image of S, (i) is equivalent to (ii) [9]. We show that (i) and (iv) are equivalent. Let $\varphi : l^1(S) \to C^*(S/\approx)$; $\delta_s \mapsto \delta_{[s]}$. This is a continuous $*$-homomorphism with range $l^1(S/\approx)$. Since $l^1(S/\approx)$ is dense in $C^*(S/\approx)$, this map lifts to a surjective $*$-homomorphism $\tilde{\varphi} : C^*(S) \to C^*(S/\approx)$. We show that $Ker(\tilde{\varphi}) = \bar{J}$, where the right hand side is the closure of J in the C^*-norm of $C^*(S)$. We have

$$\varphi(\delta_s.\delta_e - \delta_e.\delta_s) = \varphi(\delta_{se} - \delta_s) = \varphi(\delta_{se}) - \varphi(\delta_s) = \delta_{[se]} - \delta_{[s]} = 0$$

Hence $J \subseteq Ker(\tilde{\varphi})$. On the other hand, let $\varphi(\sum_{i=1}^{\infty} a_i\delta_{s_i}) = 0$. First suppose that for $i, j \in \mathbb{N}$, $s_i \approx s_j$, then clearly $\sum_{i=1}^{\infty} a_i = 0$. Now there is $e \in E$ such that $s_1e = s_2e$. Therefore

$$\begin{aligned} a_1\delta_{s_1} + a_2\delta_{s_2} &= (a_1\delta_{s_1} - a_1\delta_{s_1e}) + a_1\delta_{s_1e} + a_2\delta_{s_2} \\ &= j_1 + a_1\delta_{s_2e} + a_2\delta_{s_2} = j_1 + a_1\delta_{s_2e} + (a_2\delta_{s_2} - a_2\delta_{s_2e}) + a_2\delta_{s_2e} \\ &= j_1 + j_2 + (a_1 + a_2)\delta_{s_2e} = j + (a_1 + a_2)\delta_{s_2e} \end{aligned}$$

in which $j, j_1, j_2 \in J$. Similarly we can show that for every $n \in \mathbb{N}$, there exist $e \in E, j \in J$, such that

$$\sum_{i=1}^{n} a_i\delta_{s_i} = j + (\sum_{i=1}^{n} a_i)\delta_{s_ne}$$

Therefore since $\sum_{i=1}^{n} a_i = 0$, $\sum_{i=1}^{\infty} a_i\delta_{s_i} \in J$. Now in general if

$$\varphi(\sum_{i=1}^{\infty} a_i\delta_{s_i}) = 0,$$

by classification of s_i's into classes of mutually equivalent elements, the desired result follows by a similar argument. Hence we have

$$C^*(S)/\bar{J} \cong C^*(S/\approx). \tag{2.19}$$

Therefore, by Theorem 2.6, $C^*(S)$ is $C^*(E)$-module nuclear if and only if $C^*(S/\approx)$ is nuclear. Since $S/\approx$ is a discrete group, the latter statement is equivalent to (ii) which in turn is equivalent to (i).

Since the left regular representation of an inverse semigroup S on $\ell^2(S)$ is faithful [13], there is a dense copy of $\ell^1(S)$ in $C^*_{red}(S)$, and a similar argument as above shows that $C^*_{red}(S)/\bar{J}_{red} \cong C^*_{red}(S/\approx)$, where $\bar{J}_{red}$ is the closure of J in the reduced C^*-norm of $C^*_{red}(S)$. Therefore (iii) and (i) are equivalent. □

3 Module Injectivity

A C^*-algebras M is an injective C^*-algebra if, whenever A is a C^*-algebra, B a C^*-subalgebra of A, and $\phi : B \to M$ a completely positive contraction, then ϕ extends to a completely positive contraction $\psi : A \to M$ [6]. It is well known that M is injective if and only if for every faithful representation π of M on a Hilbert space H, there is a conditional expectation (a norm one projection) form $B(H)$ onto $\pi(M)$ [6, IV.2.1.4].

From now on let M be a C^*-algebra and a Banach $\mathfrak{A}$-module with compatible actions.

Definition 3.1. *M is module injective if, whenever A is a C^*-algebra and a Banach $\mathfrak{A}$-modules with compatible actions,, B is a C^*-subalgebra of A and $\phi : B \to M$ is a completely positive contraction preserving module actions, then ϕ extends to a completely positive contraction $\psi : A \to M$ which preserves the module action.*

Theorem 3.2. *M^{**} is module injective if and only if $(M/J)^{**}$ is injective.*

Proof: Let M^{**} be module injective and A, B be C^*-algebras as described in the above definition. We show that $(M/J)^{**}$ is injective. Let $\phi : B \to (M/J)^{**}$ be a completely positive contraction. Suppose that A, B be Banach $\mathfrak{A}$-modules with trivial actions. We know that

$$M^{**} \simeq (M/J)^{**} \oplus J^{**} \tag{3.1}$$

So we have an inclusion $i : (M/J)^{**} \to M^{**}$. Then $\tilde{\phi} = i \circ \phi$ preserves the module actions

$$\begin{aligned}\tilde{\phi}(a.\alpha) &= \tilde{\phi}(f(\alpha)a) = f(\alpha)\tilde{\phi}(a)\\ &= f(\alpha)\tilde{\phi}(a) - \tilde{\phi}(a).\alpha + \tilde{\phi}(a).\alpha\\ &= i(f(\alpha)\phi(a) - \phi(a).\alpha) + \tilde{\phi}(a).\alpha = \tilde{\phi}(a).\alpha\end{aligned}$$

and lifts to a completely positive contraction $\psi : A \to M^{**}$. Let $\pi : M^{**} \to (M/J)^{**}$ be the quotient map and put $\tilde{\psi} = \pi \circ \psi$. Then $\tilde{\psi}$ is the desired completely positive contraction from A into $(M/J)^{**}$.

Conversely let $(M/J)^{**}$ be injective and A, B be C^*-algebras as described in the above definition. Let $\phi : B \to M^{**}$ be a completely positive contraction which preserves module actions and $\pi : M^{**} \to (M/J)^{**}$ be the quotient map. Then $\tilde{\phi} = \pi \circ \phi$ is a completely positive contraction and lifts to a completely positive contraction $\psi : A \to (M/J)^{**}$. Let $i : (M/J)^{**} \to M^{**}$ be the inclusion map. Then $\tilde{\psi} = i \circ \psi$ is a completely positive contraction from A into M^{**} which preserves the module actions since

$$\tilde{\psi}(a.\alpha) = i \circ \psi(a.\alpha) = i \circ \pi(\phi(a.\alpha)) = \phi(a.\alpha) = \phi(a).\alpha \tag{3.2}$$

for each $a \in A, \alpha \in \mathfrak{A}$. □

Corollary 3.3. *M is module amenable if and only if M^{**} is module injective.*

Corollary 3.4. *Let S be an inverse semigroup with set of idempotents E which acts on S trivially from left, and by multiplication from right. Then $VN(S)$ is $C^*(E)$-module injective if and only if S is amenable.*

The concept of Module Connes amenability is introduced in [2]. A Banach algebra is called a *dual Banach algebra* if it is the dual of a Banach space, and its multiplication map is separately w^*-continuous. When A is a dual Banach algebra and a Banach $\mathfrak{A}$-module with compatible actions, X is called a *dual* Banach A-$\mathfrak{A}$-module if there is a closed A-$\mathfrak{A}$-submodule X_* of X^* such that $X = X_*^*$, or equivalently, X is the dual of a Banach space and the action maps: $X \to X$; $x \mapsto a.x$, $x \mapsto x.a$, and $x \mapsto \alpha.x$, $x \mapsto x.\alpha$ are w^*-continuous, for each $a \in A, \alpha \in \mathfrak{A}$. X is called *normal* if the action maps: $A \to X$; $a \mapsto a.x$, $a \mapsto x.a$ are w^*-continuous, for each $x \in X$.

Let A and $\mathfrak{A}$ be as above and A is a dual $\mathfrak{A}$-module, A is called module Connes amenable (as an $\mathfrak{A}$-module) if for every commutative, normal, dual Banach A-$\mathfrak{A}$-module X, each w^*-continuous module derivation $D : A \to X$ is inner.

Let J be the w^*-closed ideal generated by elements of the form $(a.\alpha)b - a(\alpha.b)$ for $a, b \in A, \alpha \in \mathfrak{A}$. The following is proved similar to [5, Theorem 2.8].

Theorem 3.5. *If $\mathfrak{A}$ acts trivially from left on A and A/J is Connes amenable then A is module Connes amenable. Furthermore if $\mathfrak{A}$ acts trivially from left on A and A is module Connes amenable and A/J has bounded approximate identity, then A/J is Connes amenable.*

Now if A is a Banach $\mathfrak{A}$-module, then A^* will have a natural $\mathfrak{A}$-module structure via

$$(f.\alpha)(a) = f(\alpha.a), \quad (\alpha.f)(a) = f(a.\alpha) \quad (a \in A, \alpha \in \mathfrak{A}, f \in A^*).$$

This in turn gives a module structure on A^{**} whose action is an extension of the module action on A and for each $\alpha \in \mathfrak{A}$, the maps : $A^{**} \to A^{**}$; $x \mapsto \alpha.x$, $x \mapsto x.\alpha$ are w^*-continuous. Therefore A^{**} is a dual Banach $\mathfrak{A}$-module.

Corollary 3.6. *If A is a C^*-algebra and a Banach $\mathfrak{A}$-module with trivial left action, then A is module amenable if and only if A^{**} is module Connes amenable.*

Proof: An argument similar to the proof of [5, Theorem 2.8] shows that module amenability of A is equivalent to amenability of A/J. Also by the above theorem, module Connes amenability of A^{**} is equivalent to Connes amenability of $A^{**}/J^{\perp\perp} = (A/J)^{**}$. Hence the result follows from the fact that for a C^*-algebra B, amenability of B is equivalent to Connes amenability of B^{**} [12]. □

Theorem 3.7. *If A is a C^*-algebra and a Banach $\mathfrak{A}$-module with trivial left action, the following are equivalent:*

(i) A is module nuclear.

(ii) A is module amenable.

(iii) A^{**} is module Connes amenable.

(iv) A^{**} is module injective.

References

[1] M. Amini, Module amenability for semigroup algebras, *Semigroup Forum* **69**(2004), 243-254.

[2] M. Amini, Module Connes amenability of hypergroup measure algebras, preprint.

[3] M. Amini, A. Bodaghi, D. Ebrahimi Bagha, Module amenability of the second dual and module topological center of semigroup algebras, *Semigroup Forum* **80**(2010), 302-312.

[4] M. Amini, M. Khoshkam, D. Cowan, On nuclearity of inverse semigroup C^*-algebras, preprint.

[5] M. Amini, R. Rezavand, Module operator amenability of the Fourier algebra of an inverse semigroup, preprint.

[6] B. Backadar, *Operator Algebras*, Springer-Verlag, Berlin-Heidelberg, 2006.

[7] S-H. Kye, *Notes on Operator Algebras*, RIM-GARC Lecture Notes Ser. No. 7, Seoul National University, Seoul, 1993.

[8] M. V. Lawson, *Inverse Semigroups: The Theory of Partial Symmetries*, World Scientific, Singapore, 1998.

[9] A. Paterson, *Groupoids, Inverse Semigroups, and their Operator Algebras*, Progress in Mathematics 70, Birkhauser, Boston, 1999.

[10] R. Rezavand, M. Amini, M. H. Sattari, D. Ebrahimi Bagha, Module Arens regularity for semigroup algebras, Semigroup Forum **77**(2008), 300-305.

[11] M.A. Rieffel, Induced representations of C^*-algebras, Adv. Math. **13**(1974), 176257.

[12] V. Runde, *Lectures on Amenability*, Springer-Verlag, Berlin-Heidelberg-New York, 2002.

[13] J. R. Wordingham, The left regular *-representation of an inverse semigroup, Proc. Amer. Math. Soc. **86** (1982), 55-58.

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