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Mathematics of multisets in the Fraenkel-Mostowski framework

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Abstract

We present multisets in the framework of the Fraenkel-Mostowski set theory. We extend the notion of multiset over a finite alphabet by considering the notion of algebraically finitely supported multiset over a possibly infinite alphabet. We study the correspondence between some properties of multisets obtained in the Fraenkel-Mostowski framework (where only finitely supported elements are allowed) and those obtained in the classical Zermelo-Fraenkel framework.

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1 Introduction

The theory of nominal sets was developed initially by Fraenkel and Mostowski in 1930s, and it is also known as the Fraenkel-Mostowski (FM) set theory. At that time, nominal sets were used to prove independence of the axiom of choice and other axioms in the classical Zermelo-Fraenkel (ZF) set theory. In computer science nominal sets offer an elegant formalism for describing λ-terms modulo α-conversion [8], automata on data words [5], or languages over infinite alphabets [4]. The FM axioms are precisely the Zermelo-Fraenkel with atoms (ZFA) axioms over an infinite set A of atoms [8], together with the special axiom of finite support which claims that for each element x in an arbitrary set we can find a finite set of atoms supporting x. Therefore in the FM universe only finitely supported elements are allowed. The Fraenkel-Mostowski set theory over an infinite set of atoms represents the mathematical model for names in syntax. Atoms have the same properties as names in computer science. Most often the precise nature of names is unimportant; what matters is their ability to identify and their distinctness. The finite support axiom is motivated by the fact that syntax can only ever involve finitely many names. The Fraenkel-Mostowski set theory have been used in the last years in algebra [1, 3], in logic [13], in topology [12], or in domain theory [14].

Multisets are used more and more in computer science for quantitative analysis and models of resources. The concept of multiset was introduced in order to capture the idea of multiplicity of appearance, or resource. Multisets are defined by assuming that for a given set Σ an element x occurs a finite number of times. Some examples of multisets are the primes in the prime factorization of a natural number, the invariants of a finite abelian group, or the processes in an operating system. A survey on the applications of multisets in computer science can be found in Section 1 of [2].

The aim of this paper is to present several ZF algebraic properties of multisets, and to translate them into the FM framework. Informally, in the FM framework we are able to replace "multiset over a finite alphabet" by "multiset over a finitely supported, possible infinite alphabet" preserving the basic mathematical properties. Since in a mathematics of the experimental sciences, where we are interested on a quantitative approach, there exists no evidence for the presence of infinite, we try to model the infinite using a more relaxed notion of finite, namely the notion of finite support. Thus the multisets over infinite alphabets are analyzed in terms of finitely supported elements. This paper was announced as a future work in [1]. Note that any ZF set is a particular nominal set equipped with a trivial permutation action (Example 1(2)). Therefore the general properties of nominal sets lead to valid properties of ZF sets. The converse is not always valid, that is, not every ZF result can be directly rephrased in the world of nominal sets, in terms of finitely supported elements according to arbitrary permutation actions. This is because, given a nominal set X, there could exist some subsets of X which fail to be finitely supported. Thus translating the ZF properties of multisets into FM is not trivial, and represents the original part of this paper.

2 Nominal Sets

The notions we present in this section are slightly modified versions of the general properties of nominal sets [8]. Let A be an infinite set of atoms characterized by the axiom "$y \in x \Rightarrow x \notin A$".

Definition 1. *A* transposition *is a function* $(a\,b) : A \to A$ *defined by* $(a\,b)(a) = b$, $(a\,b)(b) = a$, *and* $(a\,b)(n) = n$ *for* $n \neq a, b$. *A* permutation *of* A *is generated by composing finitely many transpositions.*

Let S_A be the set of all permutations of A (i.e. the set of all bijections on A which leave unchanged all but finitely many elements).

Definition 2.

1. *Let* X *be a ZF set. An* S_A-action *on* X *is a function* $\cdot : S_A \times X \to X$ *having the properties that* $Id \cdot x = x$ *and* $\pi \cdot (\pi' \cdot x) = (\pi \circ \pi') \cdot x$ *for all* $\pi, \pi' \in S_A$ *and* $x \in X$. *An* S_A-set *is a pair* $(X, \cdot)$ *where* X *is a ZF set, and* $\cdot : S_A \times X \to X$ *is an* S_A*-action on* X.
2. *Let* $(X, \cdot)$ *be an* S_A*-set. We say that* $S \subset A$ supports x *whenever for each* $\pi \in Fix(S)$ *we have* $\pi \cdot x = x$, *where* $Fix(S) = \{\pi \,|\, \pi(a) = a, \forall a \in S\}$.
3. *Let* $(X, \cdot)$ *be an* S_A*-set. We say that* X *is a* nominal set *if for each* $x \in X$ *there exists a finite set* $S_x \subset A$ *which supports* x.
4. *Let* X *be an* S_A*-set and let* $x \in X$. *If there exists a finite set supporting* x, *then there exists a least finite set supporting* x *[8] which is called* the support of x, *and is denoted by* $supp(x)$.

Proposition 1. *Let $(X, \cdot)$ be an S_A-set and let $\pi \in S_A$. If $x \in X$ is finitely supported, then $\pi \cdot x$ is finitely supported and $supp(\pi \cdot x) = \pi(supp(x))$.*

Example 1.

1. *The set A of atoms is an S_A-set with the S_A-action $\cdot : S_A \times A \to A$ defined by $\pi \cdot a := \pi(a)$ for all $\pi \in S_A$ and $a \in A$.*

2. *Any ordinary ZF set X (like $\mathbb{N}$ or $\mathbb{Z}$) is an S_A-set with the trivial S_A-action $\cdot : S_A \times X \to X$ defined by $\pi \cdot x := x$ for all $\pi \in S_A$ and $x \in X$.*

3. *If $(X, \cdot)$ is an S_A-set, then $\wp(X) = \{Y \,|\, Y \subseteq X\}$ is also an S_A-set with the S_A-action $\star : S_A \times \wp(X) \to \wp(X)$ defined by $\pi \star Y := \{\pi \cdot y \,|\, y \in Y\}$ for all π of A, and all subsets Y of X. For each nominal set $(X, \cdot)$ we denote by $\wp_{fs}(X)$ the set formed from those subsets of X which are finitely supported according to the action $\star$. $(\wp_{fs}(X), \star|_{\wp_{fs}(X)})$ is a nominal set, where $\star|_{\wp_{fs}(X)}$ represents the action $\star$ restricted to $\wp_{fs}(X)$.*

4. *Let $(X, \cdot)$ and $(Y, \diamond)$ be S_A-sets. The Cartesian product $X \times Y$ is also an S_A-set with the S_A-action $\star : S_A \times (X \times Y) \to (X \times Y)$ defined by $\pi \star (x, y) = (\pi \cdot x, \pi \diamond y)$ for all $\pi \in S_A$ and all $x \in X$, $y \in Y$. If $(X, \cdot)$ and $(Y, \diamond)$ are nominal sets, then $(X \times Y, \star)$ is also a nominal set.*

Definition 3. *Let $(X, \cdot)$ be a nominal set. A subset Z of X is called* finitely supported *if and only if $Z \in \wp_{fs}(X)$ with the notations in Example 1 (3).*

Since functions are particular relations we can present the following results. For more details see Section 2 from [1]:

Definition 4. *Let X and Y be nominal sets. A function $f : X \to Y$ is* finitely supported *if $f \in \wp_{fs}(X \times Y)$.*

Let $Y^X = \{f \subseteq X \times Y \mid f$ is a function from the underlying set of X to the underlying set of $Y\}$.

Proposition 2. *Let $(X, \cdot)$ and $(Y, \diamond)$ be nominal sets. Then Y^X is an S_A-set with the S_A-action $\star : S_A \times Y^X \to Y^X$ defined by $(\pi \star f)(x) = \pi \diamond (f(\pi^{-1} \cdot x))$ for all $\pi \in S_A$, $f \in Y^X$ and $x \in X$. A function $f : X \to Y$ is finitely supported in the sense of Definition 4 if and only if it is finitely supported with respect the permutation action $\star$.*

Proposition 3. *Let $(X, \cdot)$ and $(Y, \diamond)$ be nominal sets. Let $f \in Y^X$ and $\pi \in S_A$ be arbitrary elements. Let $\star : S_A \times Y^X \to Y^X$ be the S_A-action on Y^X defined by: $(\pi \star f)(x) = \pi \diamond (f(\pi^{-1} \cdot x))$ for all $\pi \in S_A$, $f \in Y^X$ and $x \in X$. Then $\pi \star f = f$ if and only if for all $x \in X$ we have $f(\pi \cdot x) = \pi \diamond f(x)$.*

3 Algebraic Properties of Multisets

Definition 5. *Given a finite alphabet Σ, any function $f : \Sigma \to \mathbb{N}$ is called* multiset over Σ. *The value of $f(a)$ is said to be the* multiplicity of a. *The set of all multisets over Σ is denoted by $\mathbb{N}(\Sigma)$.*

The additive structure of $\mathbb{N}$ induces an additive operation (sum) on multisets. On $\mathbb{N}(\Sigma)$ we define an additive law "$+$" $: \mathbb{N}(\Sigma) \times \mathbb{N}(\Sigma) \to \mathbb{N}(\Sigma)$ by $(f, g) \mapsto f + g$, where $f + g : \Sigma \to \mathbb{N}$ is defined pointwise by $(f + g)(a) = f(a) + g(a)$ for all $a \in \Sigma$. Since $\mathbb{N}(\Sigma)$ is formed by all functions from Σ to $\mathbb{N}$, it is clear that $(\mathbb{N}(\Sigma), +)$ is an abelian monoid, the identity being the empty multiset $\theta : \Sigma \to \mathbb{N}$, $\theta(a) = 0$ for all $a \in \Sigma$. Since $(\mathbb{N}(\Sigma), +)$ is an abelian monoid, it follows that $(\mathbb{N}(\Sigma), +)$ is an $\mathbb{N}$-semimodule i.e. a semimodule over the semiring $\mathbb{N}$, with the scalar multiplication "$\cdot$" $: \mathbb{N} \times \mathbb{N}(\Sigma) \to \mathbb{N}(\Sigma)$ defined by $(n, f) \mapsto n \cdot f$, where $n \cdot f : \Sigma \to \mathbb{N}$ is defined pointwise by $(n \cdot f)(a) = n \cdot f(a)$, for all $a \in \Sigma$ and $n \in \mathbb{N}$.

Proposition 4. $(\mathbb{N}(\Sigma), +)$ *is a free abelian* $\mathbb{N}$*-semimodule.*

Proof: If $a \in \Sigma$, let us consider the multiset $\widetilde{a} : \Sigma \to \mathbb{N}$ defined by $\widetilde{a}(b) = \begin{cases} 1 & \text{for } b = a \\ 0 & \text{for } b \in \Sigma \setminus \{a\} \end{cases}$. It is easy to check that every multiset $f \in \mathbb{N}(\Sigma)$ can be uniquely expressed as $f = \sum_{a \in \Sigma} f(a) \cdot \widetilde{a}$. Since Σ is finite, the sum is finite. □

Since $\mathbb{N}(\Sigma)$ is the free $\mathbb{N}$-semimodule with basis $\widetilde{\Sigma}$, it satisfies the universality property described in Proposition 5. We denote by $j : \Sigma \to \mathbb{N}(\Sigma)$ the function which maps each $a \in \Sigma$ into $\widetilde{a} \in \widetilde{\Sigma}$.

Proposition 5. *If M is any abelian monoid and $f : \Sigma \to M$ is an arbitrary function, then there exists a unique homomorphism of abelian monoids $g : \mathbb{N}(\Sigma) \to M$ with $g \circ j = f$, i.e. $g(\widetilde{a}) = f(a)$ for all $a \in \Sigma$.*

Definition 6. *Adjoin one element to Σ and denote it by 1. A* word on Σ *is either the element 1 or a formal expression $x_1 x_2 \dots x_n$ where $n \in \mathbb{N}$, $x_i \in \Sigma$. The* juxtaposition *of words $w = x_1 x_2 \dots x_n$ and $w' = y_1 y_2 \dots y_m$ is the word $w \# w' := x_1 x_2 \dots x_n y_1 y_2 \dots y_m$. Moreover, we define $w \# 1 = 1 \# w = w$ for all words w. The* free monoid Σ^* *is the set of words on Σ with the internal law $\#$.*

The free monoid on Σ also satisfies the so-called universality property:

Theorem 1. *For each monoid M and each function $f : \Sigma \to M$, there exists a unique homomorphism of monoids $g : \Sigma^* \to M$ with $g \circ i = f$, where $i : \Sigma \to \Sigma^*$ is the standard inclusion of Σ into Σ^* which maps each element $a \in \Sigma$ into the word a.*

We can compare multisets with vectors of natural numbers. If $\Sigma = \{a_1, \dots, a_k\}$, then $\mathbb{N}(\Sigma) \cong \mathbb{N}^k$ as $\mathbb{N}$-semimodules and hence as abelian monoids. We can connect all these views using the universal property of the free monoid Σ^*. If we replace in the statement of Theorem 1, M with $\mathbb{N}(\Sigma)$, and $f : \Sigma \to M$ with $j : \Sigma \to \mathbb{N}(\Sigma)$ where j maps each a into $\widetilde{a}$, we get a function $g : \Sigma^* \to \mathbb{N}(\Sigma)$ such that $g \circ i = j$, where $i : \Sigma \to \Sigma^*$ is the standard inclusion of Σ into Σ^* which maps each element $a \in \Sigma$ into the word a. If $w = x_1 x_2 \cdots x_m$, $x_i \in \Sigma$, $i = 1, \dots, m$, then $g(w) = j(x_1) + j(x_2) + \dots + j(x_m)$.

If for any alphabet $\{a_1, \dots, a_k\}$, $k \in \mathbb{N}$ we denote by $|w|_{a_i}$ the number of appearances of the symbol a_i in w, $g(w) = \sum_{i=1}^{k} |w|_{a_i} \cdot \widetilde{a}_i$ is a surjective morphism. From the isomorphism theorem for monoids we get $\Sigma^*/Ker\, g \cong \mathbb{N}(\Sigma)$. Note that $Ker\, g$ is a congruence on Σ^*. Two words w

and w' are in the same equivalence class with respect to $Ker\, g$ if and only if $g(w) = g(w')$; this is equivalent with $|w|_{a_i} = |w'|_{a_i}$ for all $i = 1, \ldots, k$.

If $\Sigma = \{a_1, \ldots, a_k\}$, we define the Parikh image [11] $\varphi_\Sigma : \Sigma^* \to \mathbb{N}^k$ in the following way: if $w = x_1 x_2 \ldots x_n$, then $\varphi_\Sigma(w)$ is the vector in $\mathbb{N}^k$ whose i-component is $\sum_{\substack{x_j = a_i \\ j=\overline{1,n}}} 1$ for each $i = \overline{1,k}$; if there is no j such that $x_j = a_i$ the the i-th component of the vector is defined to be 0. Informally $\varphi_\Sigma(w)$ calculates the number of "occurrences" of each element from Σ in w.

If we replace, in the statement of Proposition 5, M with $\mathbb{N}^k$ and $f : \Sigma \to M$ with the function $\varphi_\Sigma \circ i$ where where $i : \Sigma \to \Sigma^*$ is the standard inclusion of Σ into Σ^* which maps each element $a_u \in \Sigma$ into the word a_u, then there exists a unique homomorphism of abelian monoids $\psi_\Sigma : \mathbb{N}(\Sigma) \to \mathbb{N}^k$ with $\psi_\Sigma \circ j = \varphi_\Sigma \circ i$, that is $\psi_\Sigma(\widetilde{a_u}) = \varphi_\Sigma(a_u) = (0, \ldots, 0, 1, 0, \ldots, 0) = e_u$ for all $a_u \in \Sigma$, where $e_u = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the vector in $\mathbb{N}^k$ whose all components are 0 except the u-th component which is 1.

Now, because ψ_Σ maps one-to-one each element from a finite basis of $\mathbb{N}(\Sigma)$ into an element from a finite basis of $\mathbb{N}^k$, and $\mathbb{N}(\Sigma)$ and $\mathbb{N}^k$ have the same rank, we have that $\psi_\Sigma : \mathbb{N}(\Sigma) \to \mathbb{N}^k$ is an isomorphism, and $\psi_\Sigma(\sum_{i=1}^{k} f(a_i) \cdot \widetilde{a}_i) = (f(a_1), \ldots, f(a_k))$, for each $f \in \mathbb{N}(\Sigma)$. Moreover, the properties of commutative diagrams shows us that $\psi_\Sigma \circ g = \varphi_\Sigma$ where $g : \Sigma^* \to \mathbb{N}(\Sigma)$ is the homomorphism built before such that $g \circ i = j$.

Several orders can be defined on the set $\mathbb{N}(\Sigma)$. The most common is the order obtained from the definition of the subset-hood in the case of multisets.

Definition 7. *A multiset $f : \Sigma \to \mathbb{N}$ is a subset of a multiset $g : \Sigma \to \mathbb{N}$ (written $f \subseteq g$) if and only if $f(x) \leq g(x)$ for all $x \in \Sigma$.*

Clearly $(\mathbb{N}(\Sigma), \leq)$ is a partially ordered set. Another multiset order is Dershowitz and Manna order [7] which is a main tool of many orders used to prove the finite termination of programs and also of term rewriting systems [6].

Definition 8. *Let us suppose there exists a strict order $\prec$ on Σ. We define the Dershowitz and Manna (DM) strict order $\ll_{DM}$ on $\mathbb{N}(\Sigma)$ by $f \ll_{DM} g$ if and only if there exists $h, k \in \mathbb{N}(\Sigma)$ with the following properties:*

1. *$\theta \neq h \subseteq g$;*

2. *$f = (g - h) + k$;*

3. *for all $y \in \Sigma$ with $k(y) > 0$, there exists $x \in \Sigma$ with $h(x) > 0$ and $y \prec x$.*

The DM definition is difficult to use in order to prove that two multisets are not related by an inclusion. An equivalent definition is presented in [10].

Definition 9. *Let us suppose there exists a strict order $\prec$ on Σ. We define the Huet and Oppen strict order $\ll_{HO}$ on $\mathbb{N}(\Sigma)$ by $f \ll_{HO} g$ if and only if the following properties are satisfied:*

1. *$f \neq g$;*

2. *for all $y \in \Sigma$ with $f(y) > g(y)$, there exists $x \in \Sigma$ with $y \prec x$, $f(x) < g(x)$.*

Theorem 2 ([10]). *The orderings $\ll_{DM}$ and $\ll_{HO}$ are equivalent.*

Since $\ll_{DM}$ and $\ll_{HO}$ are equivalent, we denote these order relations by $\ll$.

4 Multisets over Nominal Sets

We formalize now the concept of multisets in the FM universe. Such a work have already been announced in [1]. According to Example 1 (2), we know that $\mathbb{N}$ is an S_A-set with the S_A-action $\cdot : S_A \times \mathbb{N} \to \mathbb{N}$ defined by $\pi \cdot x := x$ for all $\pi \in S_A$ and $x \in \mathbb{N}$. Also, $\mathbb{N}$ is a nominal set because for each $x \in \mathbb{N}$ we have that $\emptyset$ supports x. Moreover, $supp(x) = \emptyset$ for each $x \in \mathbb{N}$.

Proposition 6. *Let $(\Sigma, \cdot)$ be a nominal set (possible infinite), and $f : \Sigma \to \mathbb{N}$ a function such that its algebraic support $S_f \overset{def}{=} \{x \in \Sigma \,|\, f(x) \neq 0\}$ is finite. Then f is finitely supported and $supp(f) \subseteq supp(S_f)$.*

Proof: A function $f : \Sigma \to \mathbb{N}$ is finitely supported in the sense of Definition 4 if and only if it is finitely supported with respect the permutation action $\star$ defined in Proposition 2. However on $\mathbb{N}$ we have defined the trivial action $(\pi, x) \mapsto x$. Therefore the S_A-action $\star$ on $\mathbb{N}^\Sigma$ is given by $(\pi \star f)(x) = f(\pi^{-1} \cdot x)$ for all $\pi \in S_A$, $f \in \mathbb{N}^\Sigma$ and $x \in \Sigma$. Let $f : \Sigma \to \mathbb{N}$ be a algebraically finitely supported function. Let $S_f = \{a_1, \ldots, a_k\}$, and $S = supp(a_1) \cup \ldots \cup supp(a_k)$. We have to prove that S supports f. Let $\pi \in Fix(S)$. We have that $\pi \in Fix(supp(a_i))$ for each $i \in \{1, \ldots, k\}$. Therefore $\pi \cdot a_i = a_i$ for each $i \in \{1, \ldots, k\}$ because $supp(a_i)$ supports a_i for each $i \in \{1, \ldots, k\}$. However $f = \sum\limits_{a \in \Sigma} f(a) \cdot \widetilde{a} = \sum\limits_{a \in S_f} f(a) \cdot \widetilde{a}$. A simple calculus show us that $\pi \star f = \sum\limits_{a \in S_f} f(a) \cdot (\widetilde{\pi \cdot a})$. Therefore $\pi \star f = f$ for each $\pi \in Fix(S)$. □

Since $\mathbb{N}$ is a trivial nominal set with the same S_A-action as $\mathbb{Z}$, the following proposition can be proved as in Remark 3.5 from [1], by replacing $\mathbb{Z}$ with $\mathbb{N}$.

Proposition 7. *If $f : A \to \mathbb{N}$ such that S_f is finite, then $S_f = supp(f)$.*

According to Proposition 7, the notion of algebraic support of a multiset represents an extension of the notion of nominal support. In the nominal framework, the multisets over finite alphabets can be replaced by the algebraically finitely supported multisets over possible infinite alphabets.

Definition 10. *Given a nominal set $(\Sigma, \cdot)$ (possible infinite), any function $f : \Sigma \to \mathbb{N}$ with the property that S_f is finite is called* extended multiset over Σ. *The set of all extended multisets over Σ is denoted by $\mathbb{N}_{ext}(\Sigma)$.*

We remark that each function $f \in \mathbb{N}_{ext}(\Sigma)$ can be expressed as $f = \sum\limits_{a \in \Sigma} f(a) \cdot \widetilde{a}$. Since S_f is finite, the previous sum is finite. Therefore $\mathbb{N}_{ext}(\Sigma)$ is a free abelian monoid with basis Σ. According to the proof of Theorem 4 every $\widetilde{a}$ is finitely supported, and so, the the expression of f makes sense in FM. Whenever Σ is finite we have $\mathbb{N}_{ext}(\Sigma) = \mathbb{N}(\Sigma)$.

Definition 11. *A* nominal monoid *is a triple* $(M, +, \cdot)$ *such that the following conditions are satisfied:*

- $(M, +, 0)$ *is a monoid;*
- $(M, \cdot)$ *is a nominal set;*
- *for each* $\pi \in S_A$ *and* $x, y \in M$ *we have* $\pi \cdot (x + y) = \pi \cdot x + \pi \cdot y$ *and* $\pi \cdot 0 = 0$.

The relation requested on item 3 from Definition 11 is justified by the fact that in the FM universe only finitely supported elements are allowed. According to Proposition 3, this relation is equivalent to the equivariance of $+$.

According to Proposition 4 we know that $(\mathbb{N}(\Sigma), +)$ is a free abelian monoid if we work in the standard ZF set theory. Analogue $(\mathbb{N}_{ext}(\Sigma), +)$ is a free abelian monoid. In the FM framework we have the following result:

Theorem 3. $\mathbb{N}_{ext}(\Sigma)$ *is a free abelian nominal monoid whenever* $(\Sigma, \cdot)$ *is a nominal set.*

Proof: We already know that $(\mathbb{N}_{ext}(\Sigma), +)$ is a free abelian monoid. Also $\pi \star f \in \mathbb{N}_{ext}(\Sigma)$ for all $f \in \mathbb{N}_{ext}(\Sigma)$, where $\star$ is the standard S_A-action on $\mathbb{N}^{\Sigma}$. According to Proposition 6 we have that $(\mathbb{N}_{ext}(\Sigma), \star)$ is a nominal set with the S_A-action $\star\colon S_A \times \mathbb{N}_{ext}(\Sigma) \to \mathbb{N}_{ext}(\Sigma)$ defined by $(\pi \star f)(x) = f(\pi^{-1} \cdot x)$ for all $\pi \in S_A$, $f \in \mathbb{N}_{ext}(\Sigma)$ and $x \in \Sigma$. Let $f, g \in \mathbb{N}_{ext}(\Sigma)$. For each $x \in \Sigma$ we have $(\pi \star (f+g))(x) = (f+g)(\pi^{-1} \cdot x) = f(\pi^{-1} \cdot x) + g(\pi^{-1} \cdot x) = (\pi \star f)(x) + (\pi \star g)(x) = ((\pi \star f) + (\pi \star g))(x)$. Hence $\pi \star (f + g) = \pi \star f + \pi \star g$. Also, $\pi \star \theta = \theta$, where θ is the empty multiset. □

For nominal monoids we also have an universality property which is similar to Proposition 5 in the FM framework.

Theorem 4. *Let* $(\Sigma, \cdot)$ *be a nominal set. Let* $j\colon \Sigma \to \mathbb{N}_{ext}(\Sigma)$ *be the function which maps each* $a \in \Sigma$ *into* $\widetilde{a} \in \widetilde{\Sigma}$. *If* $(M, +, \diamond)$ *is an arbitrary abelian nominal monoid and* $\varphi\colon \Sigma \to M$ *is an arbitrary finitely supported function, then there exists a unique finitely supported homomorphism of abelian monoids* $\psi\colon \mathbb{N}_{ext}(\Sigma) \to M$ *with* $\psi \circ j = \varphi$, *i.e.* $\psi(\widetilde{a}) = \varphi(a)$ *for all* $a \in \Sigma$. *Moreover, if a finite set* S *supports* φ, *then the same set* S *supports* ψ. *Therefore, if* φ *is equivariant, then* ψ *is also equivariant.*

Proof: First we show that the statement of the theorem is well written in FM. We know that in the FM universe only elements with finite support are allowed. For this, we have to prove that j is finitely supported. According to Proposition 2 we know that $\mathbb{N}_{ext}(\Sigma)^{\Sigma}$ is an S_A-set with the S_A-action $\triangleright\colon S_A \times \mathbb{N}_{ext}(\Sigma)^{\Sigma} \to \mathbb{N}_{ext}(\Sigma)^{\Sigma}$ defined by $(\pi \triangleright f)(x) = \pi \star (f(\pi^{-1} \cdot x))$ for all $\pi \in S_A$, $f \in \mathbb{N}_{ext}(\Sigma)^{\Sigma}$ and $x \in \Sigma$ (the S_A-action on $\mathbb{N}_{ext}(\Sigma)$ is denoted by $\star$). A function $f\colon \Sigma \to \mathbb{N}_{ext}(\Sigma)$ is finitely supported in the sense of Definition 4 if and only if it is finitely supported with respect the permutation action $\triangleright$. We prove that j is equivariant. In the view of Proposition 3 we must prove that $j(\pi \cdot x) = \pi \star j(x)$ for each $\pi \in S_A$ and each $x \in \Sigma$. Let $\pi \in S_A$ and $x \in \Sigma$ be arbitrary elements. For each $y \in \Sigma$ we have (by the definition of j) that $(j(\pi \cdot x))(y) = \begin{cases} 1 & \text{for } \pi \cdot x = y \\ 0 & \text{for } \pi \cdot x \neq y \end{cases}$.

Also $(\pi \star j(x))(y) = j(x)(\pi^{-1} \cdot y) = \begin{cases} 1 & \text{for } x = \pi^{-1} \cdot y \\ 0 & \text{for } x \neq \pi^{-1} \cdot y \end{cases} = \begin{cases} 1 & \text{for } \pi \cdot x = y \\ 0 & \text{for } \pi \cdot x \neq y \end{cases}$. Hence $j(\pi \cdot x) = \pi \star j(x)$ for each $\pi \in S_A$ and each $x \in \Sigma$ which means that j has empty support.

The homomorphism ψ is defined in the same way as the homomorphism g in Proposition 5. As in the standard theory of abelian monoids (or free $\mathbb{N}$-semimodules) the homomorphism ψ is defined by $\psi(f) = \sum_{a \in S_f} f(a) \cdot \varphi(a)$ whenever $f = \sum_{a \in S_f} f(a) \cdot \widetilde{a}$. We must prove only that ψ is finitely supported because the other properties of ψ requested in the statement of the theorem have standard proofs as in the classical monoids theory. According to Proposition 2 we know that $M^{\mathbb{N}_{ext}(\Sigma)}$ is an S_A-set with the S_A-action $\bullet$: $S_A \times M^{\mathbb{N}_{ext}(\Sigma)} \to M^{\mathbb{N}_{ext}(\Sigma)}$ defined by $(\pi \bullet \phi)(g) = \pi \diamond (\phi(\pi^{-1} \star g))$ for all $\pi \in S_A$, $\phi \in M^{\mathbb{N}_{ext}(\Sigma)}$ and $g \in \mathbb{N}_{ext}(\Sigma)$. We prove that $S = supp(\varphi)$ supports ψ. Let $\pi \in Fix(S)$. In the view of Proposition 3, for proving that $\pi \bullet \psi = \psi$ it is enough to prove that $\psi(\pi \star g) = \pi \diamond \psi(g)$ for each $g \in \mathbb{N}_{ext}(\Sigma)$. Let $f \in \mathbb{N}_{ext}(\Sigma)$ be an arbitrary element. Then $f = \sum_{a \in S_f} f(a) \cdot \widetilde{a}$. Since j is equivariant and ψ is a monoid homomorphism we obtain $\psi(\pi \star f) = \psi(\sum_{a \in S_f} f(a) \cdot (\pi \star \widetilde{a})) = \psi(\sum_{a \in S_f} f(a) \cdot (\widetilde{\pi \cdot a})) = \sum_{a \in S_f} f(a) \cdot \psi(\widetilde{\pi \cdot a}) = \sum_{a \in S_f} f(a) \cdot \varphi(\pi \cdot a)$. Also $\pi \diamond \psi(f) = \pi \diamond (\sum_{a \in S_f} f(a) \cdot \varphi(a)) = \sum_{a \in S_f} f(a) \cdot (\pi \diamond \varphi(a)) = \sum_{a \in S_f} f(a) \cdot \varphi(\pi \cdot a)$; the second equality follows because $(M, +, \diamond)$ is a nominal monoid and the third because π fixes $supp(\varphi)$ pointwise. Therefore S supports ψ. ☐

In the previous subsection we established a connection between $\mathbb{N}(\Sigma)$ and the free monoid Σ^*. Our aim is to prove that the results obtained in the previous subsection in the ZF framework can also be proved in the FM framework. Analogue as in Proposition 3.6 of [1], we can prove the following result.

Theorem 5. *Σ^* is a nominal monoid whenever $(\Sigma, \cdot)$ is a (possible infinite) nominal set. The the S_A-action $\widetilde{\star} : S_A \times \Sigma^* \to \Sigma^*$ is defined by $\pi \widetilde{\star} x_1 x_2 \ldots x_l = (\pi \cdot x_1) \ldots (\pi \cdot x_l)$ for all $\pi \in S_A$ and $x_1 x_2 \ldots x_l \in \Sigma^* \setminus \{1\}$, and $\pi \widetilde{\star} 1 = 1$ for all $\pi \in S_A$.*

Theorem 1 which represents the universality property for Σ^* in the ZF framework has a similar result in FM. This result can be proved analogue as Theorem 3.7 from [1] by replacing "free group" with "free monoid".

Theorem 6. *Let $(\Sigma, \diamond)$ be a (possible infinite) nominal set. Let $i : \Sigma \to \Sigma^*$ be the standard inclusion of Σ into Σ^* which maps each element $a \in \Sigma$ into the word a. If $(M, \cdot, \diamond)$ is an arbitrary nominal monoid and $\varphi : \Sigma \to M$ is an arbitrary finitely supported function, then there exists a unique finitely supported homomorphism of monoids $\psi : \Sigma^* \to M$ with $\psi \circ i = \varphi$. Moreover, if a finite set S supports φ, then the same set S supports ψ. Therefore, if φ is equivariant, then ψ is also equivariant.*

Several results obtained in the previous subsection (in the ZF framework) can be translated to the FM framework. If we replace in the statement of Theorem 6, M with $\mathbb{N}_{ext}(\Sigma)$, and $\varphi : \Sigma \to M$ with $j : \Sigma \to \mathbb{N}_{ext}(\Sigma)$ where j maps each a into $\widetilde{a}$, we get an *equivariant* monoid homomorphism $\psi : \Sigma^* \to \mathbb{N}_{ext}(\Sigma)$ such that $\psi \circ i = j$, where $i : \Sigma \to \Sigma^*$ is the standard inclusion of Σ into Σ^* which maps each element $a \in \Sigma$ into the word a. Now if $w = x_1 x_2 \ldots x_n$

then we obtain that $\psi(w) = j(x_1) + j(x_2) + \ldots + j(x_n)$. Now, clearly, ψ is surjective and, from the first isomorphism theorem for monoids we have $\Sigma^*/Ker\,\psi \cong \mathbb{N}_{ext}(\Sigma)$. Moreover, in FM we have the following result:

Proposition 8. *$\Sigma^*/Ker\,\psi$ is a nominal monoid and the isomorphism Θ between the nominal monoids $\Sigma^*/Ker\,\psi$ and $\mathbb{N}_{ext}(\Sigma)$, defined by $\Theta([w]) = \psi(w)$ for each $w \in \Sigma^*$ (where $[w]$ is the equivalence class of w modulo the equivalence relation $Ker\,\psi$) is equivariant.*

Proof: We remark that Θ is defined as in the standard proof of the first isomorphism theorem for monoids. First we prove that we can define a nominal structure on $\Sigma^*/Ker\,\psi$. We know that $(\Sigma^*, \widetilde{\star})$ is a nominal set (Theorem 5). We define $\odot : S_A \times \Sigma^*/Ker\,\psi \to \Sigma^*/Ker\,\psi$ by $\pi \odot [w] = [\pi \widetilde{\star} w]$ for each $w \in \Sigma^*$ and each $\pi \in S_A$. First we show that $\odot$ is a well defined function. Let $w = x_1 x_2 \ldots x_n$ and $v = y_1 y_2 \ldots y_m$ be two elements in Σ^* such that $[w] = [v]$. This means $\psi(w) = \psi(v)$ which by the definition of ψ is the same with $j(x_1) + j(x_2) + \ldots + j(x_n) = j(y_1) + j(y_2) + \ldots + j(y_m)$. Now we have $\pi \star (j(x_1) + j(x_2) + \ldots + j(x_n)) = \pi \star (j(y_1) + j(y_2) + \ldots + j(y_m))$ for each $\pi \in S_A$ (where $\star$ represents the S_A-action on $\mathbb{N}_{ext}(\Sigma)$). Since $\mathbb{N}_{ext}(\Sigma)$ is a nominal monoid and because j is *equivariant* (see the proof of Theorem 4), in the view of Proposition 3 we have $j(\pi \cdot x_1) + j(\pi \cdot x_2) + \ldots + j(\pi \cdot x_n) = j(\pi \cdot y_1) + j(\pi \cdot y_2) + \ldots + j(\pi \cdot y_m)$ which means $\psi(\pi \widetilde{\star} w) = \psi(\pi \widetilde{\star} v)$ for each $\pi \in S_A$. Therefore $[\pi \widetilde{\star} w] = [\pi \widetilde{\star} v]$ for each $\pi \in S_A$ which means that $\odot$ is well defined. Since $\widetilde{\star}$ is an S_A-action on Σ^*, an easy calculation shows us that $\odot$ is an S_A-action on $\Sigma^*/Ker\,\psi$. Moreover, each element in $\Sigma^*/Ker\,\psi$ is finitely supported by the support of its representative. Therefore $(\Sigma^*/Ker\,\psi, \odot)$ is a nominal set. Since $(\Sigma^*, \#, \widetilde{\star})$ is a nominal monoid (the axioms in Definition 11 are satisfied) it is trivial to check that $(\Sigma^*/Ker\,\psi, \#, \odot)$ (we denote also by # the internal law on the factor monoid $\Sigma^*/Ker\,\psi$) is also a nominal monoid; the proof is an easy calculation which uses only the definition on $\odot$ and the distributivity property of $\widetilde{\star}$ over #.

We claim that Θ is equivariant. For this, in the view of Proposition 3, it is sufficient to prove that for each $\pi \in S_A$ we have $\Theta(\pi \odot [w]) = \pi \star (\Theta[w])$, $\forall w \in \Sigma^*$. Let $\pi \in S_A$ be an arbitrary element. Since ψ is equivariant we have $\Theta(\pi \odot [w]) = \Theta([\pi \widetilde{\star} w]) = \psi(\pi \widetilde{\star} w) = \pi \star \psi(w) = \pi \star (\Theta([w])$. This means Θ is equivariant. □

If $\Sigma = \{a_1, \ldots, a_k\}$, the Parikh image $\varphi_\Sigma : \Sigma^* \to \mathbb{N}^k$ is finitely supported. Indeed, $\mathbb{N}$ is an S_A-set with the S_A-action $\cdot : S_A \times \mathbb{N} \to \mathbb{N}$ defined by $\pi \cdot x := x$ for all $\pi \in S_A$ and $x \in \mathbb{N}$. From Example 1 (4) we know how an S_A-action on the Cartesian product of two nominal sets looks like. Therefore $\mathbb{N}^k$ is endowed with a trivial S_A-action defined by $\pi \cdot x := x$ for all $\pi \in S_A$ and $x \in \mathbb{N}^k$. Also $\mathbb{N}^k$ is a nominal set because for each $x \in \mathbb{N}^k$ we have that $\emptyset$ supports x. Moreover, $supp(x) = \emptyset$ for each $x \in \mathbb{N}^k$. According to Proposition 3 and Theorem 5, by easy calculation we obtain that $supp(a_1) \cup \ldots \cup supp(a_k)$ supports φ_Σ. If we replace, in the statement of Theorem 4, M with $\mathbb{N}^k$ and $\varphi : \Sigma \to M$ with the function $\varphi_\Sigma \circ i$ where $i : \Sigma \to \Sigma^*$ is the standard inclusion of Σ into Σ^*, then there exists a unique *finitely supported* homomorphism of abelian monoids $\psi_\Sigma : \mathbb{N}(\Sigma) \to \mathbb{N}^k$ with $\psi_\Sigma \circ j = \varphi_\Sigma \circ i$.

According to [14], a nominal partially ordered set $(E, \sqsubseteq)$ is a nominal set E together with a partial order relation $\sqsubseteq$ which is equivariant as a subset of $E \times E$ in the sense of Definition 3. Similarly nominal strictly ordered set $(E, <)$ is a nominal set E together with a strict (partial or total) order relation $<$ which is equivariant as a subset of $E \times E$ in the sense of Definition 3.

Proposition 9. *If $(\Sigma, \cdot)$ is a nominal set, then $(\mathbb{N}_{ext}(\Sigma), \star, \subseteq)$ is a nominal partially ordered set.*

Proof: According to Proposition 6 we have that $(\mathbb{N}_{ext}(\Sigma), \star)$ is a nominal set with the S_A-action $\star : S_A \times \mathbb{N}_{ext}(\Sigma) \to \mathbb{N}_{ext}(\Sigma)$ defined by $(\pi \star f)(x) = f(\pi^{-1} \cdot x)$ for all $\pi \in S_A$, $f \in \mathbb{N}_{ext}(\Sigma)$ and $x \in \Sigma$. Let $f, g \in \mathbb{N}_{ext}(\Sigma)$ such that $f \subseteq g$. This means $f(x) \leq g(x)$ for all $x \in \Sigma$. We should prove that $\pi \star f \subseteq \pi \star g$. Let $x \in \Sigma$. We have $(\pi \star f)(x) = f(\pi^{-1} \cdot x) \leq g(\pi^{-1} \cdot x) = (\pi \star g)(x)$. Therefore $\subseteq$ is equivariant. □

In [1] we presented an embedding theorem of Cayley-type valid for a particular class of nominal groups, namely the uniform nominal groups. In this paper we are able to prove an embedding theorem of Cayley-type that works for all nominal monoids and not only for those which are uniformly supported. This theorem leads to an interesting property of extended multisets, namely Corollary 1.

Definition 12. *Let $(M, +, \cdot)$ be a nominal monoid. A submonoid M' of M is called a* nominal submonoid *of M if $(M', \cdot|_{M'})$ is a nominal set, that is, $\pi \cdot m' \in M'$ for all $m' \in M'$ and all $\pi \in S_A$.*

Lemma 1. *Let $(X, \cdot)$ be a nominal set. The finitely supported elements from X^X form a nominal monoid.*

Proof: Since X is a nominal set, we have that X^X is an S_A-set with the S_A-action $\star$ defined as in Proposition 2. According to Proposition 1, the finitely supported elements from X^X form a nominal set. It is clear that $(X^X, \circ)$ is a monoid (where $\circ$ represent the usual composition of functions). It remains to prove that for each $f, g \in X^X$ we have $\pi \star (f \circ g) = (\pi \star f) \circ (\pi \star g)$. Indeed, for each $x \in X$, we have $(\pi \star (f \circ g))(x) = \pi \cdot (f(g(\pi^{-1} \cdot x)))$. Also, if we denote $(\pi \star g)(x) = y$ we have $y = \pi \cdot (g(\pi^{-1} \cdot x))$ and $((\pi \star f) \circ (\pi \star g))(x) = (\pi \star f)(y) = \pi \cdot (f(\pi^{-1} \cdot y)) = \pi \cdot (f((\pi^{-1} \circ \pi) \cdot g(\pi^{-1} \cdot x))) = \pi \cdot (f(g(\pi^{-1} \cdot x)))$. Therefore $\pi \star (f \circ g) = (\pi \star f) \circ (\pi \star g)$, and X^X is a nominal monoid. □

Theorem 7 (Cayley-theorem for nominal monoids)**.** *Let $(X, +, \cdot)$ be a nominal monoid. There exists an equivariant isomorphism between X and a nominal submonoid of the nominal monoid formed by the finitely supported elements from X^X.*

Proof: For each $x \in X$ we consider the function $f_x : X \to X$ defined by $f_x(y) = x + y$. Let $M = \{f_x \,|\, x \in X\}$. According to Proposition 3 and because $+$ is equivariant we have that each element of form f_x is supported by *supp*(x). Therefore each element in M is finitely supported, and M is a submonoid of X^X. Moreover, we have that M is a nominal submonoid of X^X. Indeed, if $m \in M$ we have that $m = f_x$ for some $x \in X$. Let $\pi \in S_A$. We have $(\pi \star m)(y) = \pi \cdot (f_x(\pi^{-1} \cdot y)) = \pi \cdot (x + (\pi^{-1} \cdot y)) = (\pi \cdot x) + y = f_{\pi \cdot x}(y)$. Therefore $\pi \star m \in M$ for all $\pi \in S_A$ and all $m \in M$, and M is a a nominal submonoid of X^X.

Let $T : X \to X^X$ be the function defined by $T(x) = f_x$ for each $x \in X$. As in the standard proof of Cayley's theorem for monoids, it can be proved (by direct calculation) that T is an injective monoid homomorphism whose image is M. It remains to prove that T is equivariant. It is sufficient to prove that $T(\pi \cdot x) = \pi \star T(x)$ for each $x \in X$ and $\pi \in S_A$, where by $\star$ we denoted the S_A-action on X^X. However, according to the paragraph above, we have $T(\pi \cdot x) = f_{\pi \cdot x}$, $\pi \star T(x) = \pi \star f_x$, and $\pi \star f_x = f_{\pi \cdot x}$ for all $\pi \in S_A$. □

According to Theorem 7, we have the following Cayley-type result in the FM framework.

Corollary 1. *Let Σ be a possible infinite nominal set. There exists an equivariant isomorphism between $\mathbb{N}_{ext}(\Sigma)$ and a nominal submonoid of the nominal monoid formed by the finitely supported elements from $(\mathbb{N}_{ext}(\Sigma))^{(\mathbb{N}_{ext}(\Sigma))}$.*

5 An Extension of the Framework to the FM Cumulative Hierarchy

Since a finite nominal set necessarily has a trivial permutation action (see Example 2.2.3.1 from [14]), the restriction of the results in Section 4 for $\mathbb{N}(\Sigma)$ [1] become trivial. This is the reason why we do not discuss the results in Section 4 (which are valid in the general case when Σ is an infinite nominal set) in the particular case when Σ is a finite alphabet. The triviality of the properties of $\mathbb{N}(\Sigma)$ could be avoided if we would replace "finite nominal set" by "finite set in the Fraenkel-Mostowski cumulative hierarchy". The FM cumulative hierarchy FM_A described in [8] is a nominal set with the S_A-action $\cdot : S_A \times FM_A \to FM_A$ defined inductively by $\pi \cdot a := \pi(a)$ for all atoms $a \in A$ and $\pi \cdot x := \{\pi \cdot y \mid y \in x\}$ for all $x \in FM_A$. An FM-set is a finitely supported element in FM_A; additionally an FM-set have the recursive property that all its elements are also FM-sets. We generalize the results in the previous section in order to be closer from the framework of FM-sets. However we preserve the terminology of nominal sets in order to present the following results.

Definition 4 can be generalized in the following way:

Definition 13. *Let X and Y be nominal sets, and let Z be a finitely supported subset of X. A function $f : Z \to Y$ is* finitely supported *if $f \in \wp_{fs}(X \times Y)$.*

The following result generalizes Proposition 3.

Proposition 10. *Let $(X, \cdot)$ and $(Y, \diamond)$ be nominal sets, and let Z be a finitely supported subset of X. Let $f : Z \to Y$ be a function. The function f is finitely supported in the sense of Definition 13 if and only if there exists a finite set S of atoms such that for all $x \in Z$ and all $\pi \in Fix(S)$ we have $\pi \cdot x \in Z$ and $f(\pi \cdot x) = \pi \diamond f(x)$* [2].

Proof: Suppose that f is finitely supported in the sense of Definition 13. There exists a finite set S of atoms such that $\pi \star f = f$ for all $\pi \in Fix(S)$, where $\star$ represents the S_A-action on $\wp(X \times Y)$ defined as in Example 1(3). Let $x \in Z$ and $\pi \in Fix(S)$ be arbitrary elements. Then there exists an unique $y \in Y$ such that $(x, y) \in f$. Since $\pi \star f = f$ we have $(\pi \cdot x, \pi \diamond y) \in f \subseteq (Z \times Y)$. Thus $\pi \cdot x \in Z$ and $f(\pi \cdot x) = \pi \diamond y = \pi \diamond f(x)$.

[1] In the particular case when Σ is a finite alphabet we have that $\mathbb{N}_{ext}(\Sigma)$ coincides with $\mathbb{N}(\Sigma)$.

[2] The case when Z is equivariant reduces to Proposition 3 because the equivariant subsets have similar properties as the nominal sets.

Conversely, assume that there exists a finite set S of atoms such that for all $x \in Z$ and all $\pi \in Fix(S)$ we have $\pi \cdot x \in Z$ and $f(\pi \cdot x) = \pi \diamond f(x)$. We claim that $\pi \star f = f$ for all $\pi \in Fix(S)$. Fix some $\pi \in Fix(S)$, and consider (x, y) an arbitrary element in f. We have $f(x) = y$, and so $(\pi \cdot x, \pi \diamond y) \in f$. However $\pi \triangleright (x, y) = (\pi \cdot x, \pi \diamond y) \in f$, where $\triangleright$ represents the S_A-action on $X \times Y$ defined as in Example 1(4). That means $\pi \star f = f$. ∎

According to Proposition 10 we conclude that, if X and Y are nominal sets and Z is a finitely supported subset of X, then the set of all finitely supported functions from Z to Y is a finitely supported subset of the nominal set $X \times Y$. Moreover, if $f : Z \to Y$ is a finitely supported function, then we can define the function $g : X \to Y$ by considering $g(x) = f(x)$, $\forall x \in Z$ and $g(x) = y_0$, $\forall x \in X \setminus Z$ for some fixed $y_0 \in Y$. Obviously, g is supported by $supp(f) \cup supp(y_0)$. Therefore we can extend each finitely supported function $f : Z \to Y$ to a finitely supported function $g : X \to Y$. Informally, we agree to say that the set of all finitely supported functions from Z to Y is a finitely supported subset of the nominal set formed by the collection of all finitely supported functions from X to Y.

According to Proposition 10 the results in Section 4 can be generalized in terms of "finitely supported subsets of (possible infinite) nominal sets". We present only their statements because their proofs are analogue with the original proofs presented in Section 4 (just note that instead of Proposition 3 we have to use Proposition 10).

Proposition 11. *Let $(\Sigma, \cdot)$ be a nominal set (possible infinite), $(\Sigma_0, \cdot)$ be a finitely supported subset of Σ, and $f : \Sigma_0 \to \mathbb{N}$ a function such that its algebraic support $S_f \stackrel{def}{=} \{x \in \Sigma_0 \mid f(x) \neq 0\}$ is finite. Then f is finitely supported and $supp(f) \subseteq supp(S_f) \cup supp(\Sigma_0)$.*

Corollary 2. *Let $\Sigma_0 = \{a_1, \ldots, a_k\}$ be a finite subset of a nominal set $(\Sigma, \cdot)$. Then each function $f : \Sigma_0 \to \mathbb{N}$ is supported by the set $S = supp(a_1) \cup \ldots \cup supp(a_k)$.*

Definition 14. *Given a nominal set $(\Sigma, \cdot)$ (possible infinite) and $(\Sigma_0, \cdot)$ a finitely supported subset of Σ, any function $f : \Sigma_0 \to \mathbb{N}$ with the property that S_f is finite is called* extended multiset over Σ_0 of rank 1. *The set of all extended multisets over Σ_0 of rank 1 is denoted by $\mathbb{N}^1_{ext}(\Sigma_0)$.*

If $f : \Sigma_0 \to \mathbb{N}$ is an extended multiset over Σ_0 of rank 1, then we can define the function $g : \Sigma \to \mathbb{N}$ by considering $g(x) = f(x)$, $\forall x \in \Sigma_0$ and $g(x) = 0$, $\forall x \in \Sigma \setminus \Sigma_0$. Obviously, g is an extended multiset over Σ which is supported by $supp(f)$. Informally, $\mathbb{N}^1_{ext}(\Sigma_0)$ is a finitely supported subset of the nominal set $\mathbb{N}_{ext}(\Sigma)$.

Definition 15. *A* finitely supported monoid *is a triple $(M, +, \cdot)$ such that the following conditions are satisfied:*

- *$(M, +, 0)$ is a monoid;*
- *$(M, \cdot)$ is a finitely supported subset of a nominal set;*
- *for each $\pi \in Fix(supp(M))$ and each $x, y \in M$ we have $\pi \cdot (x + y) = \pi \cdot x + \pi \cdot y$ and $\pi \cdot 0 = 0$.*

Theorem 8. *Let $(\Sigma, \cdot)$ be a nominal set and $(\Sigma_0, \cdot)$ be a finitely supported subset of Σ. The set $\mathbb{N}^1_{ext}(\Sigma_0)$ is a free abelian finitely supported monoid.*

Theorem 9. *Let $(\Sigma, \cdot)$ be a nominal set and $(\Sigma_0, \cdot)$ a finitely supported subset of Σ. Let $j\colon \Sigma_0 \to \mathbb{N}^1_{ext}(\Sigma_0)$ be the function which maps each $a \in \Sigma_0$ into $\widetilde{a} \in \widetilde{\Sigma_0}$. Then j is supported by $supp(\Sigma_0)$. If $(M, +, \diamond)$ is an arbitrary abelian finitely supported monoid and $\varphi\colon \Sigma_0 \to M$ is an arbitrary finitely supported function, then there exists a unique finitely supported homomorphism of abelian monoids $\psi\colon \mathbb{N}^1_{ext}(\Sigma_0) \to M$ with $\psi \circ j = \varphi$, i.e. $\psi(\widetilde{a}) = \varphi(a)$ for all $a \in \Sigma_0$. Moreover, if a finite set S supports φ, then the set $S \cup supp(M) \cup supp(\Sigma_0)$ supports ψ.*

Theorem 10. *Let $(\Sigma, \cdot)$ be a nominal set and $(\Sigma_0, \cdot)$ a finitely supported subset of Σ. Σ_0^* is a finitely supported monoid. Moreover, each element of form $x_1 x_2 \dots x_n$ from Σ_0^* is supported by the set $U = supp(a_1) \cup \dots \cup supp(a_k)$ when Σ_0 is a finite subset $\{a_1, \dots, a_k\}$ of Σ.*

Clearly, Σ_0^* is a finitely supported subset of Σ^* (supported by $supp(\Sigma_0)$).

Theorem 11. *Let $(\Sigma, \cdot)$ be a nominal set and $(\Sigma_0, \cdot)$ a finitely supported subset of Σ. Let $i : \Sigma_0 \to \Sigma_0^*$ be the standard inclusion of Σ_0 into Σ_0^* which maps each element $a \in \Sigma_0$ into the word a. Then i is supported by $supp(\Sigma_0)$. If $(M, \cdot, \diamond)$ is an arbitrary finitely supported monoid and $\varphi : \Sigma_0 \to M$ is an arbitrary finitely supported function, then there exists a unique finitely supported homomorphism of monoids $\psi : \Sigma_0^* \to M$ with $\psi \circ i = \varphi$. Moreover, if a finite set S supports φ, then the set $S \cup supp(M) \cup supp(\Sigma_0)$ supports ψ.*

Proposition 12. *Let $(\Sigma, \cdot)$ be a nominal set and $(\Sigma_0, \cdot)$ be a finitely supported subset of Σ. The sub-multiset order $\subseteq$ on $\mathbb{N}^1_{ext}(\Sigma_0)$ is supported (as a subset of $\Sigma \times \Sigma$) by $supp(\Sigma_0)$.*

Theorem 12. *Let $(\Sigma, \cdot)$ be a nominal set and $(\Sigma_0, \cdot)$ be a finite subset of Σ. Let $\prec$ be a strictly order relation on Σ_0. The Dershowitz and Manna order $\ll$ on $\mathbb{N}^1_{ext}(\Sigma_0)$ induced by $\prec$ is supported (as a subset of $\Sigma \times \Sigma$) by $supp(\Sigma_0) \cup supp(\prec)$.*

Proof: We know that $\prec$ be a strictly order relation on Σ_0. Since $\prec$ is a finite subset of $\Sigma_0 \times \Sigma_0$, it follows that $\prec$ is finitely supported. Let $\pi \in Fix(supp(\Sigma_0) \cup supp(\prec))$. Since Σ_0 is finite we use the HO definition of $\ll$ (see Definition 9). Let $f, g \in \mathbb{N}^1_{ext}(\Sigma_0)$ such that $f \ll g$, and $\pi \in S_A$. Since $f \neq g$ there exists $x \in \Sigma_0$ such that $f(x) \neq g(x)$. Therefore $f(x) = (\pi \star f)(\pi \cdot x) \neq (\pi \star g)(\pi \cdot x) = g(x)$. Since $\pi \cdot x \in \Sigma_0$ we have $\pi \star f \neq \pi \star g$. Let $y \in \Sigma_0$ such that $(\pi \star f)(y) > (\pi \star g)(y)$. Then $f(\pi^{-1} \cdot y) > g(\pi^{-1} \cdot y)$. According to Definition 9 there exists $x \in \Sigma_0$ with $\pi^{-1} \cdot y \prec x$ and $f(x) < g(x)$. Since $\pi \in Fix(supp(\prec))$ we have $y \prec \pi \cdot x$. Since $f(x) < g(x)$ we obtain $(\pi \star f)(\pi \cdot x) < (\pi \star g)(\pi \cdot x)$. Thus $\pi \star f \ll \pi \star g$. ∎

Theorem 13 (Cayley-theorem for finitely supported monoids)**.** *Let $(X, +, \cdot)$ be a finitely supported monoid. There exists a finitely supported isomorphism ϕ between X and a finitely supported submonoid M of the finitely supported monoid formed by the finitely supported elements from X^X. Moreover, M and ϕ are supported by $supp(X)$.*

Corollary 3. *Let Σ be a possible infinite nominal set, and Σ_0 a finitely supported subset of Σ. There exists a finitely supported isomorphism ϕ between $\mathbb{N}^1_{ext}(\Sigma_0)$ and a nominal submonoid M of the finitely supported monoid formed by the finitely supported elements from $(\mathbb{N}^1_{ext}(\Sigma_0))^{(\mathbb{N}^1_{ext}(\Sigma_0))}$. Moreover M and ϕ are supported by $supp(\Sigma_0)$.*

Note that none of the results in this section leads to a trivial corollary when we request Σ_0 to be finite. Therefore the properties of $\mathbb{N}^1_{ext}(\Sigma_0)$=$\mathbb{N}(\Sigma_0)$ are non-trivial even when Σ_0 is a finite subset of a nominal set.

6 Conclusion and Related Work

The aim of this paper is to define and study "multisets" in the framework of nominal sets. Classical multisets over finite alphabets are extended using specific nominal techniques. We define "extended nominal multisets" over possible infinite alphabets, presenting also some properties of this new concept. The analogy between the results obtained by using the FM axioms of set theory and those obtained by using the ZF axioms of set theory is emphasized by the results presented in Section 4. In Theorem 3 we proved that the set of all extended multisets over a (possible infinite) nominal set Σ is a free abelian nominal monoid, and it satisfies the universality property expressed in Theorem 4. The free monoid over Σ is also a nominal monoid according to Theorem 5, and it satisfies the universality property presented in Theorem 6. By repeatedly applying these universality properties, a connection between the extended multisets over Σ and the free monoid over Σ is given firstly in ZF approach, and secondly in FM approach, in terms of *finitely supported* homomorphisms. Several nominal order properties of multisets and extended multisets are presented in Proposition 9 and Theorem 12. An embedding theorem of Cayley-type for $\mathbb{N}_{ext}(\Sigma)$ is presented as Corollary 1. Since every ZF-set together with the discrete S_A-action is a nominal set, we conclude that the results in Section 3 can be obtained by particularizing the results in Section 4. The framework for studying multisets is again extended in Section 5 by the informal replacement of "equivariant" with "finitely supported", namely we consider a new class of multisets defined over finitely supported subsets of nominal sets instead of the class of those multisets defined on nominal sets. Several properties of the extended multisets of rank 1 are presented in Section 5.

The classical theory of nominal sets over a fixed set A of atoms is generalized in [5] to a new theory of nominal sets over arbitrary unfixed sets of data values. The notion of 'S_A-set' is replaced by the notion of 'set endowed with an action of a subgroup of the symmetric group of $\mathbb{D}$' for an arbitrary set of data values $\mathbb{D}$, and the notion of 'finite set' is replaced by the notion of 'set with a finite number of orbits according to the previous group action (orbit-finite set)'. The theory of automata have been studied in this framework. According to definitions used in [5], the set A of atoms is a single-orbit set. However the set $\mathbb{N}_{ext}(A)$ has an infinite number of orbits because if two functions from $\mathbb{N}_{ext}(A)$ are in the same orbit, then their corresponding algebraic supports must have the same cardinality. Therefore we are able to develop a nominal theory of multisets even when the alphabet is infinite, and the set of all multisets over the related alphabet is neither finite nor orbit-finite. Informally, our paper extends the framework from 'finite/orbit-finite' to 'infinite/orbit-infinite but with a finite algebraic support'. Bojanczyk introduces a notion of nominal monoid over arbitrary data symmetries [4] in order to prove that a language of data words is definable in first-order logic if and only if its syntactic monoid is aperiodic. We have a little different perspective. The notion of nominal monoid introduced in Definition 11 makes sense only when the set of atoms is fixed, and nominal monoids are used in order to obtain nominal properties of multisets and to connect such nominal properties in terms of finitely supported homomorphisms. The notion of nominal monoid presented in Definition

11 is similar to the notion of nominal group analyzed in [1]. Therefore the correspondence, isomorphism and embedding theorems presented in [1] for nominal groups can be naturally rephrased for nominal monoids. Thus the results obtained in [1] can also provide new algebraic properties of $\mathbb{N}_{ext}(\Sigma)$. Bojanczyk uses alternative definitions for nominal algebraic structures in the generalized framework of G-sets described in [5], just to study the languages of data words. However our perspective allow us to study algebra in the classical framework of nominal sets. Some original results in this direction are presented in [1, 2, 3].

Extending a ZF algebraic structure to an FM structure is not trivial, because in the FM framework only finitely supported elements are allowed. Hence several non-trivial S_A-actions had to be defined first, in order to build a nominal structure on some algebraic structures like free monoids (Theorem 5) or factor monoids (Proposition 8). After these S_A-actions were defined, we have to check whether the classical ZF results can be naturally translated in FM only by replacing "structure" with "nominal structure". If this is possible, then things go smoothly. Using nominal techniques we are able to prove a similar FM result for each ZF result presented in Section 3. However, it is not always so simple. For example, Theorem 3.3 from [3] shows that the Tarski theorem for complete lattices cannot be translated in FM using such a procedure. The FM Tarski-like theorem is not valid for all finitely supported functions defined on an FM complete lattice, but only for those functions which are equivariant. Also the nominal embedding theorems for groups presented in Section 5 from [1] can be proved only for uniform nominal groups, and not for all nominal groups. We present another example regarding the non-triviality of the translation of a ZF result into the FM framework. We know there exist models of ZF without choice that satisfy the statement "Every set can be totally ordered" named "the ordering principle"; an example of such a model is the Howard-Rubin's first model (N38 in [9]). However we claim that the statement "For every nominal set X there exists a finitely supported total order relation on X" is inconsistent with the axioms of the FM set theory. Indeed, suppose that there exists a finitely supported total order $<$ on the nominal set A. Let $a, b, c \notin supp(<)$ with $a < b$. Since $(a\,c) \in Fix(supp(<))$ we have $(a\,c)(a) < (a\,c)(b)$, so $c < b$. However we also have $(a\,b), (b\,c) \in Fix(supp(<))$, and so $((a\,b) \circ (b\,c))(a) < ((a\,b) \circ (b\,c))(b)$, that is $b < c$. We get a contradiction, and so the translation of the ordering principle in FM realized by replacing "structure" with "finitely supported structure" yields to a false statement. Other examples of results which fail in the FM settings are represented by the Stone duality [12], the determinization of finite automata [5], or the equivalence of two-way and one-way finite automata [5].

The reader might ask why we choose not to use the general nominal equivariance and finite support principle of [13] in the proof of Theorem 4 and in the similar proof of Theorem 6. Well, this principle could ensure that the function ψ is finitely supported, but we cannot conclude from this principle that any set supporting φ also supports ψ. Thus, in order to prove that $supp(\psi) \subseteq supp(\varphi)$, we need to present a constructive method of defining a set supporting ψ. A similar remark can also be formulated for Theorem 9 and Theorem 11. Moreover, since in applying the equivariance and finite support principle one must take into account all the parameters upon which a particular construction depends. The precise verification whether the conditions for applying the equivariance and finite support principle are properly satisfied is sometimes at least as difficult as a direct proof. In some cases we can prove stronger properties without involving this finite support principle. For example, each function f_x in the proof of

Theorem 7 has a non-empty finite support. The reader could say that the function T from this theorem has also a non-empty finite support. We prove something stronger: the function T is equivariant. Moreover, many times it is necessary to present a *constructive method* of defining the support in order to ensure that some structures are (uniformly) finitely supported (see [1]).

It is also worth noting that the general equivariance and finite support principle is presented using the higher-order logic (see Theorem 3.5 from [13]). Our paper is self-contained and the results we presented can be understood without using any notions regarding the higher-order logic or the category theory.

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