

Algebraic computations on the set of real intervals

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Abstract

The set of closed intervals of \mathbb{R} is provided with a semigroup structure. We complete this semigroup to obtain a 2-dimensional vector space. But neither associative nor nonassociative algebra structure satisfying a natural property on the product of intervals can be defined on this vector space. We define an embedding of this vector space into a 4-dimensional associative algebra and we compute all the arithmetic operations on intervals in this algebra. As applications, we study polynomial functions and the problem of diagonalization of matrices whose elements are intervals.

Key Words: Intervals analysis, associative algebras, matrices of intervals, polynomial equations.

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1 Introduction and motivations

The idea of replacing real numbers by intervals is not new as shown by the construction of real numbers by Dedekind cuts. This approach is widely used in real numerical analysis. A irrational real number, for example transcendental, can not be represented by a periodic number. It is then replaced by an interval containing this number. But this idea encounters the following problem: to replace the real arithmetic by an arithmetic on the intervals requires a ring structure on the set of intervals for the operations linked to the operations in \mathbb{R} . So it is natural to define binary operations on the set of intervals in the following way: let \diamond be a binary operation on \mathbb{R} ; if X and Y are two intervals, then $X \diamond Y$ is the smallest interval containing all the real numbers $x \diamond y$ with $x \in X$ and $y \in Y$. Let \mathcal{I} be the set of non empty intervals of \mathbb{R} . Then \mathcal{I} provided with the addition and multiplication could be an arithmetic model. The first problem is that the couple $(\mathcal{I}, +)$ is not a group. This has been for a long time circumvent because $(\mathcal{I}, +)$ is a semigroup and the group completion of a semigroup is classical (an example is the construction of \mathbb{Z} from \mathbb{N}). If we denote $(\widehat{\mathcal{I}}, +)$ this group then $(\widehat{\mathcal{I}}, +, \times)$ is not a ring. In fact, the multiplication is not distributive with respect to the addition and we are unable to complete this non distributive structure in a distributive one. This is a major problem. Consider for example, in classical mechanic, the fundamental momentum theorem. The main

formula is directly based on the distributivity of the momentum. If we use a non distributive method in numerical computations, it is very difficult to know how to write the fundamental formula. Some methods have been developed to avoid this problem ([2]). Interval arithmetic is used in control problems, robotic problems, computations with floating-point numbers, inverse problems, etc... Specific tools are necessary in each of these cases. The main idea of this paper is to define an algebraic embedding of the set of intervals in an associative algebra, to transfer all algebraic computations in this algebra and so obtain as result a point of this algebra and then return an interval in some natural way that we will describe.

2 A linear model for the set of intervals

Let us denote by \mathcal{I} the set of intervals $[x, y]$, that is, closed and bounded intervals of \mathbb{R} . It is in one-to-one correspondence with the convex half plane $\mathcal{P} = \{(x, y) \in \mathbb{R}^2, x \leq y\}$. There is a regular additive semigroup structure on \mathcal{I} defined by the addition $[x_1, y_1] + [x_2, y_2] = [x_1 + x_2, y_1 + y_2]$. There is also a natural action of \mathbb{R}^+ on \mathcal{I} given by $\alpha \cdot [x, y] = [\alpha x, \alpha y]$ with $\alpha \in \mathbb{R}^+$. The smallest \mathbb{R} -vector space, denoted by $\widehat{\mathcal{I}}$, containing \mathcal{I} and compatible with the previous action is linearly isomorphic with the vector space generated by the convex \mathcal{P} , that is, \mathbb{R}^2 . It is $\widehat{\mathcal{I}} = \mathcal{I} \times \mathcal{I} / \mathcal{R}$ where \mathcal{R} is the equivalence relation

$$(X_1, Y_1)\mathcal{R}(X_2, Y_2) \Leftrightarrow X_1 + Y_2 = X_2 + Y_1,$$

$X_1, X_2, Y_1, Y_2 \in \mathcal{I}$. If we denote by $\overline{(X, Y)}$ the class of the pair (X, Y) , we have

$$\widehat{\mathcal{I}} = \{\overline{(X, 0)}, \overline{(0, X)}, X \in \mathcal{I}\}.$$

In fact any (X_1, Y_1) in $\mathcal{I} \times \mathcal{I}$ is equivalent to $(X, 0)$ if $X_1 = X + Y_1$ or $(0, X)$ if we have $Y_1 = X + X_1$. Here, 0 denote the interval $[0, 0]$. If we denote by $-\overline{(X, Y)}$ the opposite for the addition of $\overline{(X, Y)}$, we have $-\overline{(X, 0)} = \overline{(0, X)}$. The semigroup \mathcal{I} is embedded in $\widehat{\mathcal{I}}$, identifying $X \in \mathcal{I}$ with $\overline{(X, 0)}$. The vector space \mathbb{R} can be considered as a subspace of $\widehat{\mathcal{I}}$: to any $a \in \mathbb{R}$ corresponds the interval $[a, a]$ which will be simply denoted by a . We have in particular $\overline{(a, 0)} = \overline{(0, -a)}$. The multiplication by a scalar in $\widehat{\mathcal{I}}$ corresponds to the embedding of the action of \mathbb{R}^+ in \mathcal{I} . It verifies

$$\begin{cases} \alpha \overline{(X, 0)} = \overline{(\alpha X, 0)} & \text{if } \alpha \geq 0, \\ \alpha \overline{(X, 0)} = \overline{(0, -\alpha X)} & \text{if } \alpha \leq 0, \\ \alpha \overline{(0, X)} = \alpha \overline{-(X, 0)} = -\alpha \overline{(X, 0)}, & \alpha \in \mathbb{R}. \end{cases}$$

Thus $\widehat{\mathcal{I}}$ is a 2-dimensional real vector space isomorphic to \mathbb{R}^2 . In the following, we will consider the norm in $\widehat{\mathcal{I}}$ given by $\|\overline{(X, 0)}\| = \|\overline{(0, X)}\| = |l(X)| + |c(X)|$ where $l(X) = y - x$ is the length of the interval $X = [x, y]$ and $c(X) = \frac{x+y}{2}$ its center. Let \mathcal{X} be in $\widehat{\mathcal{I}}$. Then there exists $X \in \mathcal{I}$ such that $\mathcal{X} = \overline{(X, 0)}$ or $\overline{(0, X)}$. We put $l(\overline{(X, 0)}) = l(X)$ and $l(\overline{(0, X)}) = -l(X)$. Likewise, we put $c(\mathcal{X}) = c(X)$ if $\mathcal{X} = \overline{(X, 0)}$ and $c(\mathcal{X}) = -c(X)$ if $\mathcal{X} = \overline{(0, X)}$.

Definition 1. Let \mathcal{X} and \mathcal{Y} be in $\widehat{\mathcal{I}}$. We say that $\mathcal{X} \geq \mathcal{Y}$ if one of the following properties is satisfied :

- i) $\exists U \in \mathcal{I} \setminus \mathbb{R}$ such that $\mathcal{X} - \mathcal{Y} = \overline{(U, 0)}$
- ii) $\exists a \in \mathbb{R}$ such that $\mathcal{X} - \mathcal{Y} = \overline{(a, 0)}$ and $c(\mathcal{X}) \geq c(\mathcal{Y})$.

This corresponds to the lexicographic order of the couple $(l(X), c(X))$.

Remarks.

1. There exists a natural functional approach of the addition of intervals. Let X and Y be in \mathcal{I} and denote by I_X and I_Y the corresponding characteristic functions. Then $X + Y$ corresponds to the image of the convolution product $I_X \star I_Y$.

2. On the union of intervals. Let $X_1 = [a, b]$ and $X_2 = [b, c]$ be two intervals with $X_1 \cap X_2 = \{b\}$. Then $X_1 \cup X_2 = [a, c]$ and in this case the union is an internal operation. We can write in term of vectorial addition: $X_1 \cup X_2 = X_1 + X_2 - [b, b]$. If $X_1 = [a, b], X_2 = [c, d]$ with $X_1 \cap X_2 = [c, b]$ then $X_1 \cup X_2 = X_1 + X_2 - [c, b]$.

3. Consider the map $s : \mathcal{I} \rightarrow \mathcal{I}$ given by $s([x, y]) = [-y, -x]$. It extends to an endomorphism of $\widehat{\mathcal{I}}$, still denoted by s , defined by $s(\overline{(X, 0)}) = \overline{(s(X), 0)}$ for any $X \in \mathcal{I}$. Since s is linear, it is differentiable at any point. This differentiable map is often used in transfer problems in interval analysis. Consider, for example, the following problem: suppose that we have a bar of length L with $L \in [2, 3]$. We cut a piece of length $A \in [0, 1]$. The new bar will have a length belonging to the interval $[2, 3] + s[0, 1] = [1, 3]$. Let us note that the operation $[2, 3] - [0, 1] = [2, 2]$ is not a appropriated with this problem but corresponds to the problem: what length should be removed to obtain a bar of length 2.

3 Algebra structure on $\widehat{\mathcal{I}}$

Recall that an algebra structure on a vector space V is given by a bilinear map on V with values in V . The first natural idea in order to define a product on $\widehat{\mathcal{I}}$ would be to consider the set definition: if X and Y are in \mathcal{I} , we could put $P(X, Y) = \{xy, x \in X, y \in Y\}$ but this map $P(X, Y)$ is not bilinear. More precisely, in [3], we have determined, up to isomorphism, all the 2-dimensional algebras on \mathbb{R} (see also [4]). It is easy to see that none of these algebras can be defined by the map P . In fact, the map P is piecewise bilinear. We will call piecewise algebra, a vector space provided with a mapping $P : V \times V \rightarrow V$ which is piecewise bilinear. This implies that the associated multiplication is not distributive with respect to the addition. Then, we have to define an associative commutative algebra whose product is a good approximation of the product $P(X, Y)$. This is not, in our opinion, a disagreement because the aim of interval computations is to present results in an interval containing possible values.

We consider the 4-dimensional commutative unitary associative real algebra \mathcal{A}_4 whose multiplication is written is a basis $\{e_1, e_2, e_3, e_4\}$ where e_1 is the unit and

$$e_2e_2 = e_2, e_2e_3 = e_2e_4 = e_3, e_3e_3 = e_3e_4 = e_2, e_4e_4 = e_1.$$

In terms of associative algebra, the algebra \mathcal{A}_4 is isomorphic to the trivial abelian 4-dimensional associative algebra \mathbb{R}^4 . In fact, if we consider the canonical basis $\{i, j, k, l\}$ of \mathbb{R}^4 , the change of basis $e_1 = i + j + k + l, e_2 = k + l, e_3 = k - l, e_4 = i - j + k - l$ gives the wanted isomorphism. It is well known that this unitary ring is not a field. The group of invertible elements is the complementary of the union of the coordinates planes. It is also not a principal domain, and it

is not factorial. Any polynomial in $\mathcal{A}_4[X]$ of degree n can have, at most, n^4 roots. This ring is not noetherian.

If we consider the intervals $E_1 = [1, 1], E_2 = [0, 1], E_3 = [-1, 0]$ et $E_4 = [-1, -1]$, the classical products $P(E_i, E_j)$ correspond to the multiplication table of \mathcal{A}_4 given by the products $e_i e_j$. Let F be the linear subspace of \mathcal{A}_4 generated by the vectors $e_1 + e_4, e_1 - e_2 + e_3$. If $X = [x, y] \in \mathcal{I}$, then we can write $X = xE_1 + (y - x)E_2$ if $x \geq 0$, or $X = -xE_3 + yE_2$ if $x \leq 0 \leq y$, or $X = (-x + y)E_3 - yE_4$ if $x \leq y \leq 0$. Let us consider the map $\Phi : \mathcal{I} \rightarrow \mathcal{A}_4$ given by

$$\Phi(X) = \begin{cases} (x, y - x, 0, 0) & \text{in the case (1),} \\ (0, y, -x, 0) & \text{in the case (2),} \\ (0, 0, -x + y, -y) & \text{in the case (3).} \end{cases}$$

We put $v_X = \Phi(X)$. The components of this vector are positive or zero. We denote by $\bar{v}_X = \pi(v_X)$ where π is the canonical projection on \mathcal{A}_4/F . We define, in this way, an injective map, also denoted by Φ of \mathcal{I} with values in \mathcal{A}_4/F . We extend Φ to $\widehat{\mathcal{I}}$ putting

$$\Phi(\overline{(X, 0)}) = \Phi(X), \quad \Phi(\overline{(0, X)}) = -\Phi(X).$$

Then any element of $\Phi(\widehat{\mathcal{I}})$ is of type $T := \{ae_1 + be_2, ae_2 + be_3, ae_3 + be_4$ with $(a, b) \in \mathbb{R}^+ \times \mathbb{R}^+$ or $(a, b) \in \mathbb{R}^- \times \mathbb{R}^-\}$. We can then identify $\widehat{\mathcal{I}}$ to the subset T of the algebra \mathcal{A}_4 .

Proposition 1. *The subset $\widehat{\mathcal{I}}$ of \mathcal{A}_4 is a multiplicative set. Moreover the set*

$$\widehat{M} = \{(\overline{[-a, a]}, 0), (0, \overline{[-a, a]}), a \in \mathbb{R}\}$$

is a multiplicative absorbing set of $\widehat{\mathcal{I}}$. Conversely, for every $\bar{v} \in \mathcal{A}_4/F$, there exists a unique vector $w_{\bar{v}} \in \mathcal{A}_4$ such that

1. $\pi(w_{\bar{v}}) = \bar{v}$,
2. $\exists ! X \in \mathcal{I}$ such that $\Phi(\overline{(X, 0)}) = w_{\bar{v}}$ or $\Phi(\overline{(0, X)}) = w_{\bar{v}}$.

Proof. We have seen that any element of $\widehat{\mathcal{I}}$ is identified with an element of type $T := \{ae_1 + be_2, ae_2 + be_3, ae_3 + be_4$ with $(a, b) \in \mathbb{R}^+ \times \mathbb{R}^+$ or $(a, b) \in \mathbb{R}^- \times \mathbb{R}^-\}$. From the multiplication table of \mathcal{A}_4 , the multiplication preserves the set of these vectors. Let $(\overline{[-a, a]}, 0)$ be in $\widehat{\mathcal{I}}$. Such a vector is written $a(e_2 + e_3)$ with $a \geq 0$. We have $(e_2 + e_3)e_i = e_2 + e_3$ for $i = 1, 2, 3, 4$ and \widehat{M} is absorbing. Now, let $v = (x_1, x_2, x_3, x_4)$ be in \mathcal{A}_4 . We will show that it is in the congruency class defined by F of a vector $v_T \in T$ with $\pi(v_T) = \pi(v)$. If $v_T = ae_i + be_{i+1}$ with $i \in 1, 2, 3$, then $X = aE_i + bE_{i+1}$ is an interval satisfying $\Phi(\overline{(X, 0)}) = \pi(v) = \pi(v_T)$ if $(a, b) \in \mathbb{R}^+ \times \mathbb{R}^+$ and $\Phi(\overline{(0, X)}) = \pi(v) = \pi(v_T)$ if $(a, b) \in \mathbb{R}^- \times \mathbb{R}^-$. The vector v is equivalent, modulo F , to the vector $(x_1 - x_3 - x_4)e_1 + (x_2 + x_3)e_2$. If $x_2 + x_3 \geq 0$ and $x_1 - x_3 - x_4 \geq 0$, then $X = (x_1 - x_3 - x_4)E_1 + (x_2 + x_3)E_2$ satisfies $\Phi(\overline{(X, 0)}) = \pi(v)$. If $x_2 + x_3 \geq 0$ and $x_1 - x_3 - x_4 \leq 0, x_1 + x_2 - x_4 \geq 0$, then $X = (x_1 + x_2 - x_4)E_2 + (x_4 - x_1 + x_3)E_3$ satisfies $\Phi(\overline{(X, 0)}) = \pi(v)$. If $x_2 + x_3 \geq 0$ and $x_1 - x_3 - x_4 \leq 0, x_1 + x_2 - x_4 \leq 0$, then for $X = (x_2 + x_3)E_3 + (-x_1 - x_2 + x_4)E_4$ we have $\Phi(\overline{(X, 0)}) = \pi(v)$. Now if $x_2 + x_3 \leq 0, x_1 - x_3 - x_4 \geq 0$ and $x_1 + x_2 - x_4 \leq 0$, we consider $X = (x_1 + x_2 - x_4)E_2 + (-x_1 + x_3 + x_4)E_3$, if $x_2 + x_3 \leq 0, x_1 - x_3 - x_4 \geq 0$ and

$x_1 + x_2 - x_4 \geq 0$, we consider $X = (x_2 + x_3)E_3 + (-x_1 - x_2 + x_4)E_4$, and if $x_2 + x_3 \leq 0$ and $x_1 - x_3 - x_4 \leq 0$, we take $X = (x_1 - x_3 - x_4)E_1 + (x_2 + x_3)E_2$. In these cases, we have $\Phi(\overline{(0, X)}) = \pi(v)$. Note that in the third first cases, we have $X = [x_1 + x_2 - x_4, x_1 - x_3 - x_4]$ and in the third last cases, $X = [x_1 - x_3 - x_4, x_1 + x_2 - x_4]$.

Consequence. Using the previous construction we are able to define a surjective map

$$\Psi : \mathcal{A}_4 \rightarrow \widehat{\mathcal{I}}.$$

We can then define a structure of associative Ψ -algebra on $\widehat{\mathcal{I}}$ in the following sense.

Let $\mathcal{X} = \overline{(X, 0)}$ and $\mathcal{Y} = \overline{(Y, 0)}$ be in $\widehat{\mathcal{I}}$. We define the product

$$\mathcal{X} \bullet \mathcal{Y} = \Psi(\Phi(\overline{(X, 0)})\Phi(\overline{(Y, 0)})).$$

This product is well defined because the set T is a multiplicative set in \mathcal{A}_4 . If $\mathcal{X} = \overline{(X, 0)}$ and $\mathcal{Y} = \overline{(0, Y)}$, we put

$$\mathcal{X} \bullet \mathcal{Y} = -(\mathcal{X} \bullet (-\mathcal{Y})).$$

Proposition 2. *The vector space $\widehat{\mathcal{I}}$ provided with the product $\mathcal{X} \bullet \mathcal{Y}$ is an associative Ψ -algebra, that is, this product is Ψ -distributive:*

$$\Psi(\Phi(\overline{(X, 0)})\Phi(\overline{(Y, 0)}) + \Phi(\overline{(Z, 0)})) = \Psi(\Phi(\overline{(X, 0)})\Phi(\overline{(Y, 0)})) + \Psi(\Phi(\overline{(X, 0)})\Phi(\overline{(Z, 0)})).$$

In fact this is a consequence of the distributivity on \mathcal{A}_4 .

This proposition shows that the multiplicative part $\widehat{\mathcal{I}}$ of \mathcal{A}_4 can be considered as a vector space provided with an associative multiplication which is Ψ -distributive. We will say that $\widehat{\mathcal{I}}$ is a Ψ -algebra. We will still denote this Ψ -algebra by $\widehat{\mathcal{I}}$.

Proposition 3. *Let \mathcal{X} and \mathcal{Y} be in $\widehat{\mathcal{I}}$. Then, for any $\mathcal{Z} \in \widehat{\mathcal{I}}$ with $\mathcal{Z} > 0$, we have*

$$\mathcal{X} < \mathcal{Y} \Rightarrow \mathcal{X} \bullet \mathcal{Z} < \mathcal{Y} \bullet \mathcal{Z}.$$

Proof. In fact $\mathcal{Y} \bullet \mathcal{Z} - \mathcal{X} \bullet \mathcal{Z} = (\mathcal{Y} - \mathcal{X}) \bullet \mathcal{Z}$ by the Ψ -distributivity. But, by hypothesis, $\mathcal{Y} - \mathcal{X} = \overline{(U, 0)}$. Since $\mathcal{Z} > 0$ we have $\mathcal{Z} = \overline{(Z, 0)}$. Thus $\overline{(U, 0)} \bullet \overline{(Z, 0)}$ is of type $\overline{(X, 0)}$ and $\mathcal{X} \bullet \mathcal{Z} < \mathcal{Y} \bullet \mathcal{Z}$.

Proposition 4. *The Ψ -algebra $\widehat{\mathcal{I}}$ is an integral domain.*

Proof. Let us note that the ring \mathcal{A}_4 is not an integral domain. But $\overline{(X, 0)} \in \widehat{\mathcal{I}}$ is represented in \mathcal{A}_4 by a vector of type $(x, y - x, 0, 0)$ with $0 \leq x \leq y$, or $(0, y, -x, 0)$ with $x \leq 0 \leq y$ or $(0, 0, -x + y, -y)$ with $x \leq y \leq 0$. The vectors of F are written $(a + b, -b, b, a)$. Computing the different cases we obtain that $\Phi(\overline{(X_1, 0)})\Phi(\overline{(X_2, 0)}) \in F$ if and only if X_1 or $X_2 = 0$.

Proposition 5. *An element $\Phi(\overline{(X, 0)})$ or $\Phi(\overline{(0, X)})$ is invertible in \mathcal{A}_4 if and only if 0 is not in the interval X .*

Proof. This is a direct consequence of the characterization of invertible elements of \mathcal{A}_4 .

Let $X = [x, y]$ be an interval. It is invertible if $0 \notin X$. Its inverse is represented by $\Phi(\overline{(0, U)}) = (0, 0, \frac{x-y}{xy}, -\frac{1}{y})$ with $U = [-\frac{1}{x}, -\frac{1}{y}]$.

4 The polynomial algebra $\widehat{\mathcal{I}}[\mathcal{X}]$

A polynomial function with coefficients in $\widehat{\mathcal{I}}$ is a function $f : \widehat{\mathcal{I}} \rightarrow \widehat{\mathcal{I}}$ of type

$$f(\mathcal{X}) = \mathcal{B}_n \bullet \mathcal{X}^n + \dots + \mathcal{B}_1 \bullet \mathcal{X} + \mathcal{B}_0$$

where $\mathcal{B}_i \in \widehat{\mathcal{I}}$ and $\mathcal{X}^p = \mathcal{X} \bullet \mathcal{X} \bullet \dots \bullet \mathcal{X}$, p times. A particular case consists in taking the coefficients \mathcal{B}_p in \mathbb{R} , this is possible because \mathbb{R} is embedded in $\widehat{\mathcal{I}}$. The set of polynomial functions with coefficients in $\widehat{\mathcal{I}}$ is a non factorial ring because \mathcal{A}_4 is not an integral domain (there are zero divisors). Nevertheless, if \mathcal{X} corresponds to an interval, that is $\mathcal{X} = \overline{(X, 0)}$, thus $\Phi(\mathcal{X})$ is not a zero divisor. In the multiplicative part $\widehat{\mathcal{I}}$ of \mathcal{A}_4 , we call irreducible the invertible elements, that is the elements $\overline{(X, 0)}$ and $\overline{(0, X)}$ with $0 \notin X$, and the elements of \widehat{M} . Then any non irreducible element admits a unique factorization up to an irreducible element. Thus $\widehat{\mathcal{I}}$, which is without zero divisor, is factorial. This permits to develop the arithmetic of $\widehat{\mathcal{I}}$, but considering $\widehat{\mathcal{I}}$ as a subset of \mathcal{A}_4 .

Algebraic equations of degree 1. Consider the equation $\mathcal{B}_0 \bullet \mathcal{X} = \mathcal{B}_1$ with $\mathcal{B}_0, \mathcal{B}_1 \in \widehat{\mathcal{I}}$. If \mathcal{B}_0 is invertible, we have a unique solution $\mathcal{X} = \mathcal{B}_1 \bullet \mathcal{B}_0^{-1}$ where \mathcal{B}_0^{-1} is the inverse of \mathcal{B}_0 calculated in \mathcal{A}_4 . We have seen that if $\mathcal{B}_0 \in \widehat{\mathcal{I}}$, then \mathcal{B}_0^{-1} also. Assume now that \mathcal{B}_0 is not invertible. If $\mathcal{B}_0 = \overline{(B_0, 0)}$, thus $0 \in B_0$ and $\Phi(\mathcal{B}_0) = (0, b, c, 0)$ with $b, c \geq 0$. In this case, since the space generated by $\{e_2, e_3\}$ is an ideal of \mathcal{A}_4 , $\mathcal{B}_0 \bullet \mathcal{X} = \mathcal{B}_1$ has no solution if $\Phi(\mathcal{B}_1)$ is not in this ideal. Assume that $\Phi(\mathcal{B}_1) = (0, b_1, c_1, 0)$ with $b_1 c_1 \geq 0$. If $\Phi(\mathcal{X}) = (x_1, x_2, x_3, x_4)$, then $\mathcal{B}_0 \bullet \mathcal{X} = \mathcal{B}_1$ gives:

$$\begin{cases} b(x_1 + x_2) + c(x_3 + x_4) = b_1 \\ b(x_3 + x_4) + c(x_1 + x_2) = c_1. \end{cases}$$

If $b^2 - c^2 \neq 0$ we obtain $x_1 + x_2 = \frac{bb_1 - cc_1}{b^2 - c^2}$, $x_3 + x_4 = \frac{bc_1 - cb_1}{b^2 - c^2}$. If $b^2 - c^2 = 0$, then $B_0 = [-b, b]$ and we have solutions if $\mathcal{B}_1 = (0, b_1, b_1, 0)$; if not there is no solution. We obtain also analogous solutions when $\mathcal{X} = \overline{(0, X)}$.

Algebraic equations of degree 2. Consider, for example, the second order algebraic equation $p(\mathcal{X}) = \mathcal{X}^2 - \mathcal{X}$. It is easy to see that $\mathcal{X}_1 = 0$, $\mathcal{X}_2 = \overline{([1, 1], 0)}$, $\mathcal{X}_3 = \overline{([0, 1], 0)}$ are roots of $p(\mathcal{X}) = 0$. This shows that the factorization theorem using roots is not valid in the ring $\widehat{\mathcal{I}}[\mathcal{X}]$. In fact we have $\mathcal{X}^2 - \mathcal{X} = \mathcal{X} \bullet (\mathcal{X} - E_1)$. This shows that 0 and $E_1 = \overline{([0, 1], 0)}$ are roots. But, since there are zero divisors, we have to look if such equation is solved by zero divisors. If $\mathcal{X} = E_2 = \overline{([0, 1], 0)}$, thus $E_2 \bullet (E_2 - E_1) = 0$. In fact, solving an algebraic equation of degree 2 in $\widehat{\mathcal{I}}$ or \mathcal{A}_4 implies to solve polynomial systems of 2 equations of degree 2 in two variables, that is to define an intersection of two conics. In general, this gives 4 points. In our example, if we consider $\Phi(\mathcal{X}) = (x, y - x, 0, 0)$ that is $\mathcal{X} = \overline{([x, y], 0)}$ with $x \geq 0$ then $\Phi(\mathcal{X}^2 - \mathcal{X}) = (x^2 - x, y^2 - x^2 - y + x, 0, 0) = 0$. We obtain $x = 0$ or 1. If $x = 0$ then $y = 0$ or 1. If $x = 1$, then $y = 1$. Let us remark that the fourth solution given by $x = 1, y = 0$ does not correspond to an interval solution. Now, let us determine the roots of $p(\mathcal{X})$ with $\Phi(\mathcal{X}) = (0, y, -x, 0)$ with $x \leq 0 \leq y$. In this case we obtain $\Phi(\mathcal{X}^2 - \mathcal{X}) = (0, x^2 + y^2 - y, x - 2xy, 0)$. The solutions are $\mathcal{X}_1, \mathcal{X}_3$ and $\mathcal{X}_4 = \overline{([-1/2, 1/2], 0)}$. To end the computation of the roots, we have to consider $\Phi(\mathcal{X}) = (0, 0, -x + y, -y), x \leq y \leq 0$. A direct calculus shows that \mathcal{X}_1 is the only solution.

Then, the second order equation $p(\mathcal{X}) = \mathcal{X}^2 - \mathcal{X} = 0$ has 4 roots of type $\mathcal{X} = \overline{(X, 0)}$ which are $\mathcal{X}_1 = 0, \mathcal{X}_2 = \overline{([1, 1], 0)}, \mathcal{X}_3 = \overline{([0, 1], 0)}$ and $\mathcal{X}_4 = \overline{(-\frac{1}{2}, \frac{1}{2}), 0}$.

The general case will be studied in similar way. If $p(\mathcal{X})$ is an algebraic equation of degree 2, considering $\Phi(p(\mathcal{X})) = 0$ we have to solve a real polynomial system of two equations with two variables. This corresponds to the determination of the intersection points of two real conics. This corresponds to, at most, 4 points.

Polynomial equations given by a transfer of real polynomials. We consider a real polynomial $P(x) = a_0 + a_1x + \dots + a_nx^n$ of degree n . There is no direct principle which gives a function on $\widehat{\mathcal{I}}$ from $P(x)$. Usually, the transfer function must satisfy the inclusion property, that is, if $\overline{P}(\mathcal{X})$ is a transfer function of $P(x)$, thus $P(X) \subset \overline{P}(\overline{(X, 0)})$. In a first approach, we can take $\overline{P}(\mathcal{X}) = \sum a_k \mathcal{X}^k$. Any root of $P(x)$ gives a root of $\overline{P}(\mathcal{X})$. In fact if a is a root of $P(x)$, that is, $P(a) = 0$, thus $\mathcal{X}_a = \overline{([a, a], 0)}$ satisfies $\overline{P}(\mathcal{X}_a) = 0$. But we can have new roots: any interval $[a, b]$ defined by two consecutive roots a, b of $P(x)$ determine a root of $\overline{P}(\mathcal{X})$, that is, $\overline{P}(\overline{([a, b], 0)}) = 0$. Such a root will be called a particular root. If we define the formal derivative of $\overline{P}(\mathcal{X})$ by $\overline{P}'(\mathcal{X}) = \sum ka_k \mathcal{X}^{k-1}$, particular roots of $\overline{P}(\mathcal{X})$ contain the roots α of $\overline{P}'(\mathcal{X})$. To recover the inclusion principle, it is sufficient to consider intervals containing particular roots and not contained in the roots of the derivative. Let us note that we also have:

$$\overline{P}(\mathcal{X}) = \overline{P}(\mathcal{X}_0) + (\mathcal{X} - \mathcal{X}_0)\overline{P}'(\mathcal{X}_0) + \frac{(\mathcal{X} - \mathcal{X}_0)^2}{2!}\overline{P}''(\mathcal{X}_0) + \dots + \frac{(\mathcal{X} - \mathcal{X}_0)^n}{n!}\overline{P}^{(n)}(\mathcal{X}_0).$$

We can apply this transferred polynomial to study the roots of a classical real polynomial.

Let $P(x) = x^n + a_1x^{n-1} + \dots + a_n$ be a real polynomial and $P(X)$ the transferred polynomial with $X \in \mathcal{I}$. Let $X_0 \in \mathcal{I}$. If $P(X_0)$ is a not invertible interval, then X_0 contains a root of $P(x)$. Consider, for example, the case of degree 3. Let $P(x) = x^3 + a_1x^2 + a_2x + a_3$ be a real polynomial with $a_3 < 0$. The equation $P(X) = 0$ admits at least a positive root. Let β be the root of $P''(x) = 0$. If $P'(\beta) \geq 0$ we have only one positive root. We consider an interval $X_1 = [0, y]$. If $P(X_1)$ is not invertible, we consider $X_{11} = [0, \frac{y}{2}]$ and $X_{12} = [\frac{y}{2}, y]$. We keep the only interval X_{1i} such that X_{1i} is not invertible. Then X_{1i} contains the positive root. We reapply this process until we get the required precision for this positive root. The conditions $P'(\beta) \geq 0$ and $a_2 > 0$ lead us to the similar computations. If $a_2 < 0$, there is only one positive root γ of the equation $P'(x) = 0$. We consider the interval $X_1 = [u, y]$ with $\gamma > u > 0$ and we apply the previous process.

5 Linear Algebra in the vector space $\widehat{\mathcal{I}}$

We apply the previous results to study linear endomorphisms of $\widehat{\mathcal{I}}^n$ and more particularly interval matrices, that is, matrices whose elements are intervals. Let $gl(n, \widehat{\mathcal{I}})$ be the vector space of square matrices of order n with coefficients in $\widehat{\mathcal{I}}$. If $A = (\mathcal{X}_{ij})_{i,j=1, \dots, n}$ and $B = (\mathcal{X}_{ij})$ are in $gl(n, \widehat{\mathcal{I}})$, we put $A \bullet B = (Z_{ij})$ with $Z_{ij} = \sum_{k=1}^n \mathcal{X}_{ik} \bullet \mathcal{Y}_{kj}$. With this multiplication, $gl(n, \widehat{\mathcal{I}})$ is an associative Φ -algebra. We can compute the determinant of a matrix of $gl(n, \widehat{\mathcal{I}})$ using the Cramer formulae.

Definition 2. A matrix $A \in gl(n, \widehat{\mathcal{I}})$ is called invertible if its determinant is an invertible element in $\widehat{\mathcal{I}}$.

If $A = (\mathcal{X}_{ij})$ is a interval matrix of intervals, we will say that a real matrix $M = (m_{ij})$ is in A if $m_{ij} \in \mathcal{X}_{ij}$ for all i, j . In particular, if c_{ij} is the center of \mathcal{X}_{ij} , we obtain a real matrix $C(A) = (c_{ij})$ which belongs to A . It is called the central real matrix of A .

Proposition 6. Let A be a interval matrix. If A is invertible, thus any real matrix $M \in A$ is invertible.

Proof. In fact, if a matrix M belonging to A is degenerate, thus $0 \in \det A$. But any interval containing 0 is not invertible. This gives a contradiction.

Examples To simplify notations, we write $[a, b]$ in place of $\overline{([a, b], 0)}$ and $-[a, b]$ in place of $(0, \overline{[a, b]})$. Consider the matrix

$$B = \begin{pmatrix} [1, 2] & [-1, 3] \\ [-1, 3] & [1, 7] \end{pmatrix}$$

Since $\det B = \overline{(0, [-7, -4])} = -[-7, -4]$, the matrix B is invertible and

$$B^{-1} = \left[\frac{1}{7}, \frac{1}{4} \right] \begin{pmatrix} [1, 7] & -[-1, 3] \\ -[-1, 3] & [1, 2] \end{pmatrix}.$$

Let A be in $gl(n, \widehat{\mathcal{I}})$. An eigenvalue of A is an element $\mathcal{X} \in \widehat{\mathcal{I}}$ such that there exists a vector $\mathcal{V} \neq 0 \in \widehat{\mathcal{I}}^n$ with $A \bullet^t \mathcal{V} = \mathcal{X} \bullet^t \mathcal{V}$. Thus \mathcal{X} is a root of the characteristic polynomial $C_A(\mathcal{X}) = \det(A - \mathcal{X}I) = 0$ with coefficients in $\widehat{\mathcal{I}}$. For example, if we consider the matrix

$$B_2 = \begin{pmatrix} [1, 2] & [1, 2] \\ [1, 3] & [2, 5] \end{pmatrix}.$$

we have

$$\det(B_2 - \mathcal{X}I) = (-\mathcal{X}) \bullet (-\mathcal{X}) - \mathcal{X} \bullet [3, 7] + [1, 4]$$

and the eigenvalues are

$$\mathcal{X}_1 = \left[\frac{3 + \sqrt{5}}{2}, \frac{7 + \sqrt{33}}{2} \right], \mathcal{X}_2 = \left[\frac{3 - \sqrt{5}}{2}, \frac{7 + \sqrt{33}}{2} \right], \mathcal{X}_3 = \left[\frac{3 - \sqrt{5}}{2}, \frac{7 - \sqrt{33}}{2} \right],$$

$$\mathcal{X}_4 = \left[\frac{-\sqrt{29} - \sqrt{37}}{4}, \frac{\sqrt{37} - \sqrt{29} + 14}{4} \right], \mathcal{X}_5 = \left[\frac{\sqrt{29} - \sqrt{37}}{4}, \frac{\sqrt{37} + \sqrt{29} + 14}{4} \right].$$

The determination of eigenvalues of type $(0, K)$ is similar. Nevertheless we have to consider only matrices with positive entries thus we study only the positive eigenvalues. The negative eigenvalues do not correspond to physical entities.

Definition 3. Let A be a matrix in $gl(n, \widehat{\mathcal{I}})$. Let $C(A)$ be the real matrix whose elements are the center of the intervals of A . We say that an eigenvalue of A is a central eigenvalue if its center is (close to) an eigenvalue of $C(A)$.

In the previous example, \mathcal{X}_1 and \mathcal{X}_3 are the central eigenvalues.

If \mathcal{X} is an eigenvalue of A , then every vector $\mathcal{V} \in \widehat{\mathcal{I}}^n$ satisfying $A \bullet^t \mathcal{V} = \mathcal{X} \bullet^t \mathcal{V}$ is an eigenvector associated with \mathcal{X} . Consider $E_{\mathcal{X}} = \{\mathcal{V} \in \widehat{\mathcal{I}}^n, A \bullet^t \mathcal{V} = \mathcal{X} \bullet^t \mathcal{V}\}$. Then $E_{\mathcal{X}}$ is a \mathbb{R} -subspace of $\widehat{\mathcal{I}}^n$ where n is the order of the matrix A . It is also a $\widehat{\mathcal{I}}$ -submodule of $\widehat{\mathcal{I}}^n$. If \mathcal{X}_1 and \mathcal{X}_2 are two distinguish eigenvalues of A , then $E_{\mathcal{X}_1} \cap E_{\mathcal{X}_2} = \{0\}$.

Proposition 7. *Let $P_A(\mathcal{X})$ be the characteristic polynomial of A . If the real polynomial $P_{C(A)}(X)$ associated with the central matrix of A is a product of factor of degree 1, then $P_A(\mathcal{X})$ admits a factorization on $\widehat{\mathcal{I}}$.*

We have seen that $P_A(\mathcal{X})$ can be have more than degree ($P_A(\mathcal{X})$) roots. If $\mathcal{X}_1, \dots, \mathcal{X}_n$ are the central roots, we have the decomposition

$$P_A(\mathcal{X}) = a_n \prod_{i=1}^n (\mathcal{X} - \mathcal{X}_i).$$

Theorem 1. *For any n -uple of roots $(\mathcal{X}_1, \dots, \mathcal{X}_n)$ such that $P_A(\mathcal{X}) = a_n \prod_{i=1}^n (\mathcal{X} - \mathcal{X}_i)$, and if for any $i = 1, \dots, n$ the dimension of $E_{\mathcal{X}_i}$ coincides with the multiplicity of \mathcal{X}_i , then we have the vectorial decomposition $\widehat{\mathcal{I}}^n = \oplus_{i \in I} E_{\mathcal{X}_i}$ where the roots $\mathcal{X}_i, i \in I$ are pairwise distinguish.*

For example, consider the first central eigenvalue \mathcal{X}_1 of B_2 . Any eigenvector associated with this eigenvalue is written $V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \in \widehat{\mathcal{I}}^2$ with

$$V_2 = - \frac{\left[\frac{1 + \sqrt{5}}{2}, \frac{3 + \sqrt{33}}{2} \right] \bullet V_1}{[1, 2]}.$$

Now we have basis to develop linear calculus, to define linear map, to solve linear systems using intervals. Other approaches have been published in [1] and [6]. Here, we have try to describe a formal framework closer to the real one.

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