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Cohomology groups of configuration spaces of Riemann surfaces

by
Haniya Azam

Abstract

The $\mathcal{S}_n$ representation theory for the Križ model for configuration spaces of 2,3 and 4 points on Riemann surfaces is studied. Betti numbers are computed for the ordered and unordered configuration spaces of the torus and surfaces of higher genus.

Key Words: Configuration spaces, Riemann surfaces, Križ model, representations of the symmetric group.
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1 Introduction

The ordered configuration space of n points $F(X,n)$, of a topological space X is defined as

$$F(X,n) = \{(x_1, x_2, \ldots, x_n) \in X^n \mid x_i \neq x_j \text{ for } i \neq j\}.$$

The symmetric group on n letters $\mathcal{S}_n$ acts freely on $F(X,n)$ and its orbit space, denoted $C(X,n)$, is called the space of unordered configurations of n points on X. In this paper we are interested in computing the Betti numbers of the ordered and unordered 2 and 3-point configuration space of Riemann surfaces $X = \mathbb{T}^2, \mathcal{M}_g$ of genus $g > 1$ and also the Betti numbers of $C(\mathbb{T}^2, 4)$. I. Kriz [9] constructed a rational model $E(X,n)$ for $F(X,n)$ which is quasi-isomorphic to the model related to Fulton-MacPherson's compactification [7] of $F(X,n)$, for any complex projective smooth variety X ($\dim_{\mathbb{C}} X = m$).

We describe the Križ model in the case of Riemann surfaces. Let the well-known cohomology bases for $\mathbb{T}^2$ and $\mathcal{M}_g$ be ordered as follows, $\mathcal{B}_{H^*(\mathbb{T}^2)} = \{1 < a < b < w\}$ and $\mathcal{B}_{H^*(\mathcal{M}_g)} = \{1 < a_1 < \ldots < a_g < b_1 < \ldots < b_g < w\}$ with generators $a, b, \{a_i\}_{i=1}^g, \{b_i\}_{i=1}^g$ in degree one and w denoting the fundamental class in both cases. The relations are given by $ab = w = -ba$ and $a_i b_j = \delta_{ij} w = -b_j a_i$. Let $p_i^* : H^*(X) \to H^*(X^n)$ and $p_{ij}^* : H^*(X^2) \to H^*(X^n)$ for $i \neq j$ be pullbacks of the projections p_i and p_{ij}. As an algebra $E(X,n)$ is isomorphic to the exterior algebra on generators G_{ij}, $1 \leq i,j \leq n$, and G_{ij} are of degree 1, with coefficients in $H^*(X)^{\otimes n}$ modulo the following relations:

$$\begin{array}{rcll} G_{ji} & = & G_{ij} & \\ p_j^*(x)G_{ij} & = & p_i^*(x)G_{ij}, & (i<j),\ x\in H^*(X) \\ G_{ik}G_{jk} & = & G_{ij}G_{jk}-G_{ij}G_{ik}, & (i<j<k). \end{array}$$

The differential d is given by

$$d|_{H^*(X)^{\otimes n}} = 0 \text{ and } d(G_{ij}) = p_{ij}^*(\Delta),$$

where Δ is the class of the diagonal:

$$\begin{array}{lll} \Delta = & 1\otimes w + b\otimes a - a\otimes b + w\otimes 1 & \in H^*(\mathbb{T}^2;\mathbb{Q})^{\otimes 2}, \\ \Delta = & 1\otimes w + \sum_{i=1}^{g} b_i\otimes a_i - \sum_{i=1}^{g} a_i\otimes b_i + w\otimes 1 & \in H^*(\mathcal{M}_g;\mathbb{Q})^{\otimes 2}. \end{array}$$

The Križ model $(E(X,n),d)$ is a differential bigraded algebra (DGBA), the exterior degree is given by the number of exterior generators G_{ij}, and the second degree is given by the total degree of the monomial. We denote by $E_q^k(X,n)$ or simply E_q^k the bigraded component of the Kriz model with total degree k and exterior degree q; the multiplication $E_q^k\otimes E_{q'}^{k'}\to E_{q+q'}^{k+k'}$ is homogeneous and the differential $d: E_q^k\to E_{q-1}^{k+1}$ has bidegree $\binom{+1}{-1}$. Using the bigrading we will write the *double Poincaré polynomial* as $P_{E_*^*(F(X,n))}(t,s) = \sum_{k,q\geq 0}(dim\, E_q^k)t^k s^q$. Fixing an ordered basis of $H^*(X;\mathbb{Q})$ for any X, Bezrukavnikov [2] described a monomial basis for the Kriz model:

$$\Big\{x_{h_1}\otimes x_{h_2}\otimes\ldots\otimes x_{h_n}G_{i_1j_1}G_{i_2j_2}\ldots G_{i_qj_q} \mid i_a<j_a,\ x_{h_a}=1 \text{ if } h_a=j_r,\ 1\leq r\leq q\Big\}$$

where x_{h_a} are monomials in the ordered basis of $H^*(X;\mathbb{Q})$ and $1\leq j_1<j_2<\ldots<j_q\leq n$. All monomial bases considered in this paper are of this form.

The group $\mathcal{S}_n$ acts on the Križ model by DGBA automorphisms, the $\mathcal{S}_n$*-action* on $E(X,n)$ (see [7],[9]) is defined as:

$$\pi((p_1^*(x_{h_1})\ldots p_n^*(x_{h_n}))G_{I_*J_*}) = (p_{\pi(1)}^*(x_{h_1})\ldots p_{\pi(n)}^*(x_{h_n}))G_{\pi(I_*)\pi(J_*)}$$

for any $\pi\in\mathcal{S}_n$. Using standard notation (see [6]) the irreducible $\mathcal{S}_n$-module corresponding to partition $\lambda\vdash n$ is denoted by $V(\lambda)$. The action leaves bihomogeneous components E_q^k invariant; its isotypical component corresponding to the partition λ of n will be denoted by $E_q^k(V(\lambda))$. The *symmetric Poincaré polynomial* for $F(X,n)$ is defined as the formal sum with polynomial coefficients

$$SP_{F(X,n)}(t,s) = \sum_{\lambda\vdash n}(\sum_{k,q} m_{q,\lambda}^k t^k s^q)V(\lambda),$$

where $m_{q,\lambda}^k$ is the multiplicity of the irreducible representation $V(\lambda)$ in the bigaraded component of cohomology $H_q^k = H_q^k(F(X,n))$.

In [3], Bödigheimer and Cohen calculated the rational cohomology of unordered configuration spaces of a surface with one puncture. Fairly recently, the integral cohomology of punctured surfaces was studied by Napolitano in [10], from the stability point of view. Brown and White [5] computed the Betti numbers of $F(X,3)$ for X-an orientable surface, and as an application gave a lower bound for the number of equilibrium positions for equally charged particles on a surface. The cohomology algebras of ordered configuration spaces of spheres with integral coefficients were computed by Feichtner and Ziegler in [8]. In this paper we describe the symmetric structure of the cohomology $H^*(F(\mathcal{M}_g,n))$ and we also obtain the Betti numbers for the unordered configuration space as the $\mathcal{S}_n$-invariant part of the rational cohomology of $F(\mathcal{M}_g,n)$. The bigraded cohomology groups of the exceptional case $g=0$, $H^*_*(F(S^2,n))$, are computed in [1].

In section 3 we describe the $\mathcal{S}_2$-structure of the Križ model for $\mathcal{M}_g$, $(g\geq 1)$ and obtain the symmetric Poincaré polynomial for $F(\mathcal{M}_g,2)$:

Proposition 1. *The symmetric Poincaré polynomial of $F(\mathcal{M}_g,2)$, $g\geq 1$, is given by*

$$SP_{H^*(F(\mathcal{M}_g,2))}(t,s)=(1+2gt+(2g^2-g)t^2)V(2)+(2gt+(2g^2+g+1)t^2+2gt^3)V(1,1).$$

In section 4 we fully describe the $\mathcal{S}_3$-structure of $E(\mathcal{M}_g,3)$ for $g\geq 1$ and compute the differentials for subcomplexes corresponding to each isotypical component $V(3)$ and $V(2,1)$ separately, to obtain the following:

Theorem 2. *The symmetric Poincaré polynomial for $F(\mathcal{M}_g,3))$ for $g\geq 2$ is:*

$$\begin{aligned}SP_{H^*(F(\mathcal{M}_g,3))}(s,t)=\;&\big[1+2gt+(2g^2-g)t^2+\tfrac{2}{3}(2g^3-3g^2+4g)t^3+\\&+((2g^2+g+1)t^3+2gt^4)s\big]\,V(3)+\\&+\big[2gt+4g^2t^2+\tfrac{2}{3}(4g^3-g)t^3\big]V(2,1)+\\&+\big[(2g^2+g)t^2+\tfrac{2}{3}(2g^3+3g^2+4g)t^3+(2g^2+g)t^4\big]\,V(1,1,1).\end{aligned}$$

For $g=1$ we have:

$$SP_{H^*(F(\mathbb{T}^2,3))}(s,t)=(1+t)^2(1+2st^2)V(3)+2t(1+t)^2V(2,1)+3t^2(1+t)^2V(1,1,1).$$

Lastly, computations for the unordered configuration space of four points on a Riemann surface of genus one give:

Proposition 3. *The double Poincaré polynomial of the unordered 4-point configuration space of $\mathbb{T}^2$ is:*

$$P_{H^*(C(\mathbb{T}^2,4))}(s,t)=1+2t+t^2+(2t^2+5t^3+4t^4+t^5)s.$$

2 Some useful results

Here we state some general results for any complex projective variety X from [1] to be used for computations in the following sections.

Proposition 4. *[1] For any $q=0,1,\ldots,n-1$ and any k in the interval $[(2m-1)q,2mn-q]$, the following is an isomorphism of $\mathcal{S}_n$-modules*

$$E^k_q(X,n)\cong E^{2mn+2q(m-1)-k}_q(X,n).$$

The following propositions give the vanishing of cohomology at the "border" of the trapezoid (see [1]) formed by plotting the bigraded components of $E(X,n)$, and describes some interior acyclic subcomplexes.

Proposition 5. *[1] The differentials in the Križ model of a projective manifold different from $\mathbb{C}P^1$ are injective for any q in the interval $[1, n-1]$:*

$$d : E_q^{q(2m-1)}(X,n) \rightarrowtail E_{q-1}^{q(2m-1)+1}(X,n).$$

Proposition 6. *[1] The top differentials in the Križ model are injective for any k in the interval $[(n-1)(2m-1), n(2m-1)+1]$:*

$$d : E_{n-1}^{k}(X,n) \rightarrowtail E_{n-2}^{k+1}(X,n).$$

Proposition 7. *[1] All cohomology groups of the subcomplex*

$$0 \to E_{n-1}^{n(2m-1)+1}(X,n) \to E_{n-2}^{n(2m-1)+2}(X,n) \to \ldots \to E_0^{2nm}(X,n) \to 0$$

are zero.

For a fixed non empty subset $A \subset \{1,2,\ldots,n\}$ of cardinality $|A| = a \geq 2$ and a fixed sequence β of length $b = n-a$, $\beta = (x_1, x_2, \ldots, x_b)$, where all the elements x_j belong to the fixed basis $\mathcal{B}$ and are different from w, we denote the increasing sequence of elements in $\{1,2,\ldots,n\} \setminus A$ by $b_1 < b_2 < \ldots < b_b$, the product $\prod_{j=1}^{b} p_{b_j}^*(x_j)$ by $p^*(\beta)$, and its degree $\sum_{j=1}^{b} \deg(x_j)$ by $|\beta|$. Now we define subspace

$$E_*^{Top}(A,\beta) = \sum_{q=0}^{a-1} E_q^{2ma-q+|\beta|}(A,\beta)$$

by $\quad E_q^{2ma-q+|\beta|}(A,\beta) = \mathbb{Q}\langle \prod_{i \in A\setminus J_*} p_i^*(w) p^*(\beta) G_{I_* J_*} \mid I_* \cup J_* \subset A, |J_*| = q \rangle.$

Proposition 8. *[1] For any A and β as above, the space $E_*^{Top}(A,\beta)$ is an acyclic subcomplex of the Križ model.*

3 Cohomology of 2-points configuration spaces

In this section we study the $\mathcal{S}_2$ action on the Križ model for $F(\mathcal{M}_g, 2)$ for genus $g \geq 1$. A simple application of Proposition 6 gives the Poincaré polynomial of $F(\mathcal{M}_g, 2)$.

Proposition 9. *The $\mathcal{S}_2$ decomposition of $E_0^k(\mathcal{M}_g, 2)$ is given as:*

k	0	1	2	3	4	
E_0^k	1	$2g$	$2g^2 - g + 1$	$2g$	1	$V(2)$
		$2g$	$2g^2 + g + 1$	$2g$		$V(1,1)$

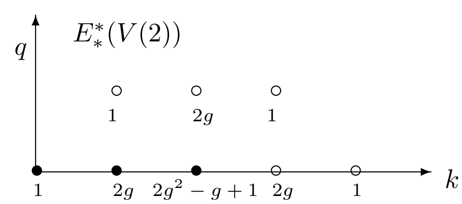
Proof: Finding explicit bases for the modules we get the required multiplicities. The component $E_0^0 \cong \langle 1 \otimes 1 \rangle \cong V(2)$. The $\mathcal{S}_2$-decomposition of E_0^1 is given by $\langle a_i \otimes 1 + 1 \otimes a_i \rangle \oplus \langle b_i \otimes 1 + 1 \otimes b_i \rangle \cong 2gV(2)$ and $\langle a_i \otimes 1 - 1 \otimes a_i \rangle \oplus \langle b_i \otimes 1 - 1 \otimes b_i \rangle \cong 2gV(1,1)$ for $1 \leq i \leq g$.
Similarly, E_0^2 is isomorphic to the direct sum of
$\langle w \otimes 1 + 1 \otimes w \rangle \oplus \langle a_i \otimes a_j - a_j \otimes a_i \rangle \oplus \langle b_i \otimes b_j - b_j \otimes b_i \rangle \oplus \langle a_i \otimes b_j - b_j \otimes a_i \rangle \cong (2g^2 - g + 1)V(2)$ and $\langle w \otimes 1 - 1 \otimes w \rangle \oplus \langle a_i \otimes a_i \rangle \oplus \langle b_i \otimes b_i \rangle \oplus \langle a_i \otimes a_j + a_j \otimes a_i \rangle \oplus \langle b_i \otimes b_j + b_j \otimes b_i \rangle \oplus \langle a_i \otimes b_j + b_j \otimes a_i \rangle \cong (2g^2 + g + 1)V(1,1)$, for $1 \leq i, j \leq g$. The $\mathcal{S}_2$-decomposition for components E_0^k for $k = 3, 4$ is obtained using Proposition 4. □

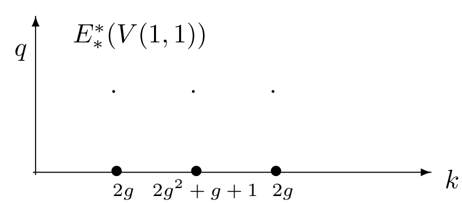
Proposition 10. *In exterior degree one, the $\mathcal{S}_2$ decomposition of $E(\mathcal{M}_g, 2)$ is given as follows:*

k	1	2	3	
E_1^k	1	$2g$	1	$V(2)$

Proof: The components are $E_1^1 \cong \langle G_{12} \rangle \cong V(2)$ and $E_1^2 \cong \langle a_i \otimes 1G_{12} \rangle \oplus \langle b_i \otimes 1G_{12} \rangle \cong 2gV(2)$, for $1 \leq i \leq g$. Also $E_1^3 \cong E_1^1$ by Proposition 4. □

The following diagrams for irreducible $\mathcal{S}_2$-submodules $V(\lambda)$, $\lambda \vdash 2$, show the corresponding multiplicities of the bigraded components of the Križ model, where dots denote the absence of the modules $V(\lambda)$ from $E_*^*(\mathcal{M}_g, 2)$, the circles denote their appearance and the bullets show components from which some part survives to contribute to cohomology.





Proof of Proposition 1.1. All differentials $E_1^k \xrightarrow{d} E_1^{k+1}$, are injective. This follows from direct computation or by using Proposition 6. □

Corollary 11. *The double Poincaré Polynomial of $C(\mathcal{M}_g, 2)$, $g \geq 1$, is*

$$P_{H^*(C(\mathcal{M}_g,2))}(t,s) = 1 + 2gt + (2g^2 - g)t^2$$

Proof: This is obtained as a consequence of the transfer theorem which equates the cohomology of $C(\mathcal{M}_g, 2)$ to the $\mathcal{S}_2$-invariant part of the cohomology of $F(\mathcal{M}_g, 2)$. ∎

Corollary 12. *The double Poincaré Polynomial of $F(\mathcal{M}_g, 2)$, $g \geq 1$, is*

$$P_{H^*(F(\mathcal{M}_g,2))}(t, s) = 1 + 4gt + (4g^2 + 1)t^2 + 2gt^3.$$

4 Cohomology of 3-points configuration spaces

In this section we describe the $\mathcal{S}_3$-structure of the Križ model $E^*_*(\mathcal{M}_g, 3)$ for the ordered 3-point configuration spaces for surfaces. Moreover we compute the Betti numbers for $H^*(F(\mathcal{M}_g, 3); \mathbb{Q})$, $H^*(C(\mathcal{M}_g, 3); \mathbb{Q})$.

We first discuss for any space X, the combinatorics of the $\mathcal{S}_3$-structure of the bigraded component corresponding to the exterior degree zero i.e. $E^*_0(X, 3) \cong H^*(X)^{\otimes 3}$. The group $\mathcal{S}_3$ acts on the $\mathbb{Q}$-span of monomials $x_a \otimes x_b \otimes x_c \in H(X)^{*\otimes 3}$ and gives the following decomposition, which depends on the degree $|x_a|$ of the factors:

Lemma 13. *We have*

1)For $a = b = c$, $\mathbb{Q}\langle x_a \otimes x_a \otimes x_a\rangle \cong \begin{cases} V(3) & \text{if } |x_a| \text{ is even} \\ V(1,1,1) & \text{if } |x_a| \text{ is odd,} \end{cases}$

2) For $a \neq b$, $\sum_{\sigma \in \mathcal{S}_3} \sigma.\mathbb{Q}\langle x_a \otimes x_a \otimes x_b\rangle \cong \begin{cases} V(3) \oplus V(2,1) & \text{if } |x_a| \text{ is even} \\ V(2,1) \oplus V(1,1,1) & \text{if } |x_a| \text{ is odd,} \end{cases}$

3) For $a \neq b \neq c$, $\sum_{\sigma \in \mathcal{S}_3} \sigma.\mathbb{Q}\langle x_a \otimes x_b \otimes x_c\rangle \cong V(3) \oplus 2V(2,1) \oplus V(1,1,1).$

Proof: For 1), we see that the generator $x_a \otimes x_a \otimes x_a$ is invariant under the action of $\mathcal{S}_3$ if the degree of x_a is even and skew-invariant if it is odd. For the 3-dimensional subspace in 2), a direct computation of characters gives us the desired decomposition. Lastly, the 6-dimensional subspace in 3) is clearly the regular representation of $\mathcal{S}_3$. ∎

4.1 The $\mathcal{S}_3$ structure for $E^*_*(F(\mathcal{M}_g, 3))$

In this section we describe the symmetric structure for the Križ model for the ordered configuration of three points on surfaces $\mathcal{M}_g$, $g \geq 1$.

Proposition 14. *The $\mathcal{S}_3$ decomposition of $E^*_0(F(\mathcal{M}_g, 3))$, $g \geq 1$, is given by the following table of multiplicities:*

k	0	1	2	3	4	5	6	
E_0^k	1	$2g$	$\binom{2g}{2}+1$	$\binom{2g}{3}+2g$	$\binom{2g}{2}+1$	$2g$	1	$V(3)$
	-	$2g$	$4g^2+1$	$2\binom{2g+1}{3}+4g$	$4g^2+1$	$2g$	-	$V(2,1)$
	-	-	$\binom{2g+1}{2}$	$\binom{2g+2}{3}+2g$	$\binom{2g+1}{2}$	-	-	$V(1,1,1)$

Proof: Using Lemma 13 and counting the number of possible types in each degree $0 \leq k \leq 6$, we obtain the result. ∎

Proposition 15. *The decomposition of $E_1^*(F(\mathcal{M}_g,3))$ into $\mathcal{S}_3$- irreducible modules is the following:*

k	1	2	3	4	5	
E_1^k	1	$4g$	$4g^2+2$	$4g$	1	$V(3)$
	1	$4g$	$4g^2+2$	$4g$	1	$V(2,1)$

Proof: The E_1^1-component of the model has generators $\{G_{12}, G_{13}, G_{23}\}$ whose sum,

$$G = G_{12} + G_{13} + G_{23}$$

spans the one-dimensional module $V(3)$ and the vectors $\{G_{12} - G_{13}, G_{12} - G_{23}\}$, form a basis of $V(2,1)$. Consider the $\mathcal{S}_3$-invariant elements in E_0^1, $\alpha_1^i = a_i \otimes 1 \otimes 1 + 1 \otimes a_i \otimes 1 + 1 \otimes 1 \otimes a_i$, for all $1 \le i \le g$. We take the orbit of $\alpha_1^i G_{12}$ under the action of $\mathcal{S}_3$ for all i. Each module, in collection of orbit spans, $\langle \alpha_1^i G_{12}, (13) \cdot \alpha_1^i G_{12}, (23) \cdot \alpha_1^i G_{12} \rangle$, for all $1 \le i \le g$, and has the $\mathcal{S}_3$ decomposition $V(3) \oplus V(2,1)$. The bases is given by

$$\{\alpha_1^i G = (a_i \otimes 1 \otimes 1 + 1 \otimes a_i \otimes 1 + 1 \otimes 1 \otimes a_i) \cdot (G_{12} + G_{13} + G_{23})\} \text{ and}$$

$$\{\alpha_1^i (G_{12} - G_{13}), \alpha_1^i (G_{12} - G_{23})\}.$$

Next we take the element $\alpha_2^i = a_i \otimes 1 \otimes 1 - 1 \otimes a_i \otimes 1$, for all $1 \le i \le g$, and consider the orbit of $\alpha_2^i G_{13}$ which is 3-dimensional: its decomposition is $V(3) \oplus V(2,1)$. For each $1 \le i \le g$, the elements:

$$\begin{aligned} \beta_1^i &= b_i \otimes 1 \otimes 1 + 1 \otimes b_i \otimes 1 + 1 \otimes 1 \otimes b_i \\ \beta_2^i &= b_i \otimes 1 \otimes 1 - 1 \otimes b_i \otimes 1. \end{aligned}$$

multiplied by G_{12} and G_{13} respectively, give two collections (for each $1 \le i \le g$) of orbits which decompose into $V(3) \oplus V(2,1)$. Thus we have $E_1^2 \cong 4gV(3) \oplus 4gV(2,1)$.
For E_1^3, one can take the following bases for irreducible modules:
$\mathbb{Q}\langle w \otimes 1 \otimes 1G_{12} + w \otimes 1 \otimes 1G_{13} + 1 \otimes w \otimes 1G_{23} \rangle \cong V(3)$
$G(w) = \mathbb{Q}\langle w \otimes 1 \otimes 1G_{12} - w \otimes 1 \otimes 1G_{13}, w \otimes 1 \otimes 1G_{12} - 1 \otimes w \otimes 1G_{23} \rangle \cong V(2,1)$,
$\mathbb{Q}\langle 1 \otimes 1 \otimes wG_{12} + 1 \otimes w \otimes 1G_{13} + w \otimes 1 \otimes 1G_{23} \rangle \cong V(3)$
$\mathbb{Q}\langle 1 \otimes 1 \otimes wG_{12} - 1 \otimes w \otimes 1G_{13}, 1 \otimes 1 \otimes wG_{12} - w \otimes 1 \otimes 1G_{23} \rangle \cong V(2,1)$,
$\mathbb{Q}\langle a_i \otimes 1 \otimes a_j G_{12} + a_i \otimes a_j \otimes 1G_{13} - a_j \otimes a_i \otimes 1G_{23} \rangle \cong V(3)$, $1 \le i, j \le g$,
$G(a_i a_j) = \mathbb{Q}\langle a_i \otimes 1 \otimes a_j G_{12} - a_i \otimes a_j \otimes 1G_{13}, a_i \otimes 1 \otimes a_j G_{12} + a_j \otimes a_i \otimes 1G_{23} \rangle \cong V(2,1)$,
$\mathbb{Q}\langle b_i \otimes 1 \otimes b_j G_{12} + b_i \otimes b_j \otimes 1G_{13} - b_j \otimes b_i \otimes 1G_{23} \rangle \cong V(3)$, $1 \le i, j \le g$,
$G(b_i b_j) = \mathbb{Q}\langle b_i \otimes 1 \otimes b_j G_{12} - b_i \otimes b_j \otimes 1G_{13}, b_i \otimes 1 \otimes b_j G_{12} + b_j \otimes b_i \otimes 1G_{23} \rangle \cong V(2,1)$,
$\mathbb{Q}\langle a_i \otimes 1 \otimes b_j G_{12} + a_i \otimes b_j \otimes 1G_{13} - b_j \otimes a_i \otimes 1G_{23} \rangle \cong V(3)$, $1 \le i, j \le g$,
$G(a_i b_j) = \mathbb{Q}\langle a_i \otimes 1 \otimes b_j G_{12} - a_i \otimes b_j \otimes 1G_{13}, a_i \otimes 1 \otimes b_j G_{12} + b_j \otimes a_i \otimes 1G_{23} \rangle \cong V(2,1)$,
$\mathbb{Q}\langle b_j \otimes 1 \otimes a_i G_{12} + b_j \otimes a_i \otimes 1G_{13} - a_i \otimes b_j \otimes 1G_{23} \rangle \cong V(3)$, $1 \le i, j \le g$,
$G(b_j a_i) = \mathbb{Q}\langle b_j \otimes 1 \otimes a_i G_{12} - b_j \otimes a_i \otimes 1G_{13}, b_j \otimes 1 \otimes a_i G_{12} + a_i \otimes b_j \otimes 1G_{23} \rangle \cong V(2,1)$.
This implies $E_1^2 \cong (4g^2+2)V(3) \oplus (4g^2+2)V(2,1)$. □

The following proposition shows the $\mathcal{S}_3$ decomposition of E_2^*

Proposition 16. *The $\mathcal{S}_3$ decomposition of $E_2^*(F(\mathcal{M}_g, 3))$ is as follows*

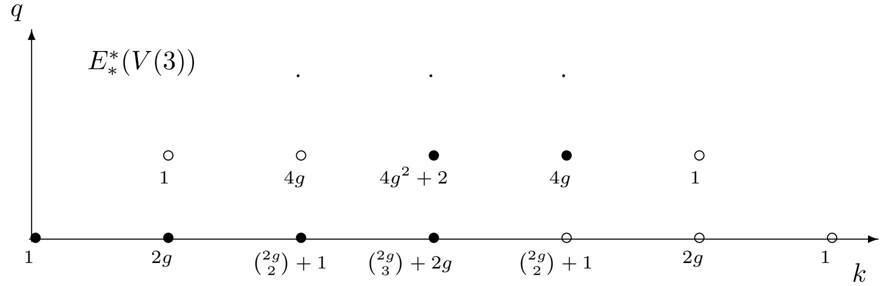
k	2	3	4	
E_2^k	1	$2g$	1	$V(2,1)$

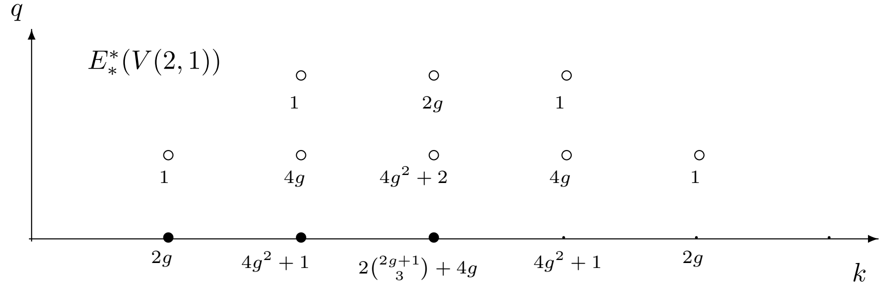
Proof: For E_2^3, we have the following description:

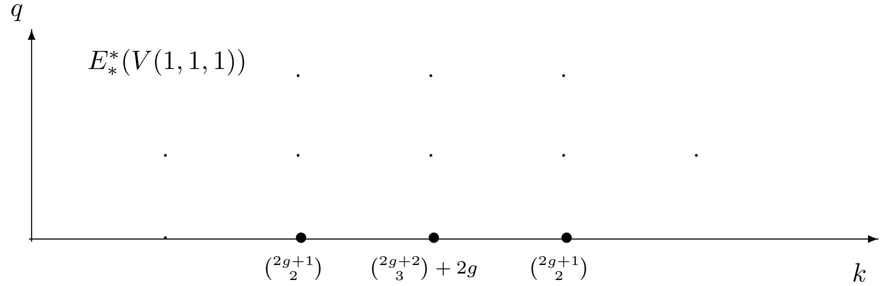
$$\begin{aligned} E_2^3 &\cong \bigoplus_i \left[(\langle \alpha_1^i \rangle \otimes E_2^2) \oplus (\langle \beta_1^i \rangle \otimes E_2^2)\right] \\ &\cong 2g(V(3) \otimes V(2,1)) \\ &\cong 2gV(2,1). \end{aligned}$$

where α_1^i and β_1^i are elements as described in the proof of the previous proposition. ∎

The following diagrams for irreducible $\mathcal{S}_3$-submodules $V(3), V(2,1)$ and $V(1,1,1)$ show the multiplicities of the irreducible components of E_*^*:







4.2 Computations of the differential

We can read off a few Betti numbers directly, without any computation, from the diagrams of multiplicities for each of the irreducible modules $V(3), V(2,1)$ and $V(1,1,1)$.

Proposition 17. *a)* $H^0_0(3) = V(3)$ *and* $H^1_0(3) \cong 2gV(3)$*;*
b) $H^1_0(2,1) \cong 2gV(2,1)$*,* $H^0_0(2,1) = H^6_0(2,1) = 0$*;*
c) $H^2_0(1,1,1) \cong H^4_0(1,1,1) \cong \binom{2g+1}{2}V(1,1,1)$*,* $H^3_0(1,1,1) \cong (\binom{2g+2}{3}+2g)V(1,1,1)$ *and* $H^k_q(1,1,1) = 0$ *otherwise.*

The next proposition gives cohomology groups as a consequence of differentials positioned on the left, top and right sides of the trapezoid (see [**?**]).

Proposition 18. *a)* $H^2_0(3) \cong \binom{2g}{2}V(3)$*,* $H^1_1(3) = H^5_1(3) = H^6_0(3) = 0$*;*
b) $H^2_0(2,1) \cong 4g^2V(2,1)$*,* $H^1_1(2,1) = H^2_2(2,1) = H^3_2(2,1) = H^4_2(2,1) = H^5_1(2,1) = 0.$

Proof: Using Proposition 5, the differential $d : E^1_1(\mathcal{M}_g, 3) \to E^2_0(\mathcal{M}_g, 3)$ is injective. By restricting this differential to the modules $V(3)$ and $V(2,1)$ we obtain the result. The differentials $d : E^k_2(\mathcal{M}_g, 3) \to E^{k+1}_1(\mathcal{M}_g, 3)$ for $k = 2,3,4$ are injective using Proposition 6 and so are their respective restrictions on the modules $V(3)$ and $V(2,1)$. Using Proposition 7, the "right side" border of the trapezoid is acyclic, which gives us the vanishing of cohomology: $H^6_0(3) = H^5_1(3) = 0$ and $H^5_1(2,1) = H^4_2(2,1) = 0$. □

For the sub-complexes corresponding to isotypical components $V(3)$ and $V(2,1)$ we compute the remaining differentials separately.

Lemma 19. *a) The differential* $E^2_1(3) \to E^3_0(3)$ *is injective.*
b) For $k = 3,4$*, the differentials* $E^k_1(3) \to E^{k+1}_0(3)$ *are surjective.*

Proof: a) The differential $E^2_1(3) \cong 4gV(3) \to E^3_0(3) \cong \frac{2}{3}(2g^3 - 3g^2 + 4g)V(3)$ is split into two halves:

$$d : E^2_1(3) \cap (a_i, G_{jk})_{(i,j,k)} \to E^3_0(3) \cap [(a_i, w) \oplus (a_i, a_j, b_j)]_{(i,j)}$$

$$d : E^2_1(3) \cap (b_i, G_{jk})_{(i,j,k)} \to E^3_0(3) \cap [(b_i, w) \oplus (a_i, b_i, b_j)]_{(i,j)}$$

Computing the differentials of the elements from the bases of the first submodule we find: $d(a_l \otimes 1 \otimes 1G_{12} + a_l \otimes 1 \otimes 1G_{13} + 1 \otimes a_l \otimes 1G_{23}) = -\sum_{\sigma\in\mathcal{S}_3} \sigma.(w \otimes a_l \otimes 1)$ and

$d(1\otimes 1\otimes a_lG_{12} + 1\otimes a_l \otimes 1G_{13} + a_l \otimes 1\otimes 1G_{23}) = -\sum_{\sigma\in\mathcal{S}_3} \sigma.(w\otimes a_l\otimes 1) + \sum_{i=1}^{g}\sum_{\sigma\in\mathcal{S}_3} \sigma.(a_i\otimes b_i\otimes a_l).$

Hence the rank of differential restricted to one half is $2g$. Similar computation for the second subspace,
$d(b_l \otimes 1 \otimes 1G_{12} + b_l \otimes 1 \otimes 1G_{13} + 1 \otimes b_l \otimes 1G_{23}) = -\sum_{\sigma\in\mathcal{S}_3} \sigma.(w \otimes b_l \otimes 1)$ and

$d(1\otimes 1\otimes b_lG_{12} + 1\otimes b_l \otimes 1G_{13} + b_l \otimes 1\otimes 1G_{23}) = -\sum_{\sigma\in\mathcal{S}_3} \sigma.(w\otimes b_l\otimes 1) + \sum_{i=1}^{g}\sum_{\sigma\in\mathcal{S}_3} \sigma.(a_i\otimes b_i\otimes b_l),$

gives the maximal rank of $d : E^2_1(3) \to E^3_0(3)$.

b) For $k=3$, we consider the differential of generators of one-dimensional modules:
$d(w\otimes 1\otimes 1G_{12}+w\otimes 1\otimes 1G_{13}+1\otimes w\otimes 1G_{23}) = w\otimes w\otimes 1+w\otimes 1\otimes w+1\otimes w\otimes w$.
$d(a_i\otimes 1\otimes a_jG_{12}+a_i\otimes a_j\otimes 1G_{13}-a_j\otimes a_i\otimes 1G_{23}) = \sum_{\sigma\in\mathcal{S}_3}\sigma.(w\otimes a_i\otimes a_j)$,
for $1\leq i<j\leq g$.
$d(b_i\otimes 1\otimes b_jG_{12}+b_i\otimes b_j\otimes 1G_{13}-b_j\otimes b_i\otimes 1G_{23}) = \sum_{\sigma\in\mathcal{S}_3}\sigma.(w\otimes b_i\otimes b_j)$,
for $1\leq i<j\leq g$.
$d(a_i\otimes 1\otimes b_jG_{12}+a_i\otimes b_j\otimes 1G_{13}-b_j\otimes a_i\otimes 1G_{23}) = \sum_{\sigma\in\mathcal{S}_3}\sigma.(w\otimes a_i\otimes b_j)$,
for $1\leq i,j\leq g$.
$d(b_j\otimes 1\otimes a_iG_{12}+b_j\otimes a_i\otimes 1G_{13}-a_i\otimes b_j\otimes 1G_{23}) = \sum_{\sigma\in\mathcal{S}_3}\sigma.(w\otimes b_j\otimes a_i)$,
for $1\leq i,j\leq g$.
The images above make a basis of the one-dimensional modules in the target space $E_0^4(3)$ corresponding to the marks (w,a_i,a_j), (w,b_i,b_j), (w,a_i,b_j), and $(w,w,1)$ respectively, hence d is surjective.

For $k=4$, the surjectivity of $d:E_1^4(3)\to E_0^5(3)$ is a consequence of the acyclicity of the subcomplexes $\bigoplus_{|A|=2}E_1^{Top}(A,a_i)$ and $\bigoplus_{|A|=2}E_1^{Top}(A,b_i)$ (see Proposition 8) for all $1\leq i,j\leq g$. □

We are now able to write the double Poincaré polynomial for $H_*^*(3)$:

Proposition 20. *The double Poincaré polynomial of $H_*^*(3)$ for $g\geq 2$ is given by*

$$P_{H_*^*(3)}(s,t) = 1+2gt+(2g^2-g)t^2+\frac{2}{3}(2g^3-3g^2-2g)t^3+((2g^2+g+1)t^3+2gt^4)s.$$

For $g=1$ we have

$$P_{H_*^*(3)}(s,t) = 1+2t+t^2+(2t^2+4t^3+2t^4)s = (1+t)^2(1+2st^2).$$

The expressions differ due to the additional modules (in the case of $g\geq 2$) in exterior degree 0 and total degree 3. An application of the transfer theorem gives us the Poincaré polynomial of the unordered configuration space for Riemann surfaces.

Corollary 21. *The double Poincaré polynomial of $C(\mathcal{M}_g,3))$ for $g\geq 2$ is:*

$$P_{H_*^*(C(\mathcal{M}_g,3))}(s,t) = 1+2gt+(2g^2-g)t^2+\frac{2}{3}(2g^3-3g^2-2g)t^3+((2g^2+g+1)t^3+2gt^4)s.$$

For $g=1$ we have

$$P_{H^*(C(\mathbb{T}^2,3)}(s,t) = (1+t)^2(1+2st^2).$$

Corollary 22. *The Poincaré polynomial of $C(\mathcal{M}_g,3))$ for $g\geq 2$ is:*

$$P_{H_*^*(C(\mathcal{M}_g,3))}(t) = 1+2gt+(2g^2-g)t^2+\frac{1}{3}(4g^3-g+3)t^3+2gt^4.$$

For $g=1$ we have

$$P_{H^*(C(\mathbb{T}^2,3)}(t) = 1+2t+3t^2+4t^3+2t^4.$$

In the subcomplex $E_*^*(2,1)$, the remaining differentials are dealt with in the following lemma.

Lemma 23. *a) The differential $E_1^2(2,1) \to E_0^3(2,1)$ is injective.*
b) For $k = 3, 4$, the differentials $E_1^k(2,1) \to E_0^{k+1}(2,1)$ are surjective.

Proof: a) As in the case for $E_1^2(3)$, the differential can be split into two halves, one defined on the subspace generated by marks a_i and another defined on the subspace generated by marks b_i, and each of them can be further split into two families: one, in which the factor of positive degree in the coefficients is in positions i-corresponding to the indices of G_{ij}, while in the other family, this factor is not positioned at i or j places. Namely, for all $1 \leq i \leq g$, we obtain a decomposition into irreducible $V(2,1)$ modules:
$V(2,1) \cong \langle a_i \otimes 1 \otimes 1G_{12} - a_i \otimes 1 \otimes 1G_{13}, a_i \otimes 1 \otimes 1G_{12} - 1 \otimes a_i \otimes 1G_{23} \rangle$
and $V(2,1) \cong \langle 1 \otimes 1 \otimes a_i G_{12} - 1 \otimes a_i \otimes 1G_{13}, 1 \otimes 1 \otimes a_i G_{12} - a_i \otimes 1 \otimes 1G_{23} \rangle$.
Computing the differential we obtain
$d(a_i \otimes 1 \otimes 1G_{12} - a_i \otimes 1 \otimes 1G_{13}) = -a_i \otimes w \otimes 1 - w \otimes a_i \otimes 1 + a_i \otimes 1 \otimes w + w \otimes 1 \otimes a_i$,
$d(a_i \otimes 1 \otimes 1G_{12} - 1 \otimes a_i \otimes 1G_{23}) = -a_i \otimes w \otimes 1 - w \otimes a_i \otimes 1 + 1 \otimes a_i \otimes w + 1 \otimes w \otimes a_i$,
$d(1 \otimes 1 \otimes a_i G_{12} - 1 \otimes a_i \otimes 1G_{13}) = -w \otimes 1 \otimes a_i - \sum_{j=1}^{g} b_j \otimes a_j \otimes a_i + \sum_{j=1}^{g} a_j \otimes b_j \otimes a_i - 1 \otimes w \otimes$
$a_i + w \otimes a_i \otimes 1 - \sum_{ij=1}^{g} b_j \otimes a_i \otimes a_j + \sum_{j=1}^{g} a_j \otimes a_i \otimes b_j + 1 \otimes a_i \otimes w$,
$d(1 \otimes 1 \otimes a_i G_{12} - a_i \otimes 1 \otimes 1G_{23}) = -w \otimes 1 \otimes a_i - \sum_{j=1}^{g} b_j \otimes a_j \otimes a_i + \sum_{j=1}^{g} a_j \otimes b_j \otimes a_i - 1 \otimes w \otimes$
$a_i + a_i \otimes w \otimes 1 + \sum_{j=1}^{g} a_i \otimes b_j \otimes a_j - \sum_{j=1}^{g} a_i \otimes a_j \otimes b_j + a_i \otimes 1 \otimes w$.
The composition

$$\bigoplus_i \mathbb{Q}\langle a_i \otimes 1 \otimes 1G_{12} - a_i \otimes 1 \otimes 1G_{13}, a_i \otimes 1 \otimes 1G_{12} - 1 \otimes a_i \otimes 1G_{23}, 1 \otimes 1 \otimes a_i G_{12} - 1 \otimes a_i \otimes 1G_{13} \rangle \xrightarrow{d}$$

$$\xrightarrow{d} E_0^3 \xrightarrow{pr} \mathbb{Q}\langle w \otimes a_i \otimes 1, w \otimes 1 \otimes a_i, a_i \otimes w \otimes 1 \rangle$$

for each component corresponding to $1 \leq i \leq g$, has the following 3×3 invertible matrix

$$\begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}.$$

b)We can compute directly the rank of the differential $d : E_1^3(2,1) \cong (4g^2+2)V(2,1) \to E_0^4(2,1) \cong (4g^2+1)V(2,1)$. The subspaces $d(G(w)), d(G(a_i a_j))$ and $d(G(b_i b_j))$ are non-zero and included in the $V(2,1)$ components of the subalgebras generated by w, by w and a_i and by w and b_i respectively, for all $1 \leq i \leq g$. The subspace $d(G(a_i b_j) \oplus G(b_j a_i))$ is not included in these subalgebras. It has dimension four, and it is isomorphic to $2V(2,1)$, because the composition:

$$\mathbb{Q}\langle a_i \otimes 1 \otimes b_j G_{12} - a_i \otimes b_j \otimes 1G_{13}, a_i \otimes 1 \otimes b_j G_{12} + b_j \otimes a_i \otimes 1G_{23}, b_j \otimes 1 \otimes a_i G_{12} - b_j \otimes a_i \otimes 1G_{13} \rangle \xrightarrow{d}$$

$$\xrightarrow{d} E_0^4 \xrightarrow{pr} \mathbb{Q}\langle w \otimes a_i \otimes b_j, w \otimes b_j \otimes a_i, a_i \otimes w \otimes b_j \rangle$$

is an isomorphism:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

For $k = 4$, the differential splits into two halves, one with values in subalgebra generated by (a_i, w) and another with values in subalgebra generated by (b_i, w). It is sufficient to show that in each half there is a non-zero differential (for each $1 \leq i \leq g$):

$$d(w \otimes 1 \otimes a_i G_{12} - w \otimes a_i \otimes 1 G_{13}) = w \otimes a_i \otimes w - w \otimes w \otimes a_i,$$

$$d(w \otimes 1 \otimes b_i G_{12} - w \otimes b_i \otimes 1 G_{13}) = w \otimes b_i \otimes w - w \otimes w \otimes b_i.$$

□

We are now able to write the double Poincaré polynomial of $H_*^*(2,1)$.

Proposition 24. *The double Poincaré polynomial of $H_*^*(2,1)$ for $g \geq 1$ is given by:*

$$P_{H_*^*(2,1)}(t,s) = 2gt + 4g^2t^2 + \frac{2}{3}(4g^3 - g)t^3.$$

Now we can conclude the proof of Theorem 1.2:

Proof of Theorem 1.2. This is a consequence of the Poincaré polynomials of $H_*^*(3)$ and $H_*^*(2,1)$ and propositions 17 and 18. □

The double Poincaré polynomial is given in the following corollary, and it coincides with the polynomial given in [4].

Corollary 25. *The double Poincaré polynomial for $g \geq 2$ is:*

$$P_{H^*(F(\mathcal{M}_g,3))}(s,t) = 1 + 6gt + 12g^2t^2 + (8g^3 + (2g^2 + g + 1)s)t^3 + (2g^2 + g + 2gs)t^4.$$

For $g = 1$ we have:

$$P_{H^*(F(\mathbb{T}^2,3))}(s,t) = 1 + 6t + 12t^2 + 10t^3 + 3t^4 + 2s(t^2 + 2t^3 + t^4) =$$

$$= (1+t)^3(1+3t) + 2(1+t)^2t^2s.$$

Corollary 26. *The Poincaré polynomial for $g \geq 2$ is:*

$$P_{H^*(F(\mathcal{M}_g,3))}(t) = 1 + 6gt + 12g^2t^2 + (8g^3 + 2g^2 + g + 1)t^3 + (2g^2 + 3g)t^4.$$

For $g = 1$ we have:

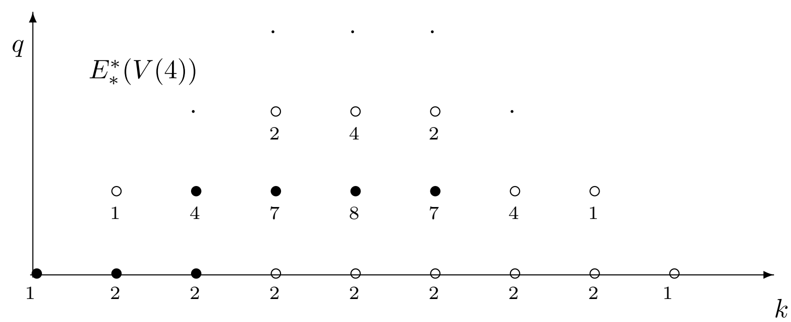
$$P_{H^*(F(\mathbb{T}^2,3))}(t) = 1 + 6t + 14t^2 + 14t^3 + 5t^4 = (1+t)^2(1 + 4t + 5t^2).$$

The polynomial above coincides with the one given in [5].

5 Cohomology of the unordered configuration spaces of 4 points on the torus

In this section we compute Betti numbers for the 4-point unordered configuration space of the torus $\mathbb{T}^2$. The structure of the $\mathcal{S}_4$-invariant part of the Križ model $E(\mathbb{T}^2, 4)$, denoted by $E^*_*(4)$, is given. In the following diagram the numbers denote the multiplicities of the irreducible $V(4)$ module (in this case, equal to dimensions of the corresponding isotypical components).

Proposition 27. *The $\mathcal{S}_4$ structure of the Križ model for the torus is given by the following diagram:*



Proof: For the component $E^*_0(4)$, this is done by computing the characters of subspaces directly and using the inner product of characters to obtain multiplicities for $V(4)$. For $E^*_1(4)$ and for $E^*_2(4)$ one can compute the sum of the $\mathcal{S}_4$-orbit of the monomials μ in the canonical basis $\sum_{\sigma\in\mathcal{S}_4} \sigma\mu$, see Propositions 29 and 30. ∎

The following contributions to cohomology are clear from the diagram (part 1) and using Propositions 5 and 7 we obtain part 2).

Proposition 28. *1) $H^0_0(4) \cong V(4)$ and $H^1_0(4) \cong 2V(4)$.*
2) $H^2_0 \cong V(4)$ and $H^7_1(4) = H^8_0(4) = 0$.

The remaining differentials for the $\mathcal{S}_4$-invariant part are computed as follows:

Proposition 29. *The differentials $d : E^k_1 \to E^{k+1}_0$, for total degree $2 \leq k \leq 6$, are surjective.*

Proof: We show surjectivity using a direct computation for each degree $2 \leq k \leq 6$. For $k = 2$, the differential is surjective because

$$d(\sum_{\sigma\in\mathcal{S}_4} \sigma(a \otimes 1 \otimes 1 \otimes 1)G_{12}) = -4 \sum_{\sigma\in\mathcal{A}_4} \sigma(a \otimes w \otimes 1 \otimes 1),$$
$$d(\sum_{\sigma\in\mathcal{S}_4} \sigma(b \otimes 1 \otimes 1 \otimes 1)G_{12}) = -4 \sum_{\sigma\in\mathcal{A}_4} \sigma(b \otimes w \otimes 1 \otimes 1).$$

In total degree $k = 3$, consider the differentials of the invariant vectors:

$$d(\sum_{\sigma\in\mathcal{S}_4} \sigma(w \otimes 1 \otimes 1 \otimes 1)G_{12}) = 4 \sum_{\sigma\in\mathcal{A}_4} \sigma(w \otimes w \otimes 1 \otimes 1),$$
$$d(\sum_{\sigma\in\mathcal{S}_4} \sigma(a \otimes 1 \otimes b \otimes 1)G_{12}) = 2 \sum_{\sigma\in\mathcal{S}_4} \sigma(a \otimes w \otimes b \otimes 1).$$

For $k=4$, taking the orbits of elements $w\otimes 1\otimes a\otimes 1G_{12}$ and $w\otimes 1\otimes b\otimes 1G_{12}$, and computing their differentials, we obtain:
$d(\sum_{\sigma\in\mathcal{S}_4}\sigma(w\otimes 1\otimes a\otimes 1)G_{12})=\sum_{\sigma\in\mathcal{S}_4}\sigma(a\otimes w\otimes w\otimes 1)$,
$d(\sum_{\sigma\in\mathcal{S}_4}\sigma(w\otimes 1\otimes b\otimes 1)G_{12})=\sum_{\sigma\in\mathcal{S}_4}\sigma(b\otimes w\otimes w\otimes 1)$.
For $k=5$, taking differential of the orbits of $w\otimes 1\otimes w\otimes 1G_{12}$ and $w\otimes 1\otimes a\otimes bG_{12}$ we have:
$d(\sum_{\sigma\in\mathcal{S}_4}\sigma(w\otimes 1\otimes w\otimes 1)G_{12})=2\sum_{\sigma\in\mathcal{A}_4}\sigma(w\otimes w\otimes w\otimes 1)$,
$d(\sum_{\sigma\in\mathcal{S}_4}\sigma(w\otimes 1\otimes a\otimes b)G_{12})=2\sum_{\sigma\in\mathcal{A}_4}\sigma(w\otimes w\otimes a\otimes b)$.
Lastly, the differential in degree $k=6$ is surjective because:
$d(\sum_{\sigma\in\mathcal{S}_4}\sigma(w\otimes 1\otimes a\otimes w)G_{12})=-\sum_{\sigma\in\mathcal{S}_4}\sigma(w\otimes w\otimes w\otimes a)$,
$d(\sum_{\sigma\in\mathcal{S}_4}\sigma(w\otimes 1\otimes b\otimes w)G_{12})=-\sum_{\sigma\in\mathcal{S}_4}\sigma(w\otimes w\otimes w\otimes b)$. □

Proposition 30. *The differentials $d:E_2^k\to E_1^{k+1}$, for total degree $3\leq k\leq 5$, are injective.*

Proof: For $k=3$, the two invariant vectors are:
$(a\otimes 1\otimes 1\otimes 1-1\otimes 1\otimes a\otimes 1)G_{12}G_{34}+(a\otimes 1\otimes 1\otimes 1-1\otimes a\otimes 1\otimes 1)G_{13}G_{24}+(a\otimes 1\otimes 1\otimes 1-1\otimes a\otimes 1\otimes 1)G_{23}G_{14}$,
$(b\otimes 1\otimes 1\otimes 1-1\otimes 1\otimes b\otimes 1)G_{12}G_{34}+(b\otimes 1\otimes 1\otimes 1-1\otimes b\otimes 1\otimes 1)G_{13}G_{24}+(b\otimes 1\otimes 1\otimes 1-1\otimes b\otimes 1\otimes 1)G_{23}G_{14}$.
Taking the differential we obtain:
$\sum_{\sigma\in\mathcal{A}_4}\sigma(a\otimes 1\otimes w\otimes 1)G_{12}+\sum_{\sigma\in\mathcal{A}_4}\sigma(a\otimes 1\otimes b\otimes a)G_{12}-\sum_{\sigma\in\mathcal{A}_4}\sigma(1\otimes 1\otimes a\otimes w)G_{12}$ $\sum_{\sigma\in\mathcal{A}_4}\sigma(b\otimes 1\otimes w\otimes 1)G_{12}+\sum_{\sigma\in\mathcal{A}_4}\sigma(b\otimes 1\otimes a\otimes b)G_{12}-\sum_{\sigma\in\mathcal{A}_4}\sigma(1\otimes 1\otimes b\otimes w)G_{12}$ respectively, hence d is injective.
Consider the differentials of the basis of invariant vectors in $E_2^4(4)$:
$d((w\otimes 1\otimes 1\otimes 1-1\otimes 1\otimes w\otimes 1)G_{12}G_{34}+(w\otimes 1\otimes 1\otimes 1-1\otimes w\otimes 1\otimes 1)G_{13}G_{24}+(w\otimes 1\otimes 1\otimes 1-1\otimes w\otimes 1\otimes 1)G_{23}G_{14})=-\frac{1}{2}\sum_{\sigma\in\mathcal{S}_4}\sigma(w\otimes 1\otimes w\otimes 1)G_{12}+\sum_{\sigma\in\mathcal{A}_4}\sigma(w\otimes 1\otimes b\otimes a)G_{12}+\frac{1}{2}\sum_{\sigma\in\mathcal{A}_4}\sigma(1\otimes 1\otimes w\otimes w)G_{12}$,
$d(a\otimes 1\otimes a\otimes 1G_{12}G_{34}+a\otimes a\otimes 1\otimes 1G_{13}G_{24}+a\otimes a\otimes 1\otimes 1G_{23}G_{14})=-\sum_{\sigma\in\mathcal{A}_4}\sigma(a\otimes 1\otimes a\otimes w)G_{12}$,
$d(b\otimes 1\otimes b\otimes 1G_{12}G_{34}+b\otimes b\otimes 1\otimes 1G_{13}G_{24}+b\otimes b\otimes 1\otimes 1G_{23}G_{14})=-\sum_{\sigma\in\mathcal{A}_4}\sigma(b\otimes 1\otimes b\otimes w)G_{12}$,
$d((a\otimes 1\otimes b\otimes 1-b\otimes 1\otimes a\otimes 1)G_{12}G_{34}+(a\otimes b\otimes 1\otimes 1-b\otimes a\otimes 1\otimes 1)G_{13}G_{24}+(a\otimes b\otimes 1\otimes 1-b\otimes a\otimes 1\otimes 1)G_{23}G_{14})=-\frac{1}{2}\sum_{\sigma\in\mathcal{S}_4}\sigma(a\otimes 1\otimes b\otimes w)G_{12}+\frac{1}{2}\sum_{\sigma\in\mathcal{S}_4}\sigma(b\otimes 1\otimes a\otimes w)G_{12}$. Clearly the rank of d is four. Lastly for total degree $k=5$, we have the following invariant vectors
$(a\otimes 1\otimes w\otimes 1-w\otimes 1\otimes a\otimes 1)G_{12}G_{34}+(a\otimes w\otimes 1\otimes 1-w\otimes a\otimes 1\otimes 1)G_{13}G_{24}+(a\otimes w\otimes 1\otimes 1-w\otimes a\otimes 1\otimes 1)G_{23}G_{14}$,
$(b\otimes 1\otimes w\otimes 1-w\otimes 1\otimes b\otimes 1)G_{12}G_{34}+(b\otimes w\otimes 1\otimes 1-w\otimes b\otimes 1\otimes 1)G_{13}G_{24}+(b\otimes w\otimes 1\otimes 1-w\otimes b\otimes 1\otimes 1)G_{23}G_{14}$.
Taking the differential we obtain:
$-\frac{1}{2}\sum_{\sigma\in\mathcal{S}_4}\sigma(w\otimes 1\otimes a\otimes w)+\frac{1}{2}\sum_{\sigma\in\mathcal{A}_4}\sigma(a\otimes 1\otimes w\otimes w)G_{12}$ and

$-\frac{1}{2}\sum_{\sigma\in\mathcal{S}_4}\sigma(w\otimes 1\otimes b\otimes w)+\frac{1}{2}\sum_{\sigma\in\mathcal{A}_4}\sigma(b\otimes 1\otimes w\otimes w)G_{12}$ respectively, hence the differential is injective. □

We can now give the double Poincaré polynomial of $C(\mathbb{T}^2,4)$:
Proof of Proposition 1.3. By an application of the transfer theorem we have $H^*_*(4)\cong H^*(C(\mathbb{T}^2,4))$. □

Corollary 31. *The Poincaré polynomial of the unordered 4-point configuration space of* $\mathbb{T}^2$ *is:*

$$P_{H^*(C(\mathbb{T}^2,4))}(s,t)=1+2t+3t^2+5t^3+4t^4+t^5.$$

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Syed Babar Ali School of Sciences and Engineering,
LUMS, Lahore-Pakistan.
E-mail: `centipedes.united@gmail.com`