

## On the partition dimension of unicyclic graphs

by

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### Abstract

The partition dimension is a graph parameter akin to the notion of metric dimension that has attracted some attention in recent years. In this paper, we obtain several tight bounds on the partition dimension of unicyclic graphs.

**Key Words:** Resolving partition, partition dimension, unicyclic graphs.

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### 1 Introduction

The concepts of resolvability and location in graphs were described independently by Harary and Melter [7] and Slater [14], to define the same structure in a graph. After these papers were published several authors developed diverse theoretical works about this topic, for instance, [1, 2, 3, 4, 5, 6, 10, 20]. Slater described the usefulness of these ideas into long range aids to navigation [14]. Also, these concepts have some applications in chemistry for representing chemical compounds [9] or to problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures [11]. Other applications of this concept to navigation of robots in networks and other areas appear in [4, 8, 10]. Some variations on resolvability or location have been appearing in the literature, like those about resolving partitions [3, 5, 6, 15, 18, 19].

Given a graph  $G = (V, E)$  and a set of vertices  $S = \{v_1, v_2, \dots, v_k\}$  of  $G$ , the *metric representation* of a vertex  $v \in V$  with respect to  $S$  is the vector  $r(v|S) = (d(v, v_1), d(v, v_2), \dots, d(v, v_k))$ , where  $d(v, v_i)$ <sup>1</sup> denotes the distance between the vertices  $v$  and  $v_i$ ,  $1 \leq i \leq k$ . We say that  $S$  is a *resolving set* if different vertices of  $G$  have different metric representations, i.e., for every pair of vertices  $u, v \in V$ ,  $r(u|S) \neq r(v|S)$ . The *metric dimension*<sup>2</sup> of  $G$  is the minimum cardinality of any resolving set of  $G$ , and it is denoted by  $\dim(G)$ .

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<sup>1</sup>To avoid ambiguity in some cases we will denote the distance between two vertices  $u, v$  of a graph  $G$  by  $d_G(u, v)$ .

<sup>2</sup>Also called locating number.

Given an ordered partition  $\Pi = \{P_1, P_2, \dots, P_t\}$  of the vertices of  $G$ , the *partition representation* of a vertex  $v \in V$  with respect to the partition  $\Pi$  is the vector

$$r(v|\Pi) = (d(v, P_1), d(v, P_2), \dots, d(v, P_t)),$$

where  $d(v, P_i)$ , with  $1 \leq i \leq t$ , represents the distance between the vertex  $v$  and the set  $P_i$ , i.e.,  $d(v, P_i) = \min_{u \in P_i} \{d(v, u)\}$ . We say that  $\Pi$  is a *resolving partition* if different vertices of  $G$  have different partition representations, i.e., for every pair of vertices  $u, v \in V$ ,  $r(u|\Pi) \neq r(v|\Pi)$ . The *partition dimension* of  $G$  is the minimum number of sets in any resolving partition for  $G$  and it is denoted by  $pd(G)$ .

The partition dimension of graphs was studied in [3, 5, 6, 13, 15, 16, 17, 18, 19]. For instance, Chappell, Gimbel and Hartman obtained several relationships between metric dimension, partition dimension, diameter, and other graph parameters [3]. Chartrand, Zhang and Salehi showed that for every nontrivial graph  $G$  it follows that  $pd(G) \leq pd(G \square K_2)$  (where  $\square$  denotes the Cartesian product of graphs) and they also showed that for an induced subgraph  $H$  of a connected graph  $G$  the ratio  $r_p = pd(H)/pd(G)$  can be arbitrarily large [5]. The partition dimension of some specific families of graphs was studied further in a number of other papers. For instance, Cayley digraphs were studied by Fehr, Gosselin and Oellermann [6], the infinite graphs  $(\mathbb{Z}^2, \xi_4)$  and  $(\mathbb{Z}^2, \xi_8)$  (where the set of vertices is the set of points of the integer lattice and the set of edges consists of all pairs of vertices whose city block and chessboard distances, respectively, are 1) were studied by Tomescu [15]. Also infinite graphs were studied in [11]. The corona product graphs were studied by Rodríguez-Velázquez, Yero and Kuziak [18] and the Cartesian product graphs were studied by Yero and Rodríguez-Velázquez [19]. Some wheel related graphs were studied in [17]. Here we study the partition dimension of unicyclic graphs. A similar study on the metric dimension was previously done by Poisson and Zhang [12].

## 2 Results

The set of all spanning trees of a connected graph  $G$  is denoted by  $\mathcal{T}(G)$ . It was shown in [4] that if  $G$  is a connected unicyclic graph of order at least 3 and  $T \in \mathcal{T}(G)$ , then

$$\dim(T) - 2 \leq \dim(G) \leq \dim(T) + 1. \quad (2.1)$$

A formula for the dimension of trees that are not paths has been established in [4, 7, 14]. In order to present this formula, we need additional definitions. A vertex of degree at least 3 in a graph  $G$  will be called a *major vertex* of  $G$ . Any pendant vertex  $u$  of  $G$  is said to be a *terminal vertex* of a major vertex  $v$  of  $G$  if  $d(u, v) < d(u, w)$  for every other major vertex  $w$  of  $G$ . The *terminal degree* of a major vertex  $v$  is the number of terminal vertices of  $v$ . A major vertex  $v$  of  $G$  is an *exterior major vertex* of  $G$  if it has positive terminal degree.

Let  $n_1(G)$  denote the number of pendant vertices of  $G$ , and let  $ex(G)$  denote the number of exterior major vertices of  $G$ . We can now state the formula for the dimension of a tree [4, 7, 14]: if  $T$  is a tree that is not a path, then  $\dim(T) = n_1(T) - ex(T)$ . Thus, by the above result and (2.1) we have that if  $G$  is a connected unicyclic graph of order at least 3 and  $T \in \mathcal{T}(G)$ , then

$$n_1(T) - ex(T) - 2 \leq \dim(G) \leq n_1(T) - ex(T) + 1. \quad (2.2)$$

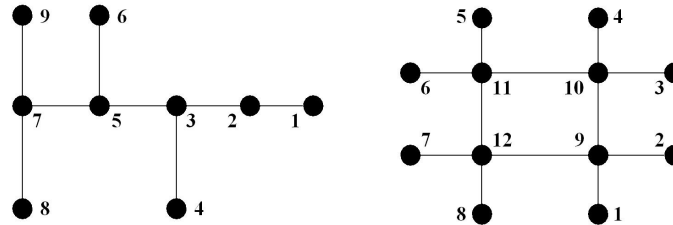


Figure 1: In the left hand side graph vertex 3 is an exterior major vertex of terminal degree two, while 1 and 4 are terminal vertices of 3. For the right hand side graph,  $\Pi = \{\{3, 5, 7, 9\}, \{1, 6, 8, 10\}, \{2, 4, 11\}, \{12\}\}$  is a resolving partition and  $\{1, 3, 5, 7\}$  is a resolving set.

**Example.** Let  $G$  be a graph obtained in the following way: we begin with a cycle  $C_4 = u_1u_2u_3u_4u_1$  and, then we add vertices  $v_1, \dots, v_k, k \geq 2$ , and edges  $u_1v_i, 1 \leq i \leq k$ . Thus,  $\dim(G) = k + 1$ . Now, let  $T \in \mathcal{T}(G)$  obtained by deleting the edge  $u_4u_1$  in the cycle. Hence, we have  $n_1(T) = k + 1$  and  $ex(T) = 1$ . So, the above upper bound is tight.

It is natural to think that the partition dimension and metric dimension are related; it was shown in [5] that for any nontrivial connected graph  $G$  we have

$$pd(G) \leq \dim(G) + 1. \tag{2.3}$$

As a consequence of (2.2), if  $G$  is a connected unicyclic graph and  $T \in \mathcal{T}(G)$ , then

$$pd(G) \leq n_1(T) - ex(T) + 2. \tag{2.4}$$

The following well-known claim is very easy to verify.

**Claim 1.** *Let  $C$  be a cycle graph. If  $x, y, u$  and  $v$  are vertices of  $C$  such that  $x$  and  $y$  are adjacent and  $d(u, x) = d(v, x)$ , then  $d(u, y) \neq d(v, y)$  and, as a consequence,  $\dim(C) = 2$ .*

Any vertex adjacent to a pendant vertex of a graph  $G$  is called a *support vertex* of  $G$ . Let  $\rho(G)$  be the number of support vertices of  $G$  adjacent to more than one pendant vertex.

**Theorem 1.** *Let  $G$  be a connected unicyclic graph. If every vertex belonging to the cycle of  $G$  has degree greater than two, then  $\dim(G) \leq n_1(G) - \rho(G)$ .*

**Proof:** Let  $C$  be the set of vertices belonging to the cycle of  $G$ . In order to show that the set of pendant vertices of  $G$  is a resolving set, we only need to show that for every  $u, v \in C$  we can find two pendant vertices,  $x, y$ , such that if  $d_G(u, x) = d_G(v, x)$ , then  $d_G(u, y) \neq d_G(v, y)$ . To begin with, for every pendant vertex  $w$  we define  $w_c$  as the vertex of  $C$  such that  $d_G(w, w_c) = d_G(w, C)$ .

We take  $x, y$  as two pendant vertices of  $G$  such that  $x_c$  and  $y_c$  are adjacent vertices. Note that in this case for every  $u, v \in C$  we have  $d_G(u, x) = d_G(u, x_c) + d_G(x_c, x)$ ,  $d_G(u, y) = d_G(u, y_c) + d_G(y_c, y)$ ,  $d_G(v, x) = d_G(v, x_c) + d_G(x_c, x)$  and  $d_G(v, y) = d_G(v, y_c) + d_G(y_c, y)$ . So,

if  $d_G(u, x) = d_G(v, x)$ , we conclude  $d_G(u, y) \neq d_G(v, y)$ . Thus, the set of pendant vertices of  $G$  is a resolving set.

If we consider pendant vertices as being equivalent if they have the same support vertex, then a resolving set of minimum cardinality should contain all but one of these pendant vertices per equivalent class. Thus, the result follows.  $\square$

The above bound is tight, it is achieved, for instance, for the right hand side graph in Figure 1. Together with the bound from Equation (2.3), we obtain:

**Corollary 1.** *Let  $G$  be a connected unicyclic graph. If every vertex belonging to the cycle of  $G$  has degree greater than two, then  $pd(G) \leq n_1(G) - \rho(G) + 1$ .*

Note that for the right hand side graph in Figure 1, Corollary 1 leads to  $pd(G) \leq 5$ , while bound (2.4) only gives  $pd(G) \leq 6$ .

For a connected unicyclic graph  $G$ , let  $\kappa(G)$  be the number of exterior major vertices of  $G$ , with terminal degree greater than one and let  $\tau(G)$  be the maximum terminal degree of any exterior major vertex of  $G$ . Note that for any  $T \in \mathcal{T}(G)$  it holds  $\kappa(G) \leq \kappa(T)$  and  $\tau(G) \leq \tau(T)$ .

**Lemma 1.** [5] *Let  $G$  be a connected graph of order  $n \geq 2$ . Then  $pd(G) = 2$  if and only if  $G \cong P_n$ .*

**Theorem 2.** *Let  $G$  be a connected unicyclic graph. (i) If  $G$  is a cycle graph or every exterior major vertex of  $G$  has terminal degree one, then  $pd(G) = 3$ . (ii) If  $G$  contains at least an exterior major vertex of terminal degree greater than one, then  $pd(G) \leq \kappa(G) + \tau(G) + 1$ .*

**Proof:** Let us prove (i). If  $G$  is a cycle graph, then by (2.3), Claim 1 and Lemma 1 we obtain  $pd(G) = 3$ . Now we consider that every exterior major vertex of  $G = (V, E)$  has terminal degree one. Then every exterior major vertex  $u$  has degree three and it belongs to the cycle  $C$  of  $G$ . Let  $\{c_0, c_1, \dots, c_{k-1}\}$  be the set of vertices of  $G$  belonging to  $C$  where  $c_i$  and  $c_{i+1}$  are adjacent (the subscripts are taken modulo  $k$ ). Without loss of generality, we can suppose that  $c_0$  has terminal degree one. For every exterior major vertex  $c_i$ ,  $W_i$  will denote the set of vertices belonging to the path starting at  $c_i$  and ending at its terminal vertex. For every  $c_j$  of degree two we assume  $W_j = \{c_j\}$ .

**Case A:  $k$  even.** For  $k$  even we claim that  $\Pi = \{W_0, A_2, A_3\}$  is a resolving partition for  $G$ , where  $A_2 = W_{\frac{k}{2}} \cup W_{\frac{k}{2}+1}$  and  $A_3 = V - (W_0 \cup W_{\frac{k}{2}} \cup W_{\frac{k}{2}+1})$ . To show this we differentiate three cases for  $x, y \in V$ .

Case 1:  $x, y \in W_0$ . Since  $d(x, c_0) \neq d(y, c_0)$ , we conclude that  $d(x, A_3) = d(x, c_0) + 1 \neq d(y, c_0) + 1 = d(y, A_3)$ .

Case 2:  $x, y \in A_2$ . If  $d(x, c_{\frac{k}{2}-1}) = d(y, c_{\frac{k}{2}-1})$ , then either  $d(x, c_{\frac{k}{2}+2}) = d(x, c_{\frac{k}{2}-1}) - 1 = d(y, c_{\frac{k}{2}-1}) - 1 = d(y, c_{\frac{k}{2}+2}) - 2ord(x, c_{\frac{k}{2}+2}) = d(x, c_{\frac{k}{2}-1}) + 1 = d(y, c_{\frac{k}{2}-1}) + 1 = d(y, c_{\frac{k}{2}+2}) + 2$ . Thus, since  $c_{\frac{k}{2}+2} \in A_3$  for  $k \geq 6$  and  $c_{\frac{k}{2}+2} \in W_0$  for  $k = 4$ , we have  $d(x, A_3) \neq d(y, A_3)$  or  $d(x, W_0) \neq d(y, W_0)$ . On the other hand, since  $c_{\frac{k}{2}-1} \in A_3$ , if  $d(x, c_{\frac{k}{2}-1}) \neq$

$d\left(y, c_{\frac{k}{2}-1}\right)$  and  $d\left(x, c_{\frac{k}{2}+1}\right) = d\left(y, c_{\frac{k}{2}+1}\right)$ , then  $d(x, A_3) \neq d(y, A_3)$ . In the case  $d\left(x, c_{\frac{k}{2}+1}\right) \neq d\left(y, c_{\frac{k}{2}+1}\right)$  we have  $d(x, W_0) = d\left(x, c_{\frac{k}{2}+1}\right) + d\left(c_{\frac{k}{2}+1}, c_0\right) \neq d\left(y, c_{\frac{k}{2}+1}\right) + d\left(c_{\frac{k}{2}+1}, c_0\right) = d(y, W_0)$ .

Case 3:  $x, y \in A_3$ . Let  $x \in W_i$  and  $y \in W_j$ . If  $i = j$ , then  $d(x, W_0) \neq d(y, W_0)$  and  $d(x, A_2) \neq d(y, A_2)$ . Now we consider several subcases.

Subcase 3.1:  $0 < i < j < k/2$ . If  $d(y, A_2) = d(x, A_2)$ , then we have  $d(y, c_j) + d(c_j, c_{k/2}) = d(y, c_{k/2}) = d(x, c_{k/2}) = d(x, c_i) + d(c_i, c_j) + d(c_j, c_{k/2})$ . So,  $d(y, c_j) = d(x, c_i) + d(c_i, c_j)$  and we obtain the following.  $d(x, c_0) = d(x, c_i) + d(c_i, c_0) = d(y, c_j) - d(c_i, c_j) + d(c_i, c_0) \neq d(y, c_j) + d(c_j, c_i) + d(c_i, c_0) = d(y, c_0)$ . Thus,  $d(x, W_0) \neq d(y, W_0)$ .

Subcase 3.2:  $\frac{k}{2} + 1 < i < j \leq k - 1$ . Proceeding analogously to Case 3.1, if  $d(y, A_2) = d(x, A_2)$ , then we obtain that  $d(x, W_0) \neq d(y, W_0)$ .

Subcase 3.3:  $0 < i < k/2$  and  $\frac{k}{2} + 1 < j \leq k - 1$ . If  $d(x, A_2) = d(y, A_2)$ , then we have  $d(x, c_i) + d(c_i, c_{k/2}) = d(x, c_{k/2}) = d(y, c_{k/2+1}) = d(y, c_j) + d(c_j, c_{\frac{k}{2}+1})$ . Thus,  $d(x, c_0) = d(x, c_i) + d(c_i, c_0) = d(y, c_j) + d(c_j, c_{\frac{k}{2}+1}) - d(c_i, c_{k/2}) + d(c_i, c_0) = d(y, c_j) + d(c_0, c_{\frac{k}{2}+1}) - d(c_0, c_j) - d(c_i, c_{k/2}) + d(c_i, c_0) = d(y, c_j) + d(c_0, c_j) + d(c_0, c_{\frac{k}{2}+1}) - 2d(c_0, c_j) - d(c_i, c_{k/2}) + d(c_i, c_0) = d(y, c_0) + \left(\frac{k}{2} - 1\right) - 2(k - j) - \left(\frac{k}{2} - i\right) + i = d(y, c_0) + 2(i + j) - 2k - 1$ .

Hence, if  $i + j \leq k$ , then  $2(i + j) - 2k - 1 < 0$  and, as a consequence,  $d(x, c_0) < d(y, c_0)$ . Analogously, if  $i + j \geq k + 1$ , then  $2(i + j) - 2k - 1 > 0$ , so we have  $d(x, c_0) > d(y, c_0)$ . As a result,  $d(x, W_0) \neq d(y, W_0)$ .

**Case B:  $k$  odd.** On the other hand, suppose  $k$  is odd. If  $k = 3$ , then it is straightforward to check that  $\{W_0, W_1, W_2\}$  is a resolving partition for  $G$ . So we assume  $k \geq 5$  and we claim that  $\Pi = \{B_1, B_2, B_3\}$  is a resolving partition for  $G$ , where  $B_1 = W_0 \cup W_1$ ,  $B_2 = W_{\lfloor k/2 \rfloor} \cup W_{\lceil k/2 \rceil}$  and  $B_3 = V - (B_1 \cup B_2)$ . To show this we consider two different vertices  $x, y \in V$  and as above we take  $x \in W_i$  and  $y \in W_j$ . If  $i = j$ , then  $x, y \in B_l$  for some  $l \in \{1, 2, 3\}$  and  $d(x, B_r) \neq d(y, B_r)$  for any  $r \in \{1, 2, 3\} - \{l\}$ . Now on we assume  $i < j$  and we differentiate the following three cases.

Case 1':  $x, y \in B_1$ . Since  $i < j$  and  $B_1 = W_0 \cup W_1$  we have  $i = 0$  and  $j = 1$ . If  $k = 5$ , then  $d(x, B_3) = d(y, B_3)$  implies  $d(x, B_2) = d(y, B_2) + 2$ . So we consider  $k \geq 7$ . Now  $d(x, B_3) = d(y, B_3)$  implies  $d(x, c_0) = d(y, c_1)$ . Thus,  $d(x, c_{\lceil k/2 \rceil}) = d(x, c_0) + d(c_0, c_{\lceil k/2 \rceil}) = d(x, c_0) + d(c_0, c_{\lfloor k/2 \rfloor})d(x, c_0) + d(c_1, c_{\lfloor k/2 \rfloor}) + 1d(y, c_1) + d(c_1, c_{\lfloor k/2 \rfloor}) + 1 = d(y, c_{\lfloor k/2 \rfloor}) + 1 > d(y, c_{\lfloor k/2 \rfloor})$ . Hence, we obtain that  $d(x, B_2) \neq d(y, B_2)$ .

Case 2':  $x, y \in B_2$ . Proceeding analogously to Case 1', we obtain that if  $d(x, B_3) = d(y, B_3)$ , then  $d(x, B_2) \neq d(y, B_2)$ .

Case 3':  $x, y \in B_3$ . Now we consider the following subcases.

Subcase 3'.1:  $1 < i < j < \lfloor k/2 \rfloor$ . If  $d(y, B_2) = d(x, B_2)$ , then we have  $d(y, c_j) + d(c_j, c_{\lfloor k/2 \rfloor}) = d(y, c_{\lfloor k/2 \rfloor}) = d(x, c_{\lfloor k/2 \rfloor}) = d(x, c_i) + d(c_i, c_j) + d(c_j, c_{\lfloor k/2 \rfloor})$ . So,  $d(y, c_j) = d(x, c_i) + d(c_i, c_j)$  and we obtain  $d(y, c_1) = d(y, c_j) + d(c_j, c_1) + d(c_i, c_1) = d(x, c_i) + 2d(c_j, c_1) + d(c_i, c_1) = d(x, c_1) + 2d(c_j, c_1)$ . Thus,  $d(x, B_1) \neq d(y, B_1)$ .

Subcase 3'.2:  $\lceil k/2 \rceil < i < j \leq k - 1$ . Proceeding as in Case 3'.1, we have that if  $d(y, B_2) =$

$d(x, B_2)$ , then we obtain that  $d(x, B_1) \neq d(y, B_1)$ .

Subcase 3'.3:  $1 < i < \lfloor k/2 \rfloor$  and  $\lceil k/2 \rceil < j \leq k-1$ . If  $d(x, B_2) = d(y, B_2)$ , then we have  $d(x, c_i) + d(c_i, c_{\lfloor k/2 \rfloor}) = d(x, c_{\lfloor k/2 \rfloor}) = d(y, c_{\lceil k/2 \rceil}) = d(y, c_j) + d(c_j, c_{\lceil k/2 \rceil})$ . Thus,  $d(x, c_1) = d(x, c_i) + d(c_i, c_1) = d(y, c_j) + d(c_j, c_{\lceil k/2 \rceil}) - d(c_i, c_{\lfloor k/2 \rfloor}) + d(c_i, c_1) = d(y, c_j) + d(c_0, c_{\lceil k/2 \rceil}) - d(c_0, c_j) - d(c_i, c_{\lfloor k/2 \rfloor}) + d(c_i, c_1) = d(y, c_j) + d(c_0, c_j) + d(c_0, c_{\lceil k/2 \rceil}) - 2d(c_0, c_j) - d(c_i, c_{\lfloor k/2 \rfloor}) + d(c_i, c_1) = d(y, c_0) + \lfloor k/2 \rfloor - 2(k-j) - (\lfloor k/2 \rfloor - i) + (i-1) = d(y, c_0) + 2(i+j-k) - 1$ .

Hence, if  $i+j \leq k$ , then  $2(i+j-k) - 1 < 0$  and, as a consequence,  $d(x, c_1) < d(y, c_0)$ . Analogously, if  $i+j \geq k+1$ , then  $2(i+j-k) - 1 > 0$ , so we have  $d(x, c_1) > d(y, c_0)$ . As a result,  $d(x, B_1) \neq d(y, B_1)$ .

Therefore, for every  $x, y \in V$ ,  $x \neq y$ , we have  $r(x|\Pi) \neq r(y|\Pi)$  and, as a consequence,  $pd(G) \leq 3$ . By Lemma 1 we know that for every graph  $G$  different from a path we have  $pd(G) \geq 3$ , hence we obtain  $pd(G) = 3$ .

Now, let us prove (ii). Let  $S = \{s_1, s_2, \dots, s_{\kappa(G)}\}$  be the set of exterior major vertices of  $G$  with terminal degree greater than one. Given an arbitrary  $s_l \in S$  we take  $u \in V$  as a vertex of the cycle  $C$  in  $G$ , such that  $d(u, s_l) = \min_{v \in C} \{d(v, s_l)\}$ . Let  $v \in C$  such that  $u$  is adjacent to  $v$ . Now, for every  $s_i \in S$ , we denote by  $\{s_{i1}, s_{i2}, \dots, s_{il_i}\}$  the set of terminal vertices of  $s_i$  and by  $S_{ij}$  the set of vertices of  $G$ , different from  $s_i$ , belonging to the  $s_i - s_{ij}$  path. If  $l_i < \tau(G)$ , we assume  $S_{ij} = \emptyset$  for every  $j \in \{l_i+1, \dots, \tau(G)\}$ . Now, let  $A = \{v\}$  and  $B = C - \{v\}$ . Let  $A_i = S_{i1}$ , for every  $i \in \{1, \dots, \kappa(G)\}$  and if  $\tau(G) \geq 3$ , then let  $B_j = \bigcup_{i=1}^{\kappa(G)} S_{ij}$ , for every  $j \in \{2, \dots, \tau(G)-1\}$ . Now we will show that the partition  $\Pi = \{A, B, A_1, A_2, \dots, A_{\kappa(G)}, B_2, B_3, \dots, B_{\tau(G)-1}, R\}$ , with  $R = V(G) - A - B - \bigcup_{i=1}^{\kappa(G)} A_i - \bigcup_{i=2}^{\tau(G)-1} B_i$ , is a resolving partition for  $G$ . Notice that the sets  $B_j$  could not exist in the case when  $\tau(G) = 2$ . Hence,  $R$  collects all major vertices of terminal degree one and the attached terminals. Let  $x, y \in V$  be two different vertices in  $G$ . We have the following cases.

Case 1: If  $x, y \in A_i$ , then  $d(x, R) \neq d(y, R)$ . Namely, any path from  $x$  or  $y$  to  $R$  must contain  $s_i$ .

Case 2: Let  $x, y \in B$ . If  $d(x, v) \neq d(y, v)$ , then  $d(x, A) \neq d(y, A)$ . On the contrary, if  $d(x, v) = d(y, v)$ , then  $d(x, u) \neq d(y, u)$  due to Claim 1. So, for  $s_l \in S$  we have  $A_l = S_{l1}$  and we obtain that  $d(x, A_l) = d(x, u) + d(u, S_{l1}) \neq d(y, u) + d(u, S_{l1}) = d(y, A_l)$ .

Case 3: Let  $x, y \in B_j$ . If  $x, y \in S_{ij}$ , then  $x$  belongs to the  $y - s_i$  path or  $y$  belongs to the  $x - s_i$  path. In both cases we have  $d(x, A_i) = d(x, s_i) + 1 \neq d(y, s_i) + 1 = d(y, A_i)$ . On the contrary, if  $x \in S_{ij}$  and  $y \in S_{kj}$ ,  $i \neq k$ , then let us suppose  $d(x, A_i) = d(y, A_i)$ . So, we have  $d(x, A_k) = d(x, s_i) + d(s_i, s_k) + 1 = d(x, A_i) + d(s_i, s_k) = d(y, A_i) + d(s_i, s_k) = d(y, s_k) + 2d(s_i, s_k) + 1 = d(y, A_k) + 2d(s_i, s_k) > d(y, A_k)$ .

Case 4: Let  $x, y \in R$ . Let  $a, b \in C$  such that  $d(x, a) = \min_{c \in C} \{d(x, c)\}$  and  $d(y, b) = \min_{c \in C} \{d(y, c)\}$ . If  $d(x, a) \neq d(y, b)$  and  $a, b \neq v$ , then  $d(x, B) \neq d(y, B)$ . Also, if  $d(x, a) = d(y, b)$  and  $(a = v \text{ or } b = v)$ , then we have either  $d(x, A) \neq d(y, A)$  or  $d(x, B) \neq d(y, B)$ . Now, let us suppose  $d(x, a) = d(y, b)$ . We have the following subcases.

Subcase 4.1:  $a = b$ . Hence, we consider a terminal vertex  $s_{i1}$ , such that  $d(x, s_{i1}) + d(y, s_{i1}) = \min_{l \in \{1, \dots, \kappa(G)\}} \{d(x, s_{l1}) + d(y, s_{l1})\}$ . Let the vertices  $c, d$  belonging to the  $a - s_{i1}$  path  $P$ , with  $d(x, c) = \min_{w \in P} \{d(x, w)\}$  and  $d(y, d) = \min_{w \in P} \{d(y, w)\}$ . If  $c = d$ , then there exists a

terminal vertex  $s_{j1}$  such that either  $x$  belongs to the  $y - s_{j1}$  path or  $y$  belongs to the  $x - s_{j1}$  path and we have either  $d(x, A_j) < d(y, A_j)$  or  $d(y, A_j) < d(x, A_j)$ . If there exists not such a terminal vertex  $s_{j1}$ , then we have that  $x \in S_{i\tau(G)}$  and  $y \in S_{j\tau(G)}$  for some  $i \neq j$ . Thus we have the following.  $d(x, A_i) = d(x, s_i) + 1 = d(x, a) - d(s_i, a) + 1 = d(y, a) - d(s_i, a) + 1 = d(y, a) + d(a, s_i) - 2d(s_i, a) + 1 = d(y, A_i) - 2d(s_i, a) < d(y, A_i)$ . On the other hand, if  $c \neq d$ , then we have either  $d(x, a) = d(x, c) + d(c, d) + d(d, a)$  and  $d(y, a) = d(y, d) + d(d, a)$ , or  $d(y, a) = d(y, d) + d(d, c) + d(c, a)$  and  $d(x, a) = d(x, c) + d(c, a)$ .

Let us suppose, without loss of generality, that the first case holds.

Thus, we have  $d(x, A_i) = d(x, c) + d(c, A_i) = d(x, a) - d(c, d) - d(d, a) + d(c, A_i) = d(y, a) - d(c, d) - d(d, a) + d(c, A_i) = d(y, d) + d(d, a) - d(c, d) - d(d, a) + d(c, A_i) = d(y, d) - d(c, d) + d(c, A_i) = d(y, d) + d(d, c) + d(c, A_i) - 2d(c, d) = d(y, A_i) - 2d(c, d) < d(y, A_i)$ . Subcase 4.2:  $a \neq b$ . If  $a = u$  or  $b = u$ , then let us suppose, for instance  $b = u$ . Let  $Q$  be a shortest path between  $a$  and  $s_{l1}$ . Let  $c$  belonging to  $Q$ , such that  $d(y, c)$  is the minimum value between the distances from  $y$  to any vertex of  $Q$ . So, we have  $d(x, A_l) = d(x, a) + d(a, u) + d(u, c) + d(c, A_l) = d(y, b) + d(a, u) + d(u, c) + d(c, A_l) = d(y, c) + d(c, u) + d(a, u) + d(u, c) + d(c, A_l) = d(y, A_l) + 2d(c, u) + d(a, u) > d(y, A_l)$ . Now, let us suppose  $a \neq u$  and  $b \neq u$ . If  $d(a, v) \neq d(b, v)$ , then  $d(x, A) \neq d(y, A)$ . On the contrary, if  $d(a, v) = d(b, v)$ , then  $d(a, u) \neq d(b, u)$  (due to Claim 1). So, we have  $d(x, A_l) = d(x, a) + d(a, u) + d(u, A_l) = d(y, b) + d(a, u) + d(u, A_l) \neq (y, b) + d(b, u) + d(u, A_l) = d(y, A_l)$ .

Therefore, for every different vertices  $x, y \in V$  we have  $r(x|\Pi) \neq r(y|\Pi)$  and  $\Pi$  is a resolving partition for  $G$  and, as a consequence, (ii) follows.  $\square$

In order to give an example where we compare the above bound with all the previous results, we present the following known result. It was shown in [5] that for any (not necessarily unicyclic) graph of order  $n \geq 3$  and diameter  $d$ ,

$$g(n, d) \leq pd(G) \leq n - d + 1, \tag{2.5}$$

where  $g(n, d)$  is the least positive integer  $k$  for which  $(d + 1)^k \geq n$ .

**Example.** Let  $G$  be a graph obtained in the following way: we begin with a cycle  $C_k$ ,  $k \geq 4$ , and, then for each vertex  $v$  of the cycle we add  $k$  vertices  $v_1, v_2, \dots, v_k$  and edges  $vv_i$ ,  $1 \leq i \leq k$ . Thus,  $G$  has  $k^2$  vertices of degree one and  $k$  exterior major vertices of terminal degree  $k$ . Notice that Theorem 2 (ii) leads to  $pd(G) \leq 2k + 1$  while (2.4) gives  $pd(G) \leq k^2 - k + 2$ , Corollary 1 gives  $pd(G) \leq k^2 - k + 1$  and (2.5) gives  $pd(G) \leq k^2 + \lceil \frac{k}{2} \rceil - 1$ .

For a connected unicyclic graph  $G$ , let  $\varepsilon(G)$  be the minimum number of leaves in any spanning tree of  $G$ , i.e.,  $\varepsilon(G) = \min_{T \in \mathcal{T}(G)} \{n_1(T)\}$ .

**Corollary 2.** *Let  $G$  be a connected unicyclic graph. For every  $T \in \mathcal{T}(G)$  such that  $\varepsilon(G) = n_1(T)$ ,  $pd(G) \leq \kappa(T) + \tau(T) + 1$ .*

For the unicyclic graph  $G$  and a spanning tree  $T \in \mathcal{T}(G)$ , let  $\xi(T)$  be the number of support vertices of  $T$  and  $\theta(T)$  be the maximum number of leaves adjacent to any support vertex of  $T$ . As a consequence of the above corollary we obtain the following result.

**Remark 1.** *If  $T$  is a spanning tree of a unicyclic graph  $G$  such that  $\varepsilon(G) = n_1(T)$ , then  $pd(G) \leq \xi(T) + \theta(T) + 1$ .*

**Proof:** If  $T$  is a path, then  $\xi(T) = 2$  and  $\theta(T) = 1$ , so the result follows. Now we suppose  $T$  is not a path. Let  $v$  be an exterior major vertex of terminal degree  $\tau(T)$  in  $T$ . Let  $x$  be the number of leaves of  $T$  adjacent to  $v$  and let  $y = \tau(T) - x$ . Since  $\kappa(T) + y \leq \xi(T)$  and  $x \leq \theta(T)$ , we deduce  $\kappa(T) + \tau(T) \leq \xi(T) + \theta(T)$ . Thus the result follows from Corollary 2.  $\square$

As the next theorem shows, the above result can be improved.

**Theorem 3.** *If  $T$  is a spanning tree of a unicyclic graph  $G$  such that  $\varepsilon(G) = n_1(T)$ , then  $\theta(T) - 1 \leq pd(G) \leq \xi(T) + \theta(T)$ .*

**Proof:** The result follows for the cycle graphs  $G = C_n$ , so we suppose  $G \neq C_n$ . Notice that different leaves adjacent to the same support vertex must belong to different sets of a resolving partition. Also, as  $\varepsilon(G) = n_1(T)$  we have  $pd(G) \geq \theta(T) - 1$ . Thus, the lower bound follows.

To obtain the upper bound, let  $T \in \mathcal{T}(G)$  be such that  $n_1(T) = \varepsilon(G)$ . Let  $C$  be the set of vertices belonging to the cycle of  $G = (V, E)$  and let  $uv \in E$ , such that  $u, v \in C$  and  $T = G - \{uv\}$ . Since  $n_1(T) = \varepsilon(G)$ , we have  $\delta_G(v) \geq 3$  or  $\delta_G(u) \geq 3$ , where  $\delta_G(u)$  represents the degree of the vertex  $u$  in  $G$ . Now, let  $S = \{s_1, s_2, \dots, s_{\xi(T)}\}$  be the set of support vertices of  $T$ , and for every  $s_i \in S$ , let  $\{s_{i1}, s_{i2}, \dots, s_{il_i}\}$  be the set of leaves of  $s_i$  and let  $\theta(T) = \max_{i \in \{1, \dots, \xi(T)\}} \{l_i\}$ .

Let now  $A_i = \{s_{i1}\}$ , for every  $i \in \{1, \dots, \xi(T)\}$ . Let  $M_{ij} = \{s_{ij}\}$ , for every  $j \in \{2, \dots, l_i\}$ . If  $l_i < \xi(T)$ , then we assume  $M_{ij} = \emptyset$ , for every  $j \in \{l_{i+1}, \dots, \theta(T)\}$ . Let  $B_j = \bigcup_{i=1}^{\xi(T)} M_{ij}$ , for every  $j \in \{2, \dots, \theta(T)\}$ . We will show that the partition

$$\Pi = \{A, A_1, A_2, \dots, A_{\xi(T)}, B_2, B_3, \dots, B_{\theta(T)}\},$$

with  $A = V(G) - \bigcup_{i=1}^{\xi(T)} A_i - \bigcup_{i=2}^{\theta(T)} B_i$ , is a resolving partition for  $G$ . Let  $x, y \in V$  be two different vertices in  $G$ . We have the following cases:

Case 1:  $x \notin C$  and  $y \in C$ . If  $\delta_G(u) = 2$  and  $\delta_G(v) \geq 3$ , then  $u$  is a leaf in  $T$  and we can suppose, without loss of generality, that  $u = s_{i1}$ , for some  $i \in \{1, \dots, \xi(T)\}$ , so  $A_i = \{u\}$ . Hence, if  $y = u$  or  $x$  is a leaf, then  $x$  and  $y$  belong to different sets of  $\Pi$ . On the contrary, if  $y \neq u$  and  $x$  is not a leaf, then there exists a leaf  $s_{l_1}$  such that  $x$  belongs to a minimum  $y - s_{l_1}$  path, thus  $d_G(y, A_l) > d_G(x, A_l)$ . Now, if  $\delta_G(u) \geq 3$  and  $\delta_G(v) \geq 3$ , then let  $a \in C$  such that  $d_G(x, a) = \min_{b \in C} \{d_G(x, b)\}$ . Hence, there exists a leaf  $s_{j1}$  such that  $x$  belongs to the  $a - s_{j1}$  path. So, we have  $d_G(y, A_j) = d_G(y, a) + d_G(a, A_j) > d_G(y, a) + d_G(x, A_j) \geq d_G(x, A_j)$ .

Case 2:  $x \notin C$  and  $y \notin C$ . If  $x, y \in B_j$ , for some  $j \in \{2, \dots, \theta(T)\}$ , then  $x = s_{ij}$  and  $y = s_{kj}$ , with  $1 \neq j \neq k \neq 1$ . So, we have  $d_G(y, A_i) = d_G(y, s_k) + d_G(s_k, s_i) + 1 \geq d_G(y, s_k) + 2 = d_G(y, s_k) + d_G(x, A_i) > d_G(x, A_i)$ . On the other hand, if  $x, y \in A$ , then there exists a leaf  $s_{i1}$  such that either,  $x$  belongs to one  $y - s_{i1}$  path or  $y$  belongs to one  $x - s_{i1}$  path. So, we have  $d_G(x, A_i) \neq d_G(y, A_i)$ .

Case 3:  $x, y \in C$ . Now we have the following subcases.

Subcase 3.1:  $\delta_G(u) \geq 3$  and  $\delta_G(v) \geq 3$ . Let  $s_{k1}$  and  $s_{j1}$ ,  $j \neq k$  be two leaves, such that the  $v - s_{k1}$  path shares with cycle  $C$  only the vertex  $v$  and the  $u - s_{j1}$  path shares with cycle  $C$  only the vertex  $u$ . If  $d_G(x, u) \neq d_G(y, u)$ , then we have  $d_G(x, A_j) = d_G(x, u) + d_G(u, s_{j1}) \neq d_G(y, u) + d_G(u, s_{j1}) = d_G(y, A_j)$ . On the contrary, if  $d_G(x, u) = d_G(y, u)$ , then  $d_G(x, v) \neq d_G(y, v)$  and



we have  $d_G(x, A_k) = d_G(x, v) + d_G(v, s_{k1}) \neq d_G(y, v) + d_G(v, s_{k1}) = d_G(y, A_k)$ .

Subcase 3.2: Without loss of generality, assume  $\delta_G(u) = 2$  and  $\delta_G(v) \geq 3$ . Hence,  $u$  is a leaf in  $T$  and we can suppose, without loss of generality, that  $u = s_{i1}$ , for some  $i \in \{1, \dots, \xi(T)\}$ , so  $A_i = \{u\}$ . If  $x = u$  or  $y = u$ , then  $x, y$  belong to different sets of  $\Pi$ . If  $d_G(x, u) \neq d_G(y, u)$ , then  $d_G(x, A_i) \neq d_G(y, A_i)$ . On the other hand, if  $d_G(x, u) = d_G(y, u)$ , then  $d_G(x, v) \neq d_G(y, v)$ . Now, let  $s_{k1}$  be a leaf, such that the  $v - s_{k1}$  path shares with cycle  $C$  only the vertex  $v$ . Hence, we have  $d_G(x, A_k) = d_G(x, v) + d_G(v, s_{k1}) \neq d_G(y, v) + d_G(v, s_{k1}) = d_G(y, A_k)$ . Therefore, for every different vertices  $x, y \in V$  we have  $r(x|\Pi) \neq r(y|\Pi)$  and  $\Pi$  is a resolving partition for  $G$ . □

Note that the above upper bound is achieved for unicyclic graphs having at most two exterior major vertices and each one of them has terminal degree one. In such a case,  $pd(G) = 3$ . The lower bound is achieved, for instance, for the graph shown in Figure 2.

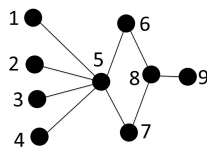


Figure 2: A graph  $G$  for which  $\Pi = \{\{1, 8\}, \{2, 5, 9\}, \{3, 6\}, \{4, 7\}\}$  is a resolving partition. For the tree  $T$  obtained by removing the edge  $\{6, 8\}$  from  $G$  we have that  $\varepsilon(G) = n_1(T) = 6$  and  $\theta(T) = 5$ . Thus, by Theorem 3 we conclude that  $pd(G) = \theta(G) - 1 = 4$ .

The following conjecture, if true, would be completely analogous to the estimate known for the metric dimension.

**Conjecture 1.** *If  $T$  is a spanning tree of a unicyclic graph  $G$ , then  $pd(G) \leq pd(T) + 1$ .*

According to (2.1), Lemma 1 and Theorem 2 (i), the above conjecture is true for every cycle graph and for every unicyclic graph where every exterior major vertex has terminal degree one. Even so, the previous conjecture seems to be very hard to prove. We therefore present the following weakened version.

**Proposition 1.** *If  $T$  is a spanning tree of a unicyclic graph  $G$ , then  $pd(G) \leq pd(T) + 3$ .*

**Proof:** Arbitrarily cut the cycle  $C = \{c_0, \dots, c_{k-1}\}$  by deleting, without loss of generality,  $c_0c_1$ . This results in a (spanning) tree  $T$ . Let  $\Pi$  be an optimum resolving partition for  $T$ , i.e.,  $\Pi = \{A_1, \dots, A_{pd(T)}\}$ . Let  $D = \{c_0, c_1, c_{\lfloor \frac{k}{2} \rfloor}\}$  and define  $A_i^G = A_i - D$ . We claim that  $\Pi^G = \{A_1^G, \dots, A_{pd(T)}^G, \{c_0\}, \{c_1\}, \{c_{\lfloor \frac{k}{2} \rfloor}\}\}$  is a resolving partition for  $G$ , where we only take the nonempty sets  $A_i^G = A_i - D$ . So, the order of this partition may be less than  $pd(T) + 3$ .

For every  $i \in \{0, 1, \dots, k - 1\}$ , let  $T_i = (V_i, E_i)$  be the subtree of  $T$  rooted at  $c_i$ . Note that may occur that  $V_i = \{c_i\}$ . We differentiate between two cases for  $x, y \in V(G)$ ,  $x \neq y$ .

Case 1.  $x, y \in V_i$ . If  $d_G(x, c_i) \neq d_G(y, c_i)$ , then  $d_G(x, c_0) \neq d_G(y, c_0)$ . Now, if  $d_G(x, c_i) =$

$d_G(y, c_i)$ , then for every  $v \in V(G) - V_i$ , it follows that  $d_G(x, v) = d_G(y, v)$  (notice that  $d_T(x, v) = d_T(y, v)$ ). Thus, for  $A_j \in \Pi$  such that  $d_T(x, A_j) \neq d_T(y, A_j)$ , there exist  $a, b \in A_j \cap V_i$  such that  $d_T(x, A_j) = d_T(x, a) \neq d_T(y, b) = d_T(y, A_j)$ . Hence,  $d_G(x, A_j^G) = d_T(x, A_j) \neq d_T(y, A_j) = d_G(y, A_j^G)$ .

Case 2.  $x \in V_i, y \in V_j, i \neq j$ . We claim that there exists a number  $r \in \{0, 1, \lfloor \frac{k}{2} \rfloor\}$  such that  $d_G(x, c_r) \neq d_G(y, c_r)$ . We proceed by deriving a contradiction. To this end, suppose that  $d_G(x, c_r) = d_G(y, c_r)$  for every  $r \in \{0, 1, \lfloor \frac{k}{2} \rfloor\}$ . In this case we obtain the following three equalities.  $d_G(x, c_i) + d_G(c_i, c_r) = d_G(y, c_j) + d_G(c_j, c_r), \quad r \in \{0, 1, \lfloor \frac{k}{2} \rfloor\}$ , or equivalently,

$$d_G(x, c_i) - d_G(y, c_j) = d_G(c_j, c_r) - d_G(c_i, c_r), \quad r \in \left\{0, 1, \left\lfloor \frac{k}{2} \right\rfloor\right\}. \quad (2.6)$$

Now we distinguish the following subcases:

Subcase 2.1.  $1 < i < j < \lfloor \frac{k}{2} \rfloor$  or  $\lfloor \frac{k}{2} \rfloor < i < j < k$ . For  $r = 1$  and  $r = \lfloor \frac{k}{2} \rfloor$  in (2.6) we deduce  $j - i = i - j$ , which is a contradiction.

Subcase 2.2.  $1 < i < \lfloor \frac{k}{2} \rfloor$  and  $\lfloor \frac{k}{2} \rfloor < j < k$ . For  $r = 0$  and  $r = 1$  in (2.6) we deduce  $k - j - i = k - i - j + 2$ , which is a contradiction.

Therefore,  $\Pi^G$  is a resolving partition for  $G$ . □

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