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Invariants and rigidity of projective hypersurfaces

by
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Abstract

We compute some invariants based on Hilbert-Poincaré series associated to Milnor algebras. These computations involve some of the classical surfaces and 3-folds with different configurations of isolated singularities. As a by-product of a recent result of E. Sernesi, we give examples of classical hypersurfaces which are (or are not) projectively rigid. We also include a Singular program to compute these invariants and to decide if a singular projective hypersurface is nodal and projectively rigid.

Key Words: Projective hypersurfaces, singularities, Milnor algebra, Hilbert-Poincaré series, rigid hypersurface.
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1 Introduction

Let $S = \mathbb{C}[x_0, ..., x_n]$ be the graded ring of polynomials in $x_0, , ..., x_n$ with complex coefficients and denote by S_r the vector space of homogeneous polynomials in S of degree r. For any polynomial $f \in S_r$ we define the *Jacobian ideal* $J_f \subset S$ as the ideal spanned by the partial derivatives $f_0, ..., f_n$ of f with respect to $x_0, ..., x_n$. For $n = 2$ we use x, y, z instead of x_0, x_1, x_2 and f_x, f_y, f_z instead of f_0, f_1, f_2.

The Hilbert-Poincaré series of a graded S-module M of finite type is defined by

$$HP(M)(t) = \sum_{k \geq 0} \dim M_k \cdot t^k \tag{1.1}$$

and it is known to be a rational function of the form

$$HP(M)(t) = \frac{P(M)(t)}{(1-t)^{n+1}} = \frac{Q(M)(t)}{(1-t)^d}. \tag{1.2}$$

For any polynomial $f \in S_r$ we define the corresponding graded *Milnor* (or *Jacobian*) *algebra* by

$$M = M(f) = S/J_f. \tag{1.3}$$

In fact, such a Milnor algebra can be seen (up to a twist in grading) as the first (or the last) homology (or cohomology) of the Koszul complex of the partial derivatives $f_0, ..., f_n$ in S, see [2] or [3, Chapter 6].

One of our research aims will be to improve the bounds in Choudary-Dimca Theorem from [2], to get sharp estimates in many cases.

Choudary-Dimca Theorem Let $D : f = 0$ be a hypersurface in $\mathbb{P}^n$ of degree d with only isolated singularities, say located at the points $a_1, ..., a_p$ in $\mathbb{P}^n$. For any $q \geq T+1, T = (n+1)(d-2)$, one has

$$\dim M(f)_q = \tau(D) = \sum_{j=1,p} \tau(D, a_j)$$

where $\tau(D)$ is the global Tjurina number of the hypersurface D. In particular, the Hilbert polynomial $H(M(f))$ is constant and this constant is $\tau(D)$.

For a hypersurface $D : f = 0$ in $\mathbb{P}^n$ with isolated singularities as above we recall **four** invariants, introduced in [7] and add a new one, as follows.

Definition 1. *(i) The coincidence threshold $ct(D)$ defined as*

$$ct(D) = \max\{q \quad : \quad \dim M(f)_k = \dim M(f_s)_k \text{ for all } k \leq q\},$$

with f_s a homogeneous polynomial in S of degree $d = \deg f$ such that $D_s : f_s = 0$ is a smooth hypersurface in $\mathbb{P}^n$.

(ii) The stability threshold $st(D)$ defined as

$$st(D) = \min\{q \quad : \quad \dim M(f)_k = \tau(D) \text{ for all } k \geq q\}$$

where $\tau(D)$ is the total Tjurina number of D, i.e. the sum of all the Tjurina numbers of the singularities of D.

(iii) The minimal degree of a nontrivial syzygy $mdr(D)$ defined as

$$mdr(D) = \min\{q \quad : \quad H^n(K^*(f))_{q+n} \neq 0\}$$

where $K^*(f)$ is the Koszul complex of $f_0, ..., f_n$ with the natural grading.

(iv) We define the integer $def(D)$ =defect of D as

$$def(D) = \text{ the first non zero coefficient of the difference } S(t) - F(t)$$

where $S(t)$ (resp. $F(t)$) are the corresponding Hilbert-Poincaré series of Milnor Algebras $M(f)$ (resp. $M(f_s)$).

(v) A finite sequence of strictly positive real numbers $a_0, ..., a_q$ is said to be log-concave if $a_{i-1}a_{i+1} \leq a_i^2$ for all $i = 1, 2, ..., q-1$.

We define a new integer $lc(D)$ = the log-concavity of D as the maximal integer q such that the sequence $\dim M(f)_0, \ldots, \dim M(f)_q$ is a log-concave sequence. If $q \geq st(D)+1$ we put $lc(D) = \infty$.

Recall also that, for a finite set of points $\mathcal{N} \subset \mathbb{P}^n$, we denote by

$$\operatorname{def} S_m(\mathcal{N}) = |\mathcal{N}| - \dim S_m/\{h \in S_m \mid h(a) = 0 \text{ for any } a \in \mathcal{N}\},$$

the *defect (or superabundance) of the linear system of polynomials in S_m vanishing at the points in $\mathcal{N}$*, see [3], p. 207. This positive integer is called the *failure of $\mathcal{N}$ to impose independent conditions on homogeneous polynomials of degree m.*

This paper continues our research [6, 7, 8] by computing some invariants based on Hilbert-Poincaré series associated to Milnor algebras. Except for the explicit computations and the final Singular program, our main results are Proposition 1, Proposition 3 and Corollary 1 and 2 relating our invariants to the question of projective rigidity of hypersurfaces with isolated singularities.

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2 Computations for some projective hypersurfaces

In this section, in the first part we analyze two curves, of degree 4, one nodal and one cuspidal, in order to explain in detail our approach.
For each curve $C : f = 0$, we find and classify all the singularities, give the genus and compute the Hilbert-Poincaré series and our invariants.

Firstly, we will find all multiple points of $C : f(x, y, z) = 0$ by solving the system of equations:

$$f(x, y, z) = 0, \frac{\partial f}{\partial x}(x, y, z) = 0, \frac{\partial f}{\partial y}(x, y, z) = 0, \frac{\partial f}{\partial z}(x, y, z) = 0$$

Because f is homogenous of degree d, we have:

$x\frac{\partial f}{\partial x}(x, y, z) + y\frac{\partial f}{\partial y}(x, y, z) + z\frac{\partial f}{\partial z}(x, y, z) = d \cdot f(x, y, z).$

Obviously, if all partial derivatives of polynomial f vanish at p, then the polynomial f vanishes at p too. Therefore, for finding singular points it is enough to solve the system of equations:

$\frac{\partial f}{\partial x}(x, y, z) = 0, \frac{\partial f}{\partial y}(x, y, z) = 0, \frac{\partial f}{\partial z}(x, y, z) = 0.$

In this computation we will use the δ-invariant of a plane curve singularity. To determine this invariant, the simplest way is to use the Milnor-Jung formula:

$$\mu = 2\delta - r + 1, \tag{2.1}$$

where μ is the Milnor number and r the number of branches of the singularity. For example, for the A_3 singularity $x^2 - y^4 = 0$ we have $\mu = 3$, $r = 2$ and hence $\delta = 2$.

The Hilbert-Poincaré series is computed with two methods: combinatorial and based on a free resolution.

To compute invariants ct and def, we need the Hilbert-Poincaré series $F(t)$ for d degree Fermat curve: $x^d + y^d + z^d = 0$ (the same for any smooth d degree curve):
$F_d(t) = (1 + t + \ldots + t^{d-2})^3$ and for $d = 4$, $F_4(t) = 1 + 3t + 6t^2 + 7t^3 + 6t^4 + 3t^5 + t^6$.

It is a very hard work to obtain manually these informations, even if we consider curves and surfaces with low degree.

In the secont part, all the computations was made by our Singular program.

2.1 Lemniscate of Bernoulli and Cardioid

The Lemniscate of Bernoulli and the Cardioid are splendid curves of the fourth degree, with different topological type of singularity (but with the same singular points).

The **Lemniscate of Bernoulli** is a nodal curve with 3 nodes (type $3A_1$). The affine equation of the Lemniscate of Bernoulli looks like

$$F(x, y) = (x^2 + y^2)^2 - 2(x^2 - y^2).$$

Next, we will homogenize the defining polynomial and we get the defining polynomial of its associated projective curve:

$$L : f(x, y, z) = (x^2 + y^2)^2 - 2(x^2 - y^2)z^2 = 0.$$

Find the position of the singularities, i.e. solve the system:
$\frac{\partial f}{\partial x}(x, y, z) = 4x(x^2 + y^2) - 4xz^2 = 0$
$\frac{\partial f}{\partial y}(x, y, z) = 4yz^2 + 4y(x^2 + y^2) = 0$
$\frac{\partial f}{\partial z}(x, y, z) = -4(x^2 - y^2)z = 0$

For $z = 1$ we obtain the double point a=(0:0:1). For $z = 0$ (singularities on the line at infinity $L_\infty : z = 0$) we get two complex double points $b = (1 : i : 0)$ and $c = (1 : -i : 0)$. Hence 3 singularities a, b, c with multiplicity 2.

Determine the type of singularities We start with $a = (0 : 0 : 1)$.
With local coordinates at a: $u = \frac{x}{z}, v = \frac{y}{z}$, local equation at a is:
$g = 2(u^2 - v^2) - (u^2 + v^2)^2 = 0$.
Use the weights $w_1 = wt(u) = 1$ and $w_2 = wt(v) = 1$
Then g is semi-weighted homogenous of type (w_1, w_2, d) with $d = 2$
Prop (7.37), p 116 and Prop (7.27), p 112 [4] imply that $\mu(g) = \mu(u^2 - v^2) = 1$ so $g \equiv A_1$ is a node (the only singularity with $\mu = 1$).
For nodes it is known that $\delta(A_1) = 1$ and $r(A_1) = 2$ (two branches $u - v = 0$ and $u + v = 0$).
Let's treat now $b = (1 : i : 0)$
Local coordinates $u = \frac{x+iy}{x}, v = \frac{z}{x}$.
$\frac{y}{x} = -i(u - 1) \Leftrightarrow u = 1 + i\frac{y}{x}, v = \frac{z}{x}$.
we divide by $x^4 : (1 + (\frac{y}{x})^2)^2 - 2(1 - (\frac{y}{x})^2)(\frac{z}{x})^2 = 0$
Local equation at b: $g = \left[1 - (u - 1)^2\right]^2 - 2(1 + (u - 1)^2)v^2 =$

$= (1 - u^2 + 2u - 1)^2 - 2v^2 - 2v^2(1 - 2u + u^2) =$
$= 4u^2 - 4v^2 +$ higher degree terms.
Hence with the same weight as before g is semi-weighted homogenous and
$\mu(g) = \mu(4(u^2 - v^2)) = 1 \Rightarrow g \equiv A_1$.
Same works for c. *Computation of genus*
Genus of a smooth curve of degree $d = 4$ is:
$g_s = \frac{(d-1)(d-2)}{2} = \frac{6}{2} = 3$.
Genus of our singular curve is:
$g(L) = g_s - \sum_{x \in \{a,b,c\}} \delta(x) = 3 - 3 = 0$.
Hence L is rational (we have to check also that f is irreducible i.e. $f \neq f_1(x, y, z) \cdot f_2(x, y, z)$ with f_1, f_2 homogenous of degree $d_1 > 0, d_2 > 0, d_1 + d_2 = 4$).
Hence, the genus of L is 0 and L is irreducible. Thus, the curve L is rational parameterizable and:
$x(t) = 2(-3 - 2t + 2t^3 + 3t^4)/(5 + 12t + 30t^2 + 12t^3 + 5t^4)$
$y(t) = -2(-1 - 6t + 6t^3 + t^4)/(5 + 12t + 30t^2 + 12t^3 + 5t^4)$
is a rational parametrization for the affine equation.
To compute the Hilbert-Poincaré series, first recall that the quotient rings S/I and $S/LT(I)$ have the same series, for any monomial ordering, where LT(I) is the ideal of leading terms of the ideal I. The leading ideal of the jacobian J_f is:

$LI =< yz^4, y^2z^3, y^3z, xz^4, xyz^2, xy^2z, x^2z, x^2y, x^3 >$

For the graded Milnor algebra, $M = \oplus_{k \geq 0} M_k$ we show the bases for the homogeneous components: $M_0 = \{1\}, M_1 = \{z, y, x\}$,
$M_2 = \{z^2, yz, xz, y^2, xy, x^2\}, M_3 = (z^3, yz^2, xz^2, y^2z, xyz, y^3, xy^2\}$,
$M_4 = (z^4, yz^3, xz^3, y^2z^2, y^4, xy^3\}$ and $M_k = \{z^k, y^k, xy^{k-1}\}$ for all $k \geq 5$.

Finally, if we count the number of monomials in each homogeneous bases, we find the Hilbert-Poincaré series $S(t) = 1 + 3t + 6t^2 + 7t^3 + 6t^4 + 3(t^5 + \ldots$
Our invariants are: $\tau = 3$, $ct = 5$, $st = 5$, $mdr = 3$, $def = 2$ $lc = 5$ and because $mdr = 3$, we show three nontrivial linear independent relations between derivatives (syzygies), with polynomial coefficients of degree 3:
$(2y^3 + yz^2)f_x + (xz^2 - 2xy^2)f_y + 2xyzf_z = 0$
$(2xy^2 + xz^2)f_x + (yz^2 - 2x^2y)f_y + (x^2z - y^2z - z^3)f_z = 0$
$(2x^2y + yz^2)f_x + (xz^2 - 2x^3)f_y - 2xyzf_z = 0$

The **Cardioid** is a cuspidal curve, with 3 cusps, (type $3A_2$)
The affine equation of Cardioid is:
$F(x, y) = (x^2 + y^2 + x)^2 - (x^2 + y^2) = 0$

The projective ecuation is: $C : f(x, y, z) = (x^2 + y^2 + xz)^2 - (x^2 + y^2)z^2 = 0$.

Find the position of the singularities, i.e. solve the system:
$\frac{\partial f}{\partial x}(x, y, z) = 4x(x^2 + y^2 + xz) + 2x^2z + 2y^2z = 0$
$\frac{\partial f}{\partial y}(x, y, z) = 2y(2x^2 + 2y^2 + 2xz - z^2) = 0$
$\frac{\partial f}{\partial z}(x, y, z) = 2(x^2 + y^2 + xz)x - 2z(x^2 + y^2) = 0$

For $y = 0$ we obtain the double point $a = (0 : 0 : 1)$. For $z = 0$ we get two complex double points $b = (1 : i : 0)$ and $c = (1 : -i : 0)$. Hence 3 singularities a, b, c.

Determine the type of singularities

Consider the case $a = (0 : 0 : 1)$.
With local coordinates at a: $u = \frac{x}{z}, v = \frac{y}{z}$, local equation at a is:
$g = 2(u^2 + v^2 + u)^2 - (u^2 + v^2) = 2u^3 - v^2 + (u^4 + v^4 + 2u^2v^2 + 2uv^2)$..
Use the weights $w_1 = wt(u) = 2$ and $w_2 = wt(v) = 3$.
Then g is semi-weighted homogenous of type (w_1, w_2, d) with $d = 6$.
Prop (7.37), p 116 and Prop (7.27), p 112 [4] imply that $\mu(g) = \mu(2u^3 - v^2) = 2$ so $g \equiv A_2$ is a cusp.
Similar computations for b and c. It is known that $\delta(A_2) = 1$ and $r(A_2) = 1$
Computation genus as before, $g(C) = g_s - \sum_{x\in\{a,b,c\}} \delta(x) = 3 - 3 = 0$. Hence, the genus is 0, since the Cardioid is irreducible. Thus, the curve C is rational parameterizable and:
$x(t) = 2(-1 + 4t^2)/(1 + 4t^2)^2$
$y(t) = -8t/(1 + 4t^2)^2$ is a rational parametrization for the affine equation.
Here is the minimal graded free resolution of Milnor algebra M:

$$0 \to R_3 \xrightarrow{C} R_2 \xrightarrow{B} R_1 \xrightarrow{A} R_0 \to M \to 0 \tag{2.2}$$

where $R_0 = S$, $R_1 = S^3(-3)$, $R_2 = S^3(-5)$ and $R_3 = S(-6)$ and the matrix are:

A is 1×3 matrix

$$\begin{pmatrix} 2x^3 + 2xy^2 - 2y^2z & 4x^2y + 4y^3 + 4xyz - 2yz^2 & 6x^2z + 6y^2z \end{pmatrix}$$

B is 3×3 matrix

$$\begin{pmatrix} -2y^2 - 1/2z^2 & -2xy + yz & 1/2xz - 1/4z^2 \\ xy + 1/2yz & x^2 & 1/4yz \\ -y^2 + 1/6xz & -xy + 1/3yz & -1/6x^2 - 1/6y^2 + 1/12xz \end{pmatrix}$$

C is 3×1 matrix

$$\begin{pmatrix} -1/2x + 1/4z \\ 1/2y \\ -1/2z \end{pmatrix}$$

To get the formulas for the Hilbert-Poincaré series, we start with the resolution (2.2) and get

$$HP(M)(t) = HP(R_0)(t) - HP(R_1)(t) + HP(R_2)(t) - HP(R_3)(t).$$

Then we use the well-known formulas

$$HP(N \oplus N')(t) = HP(N)(t) + HP(N')(t), HP(N(-r))(t) = t^r HP(N)(t), HP(S)(t) = \frac{1}{(1-t)^3}$$

and we obtain:

$$HP(M)(t) = \frac{1 - 3t^3 + 3t^5 - t^6}{(1-t)^3} = \frac{1 + 2t + 3t^2 + t^3 - t^4}{1-t} = 1 + 3t + 6t^2 + 7t^3 + 6(t^4 + \dots$$

Our invariants are: $\tau = 6$, $ct = 4$, $st = 4$, $mdr = 2$, $def = 3$ $lc = 4$ and because $mdr = 2$, we show three nontrivial linear independent relations between derivatives (syzygies), with polynomial coefficients of degree 2:
$(xz - 6y^2)f_x + (6xy + 3yz)f_y - (3z^2 + 2xz)f_z = 0$
$(yz - 3xy)f_x + 3x^2 f_y + yzf_z = 0$
$(xz - 2x^2 - 2y^2)f_x + 3yzf_y + (4x^2 + 4y^2 + 4xz - 3z^2)f_z = 0.$

2.2 Computations for higher dimensional hypersurfaces

The classification of singular cubic surfaces is a well-established subject, and can be traced back to the work of Cayley and Schläfli over a century ago.

We use the modern notation for the combination of singularities on a given surface, following Bruce and Wall [1]. For each type of singularities, we give an equation defining a surface with this type and we compute the corresponding invariants for this equation.

Up to projective isomorphism there is exactly one surface of degree three with 4 nodes, type $4A_1$, see [1]. This surface is called Cayley cubic and satisfies the following equation: $4(x^3 + y^3 + z^3 + w^3) - (x + y + z + w)^3 = 0$ or $w(xy + xz + yz) + xyz = 0$.

Up to projective isomorphism there is also a unique cubic surface of type $3A_2$ (see [1]), one representative with projective equation: $x^3 + yzw = 0$.

We *conjecture that for cubic surfaces the invariants do not depend on the chosen equation*, but for the moment we have no proof for this claim.

For quartic surfaces, see example below.

Example 1. ***Two quartic surfaces of type*** $3A_3$ ***and different invariants***
Consider the quartic surfaces defined by $D_1 : f_1 = x(x^3 + y^3 + z^3) + w^4 = 0$ *and*
$D_2 : f_2 = x^2y^2 + y^2z^2 + z^2x^2 + w^4 = 0.$
These surfaces are both irreducible, not projectively rigid, have three singularities of type A_3 *as their singular loci and distinct Hilbert-Poincaré series and invariants.*
$S_1(t) = 1 + 4t + 10t^2 + 16t^3 + 19t^4 + 17t^5 + 13t^6 + 10t^7 + 9(t^8 + \dots$
$\tau = 9$, $ct = 4$, $st = 8$, $mdr = 2$, $def = 1$, $lc = 6$.
$S_2(t) = 1 + 4t + 10t^2 + 16t^3 + 19t^4 + 16t^5 + 12t^6 + 9(t^7 + \dots$
$\tau = 9$, $ct = 5$, $st = 7$, $mdr = 3$, $def = 2$, $lc = 7$.

• **Singularities of Cubic Surface in** P^3

In the following list, the cubic surfaces are arranged in order given by the type of the singularities.

$A_1 : f = (x^2 + y^2 + z^2 + xy + xz + yz)w + 2xyz = 0$
$S(t) = 1 + 4t + 6t^2 + 4t^3 + (t^4 + \dots$
$\tau = 1$, $ct = 4$, $st = 4$, $mdr = 3$, $def = 1$, $lc = 4$.
$2A_1 : f = xzw + (z + w)y^2 + x^3 + x^2y + xy^2 + y^3 = 0$
$S(t) = 1 + 4t + 6t^2 + 4t^3 + 2(t^4 + \dots$
$\tau = 2$, $ct = 3$, $st = 4$, $mdr = 2$, $def = 1$, $lc = 4$.
$A_2 : f = (x + y + z)(x + 2y + 3z)w + xyz = 0$
$S(t) = 1 + 4t + 6t^2 + 4t^3 + 2(t^4 + \dots$
$\tau = 2$, $ct = 3$, $st = 4$, $mdr = 2$, $def = 1$, $lc = 4$.

$3A_1 : f = y^3 + y^2(x+z+w) + 4xzw; = 0$
$S(t) = 1 + 4t + 6t^2 + 4t^3 + 3(t^4 + \dots$
$\tau = 3,\ \ ct = 3,\ \ st = 4,\ \ mdr = 2,\ \ def = 2,\ \ lc = 3.$
$A_1 + A_2 : f = x^3 + y^3 + x^2y + xy^2 + y^2z + xzw = 0$
$S(t) = 1 + 4t + 6t^2 + 4t^3 + 3(t^4 + \dots$
$\tau = 3,\ \ ct = 3,\ \ st = 4,\ \ mdr = 2,\ \ def = 2,\ \ lc = 3.$
$A_3 : f = xzw + (x+z)(y^2 - x^2 - z^2) = 0$
$S(t) = 1 + 4t + 6t^2 + 4t^3 + 3(t^4 + \dots$
$\tau = 3,\ \ ct = 3,\ \ st = 4,\ \ mdr = 2,\ \ def = 2,\ \ lc = 3.$
$4A_1$ **(Cayley surface) :** $f = w(xy + xz + yz) + xyz = 0$
$S(t) = 1 + 4t + 6t^2 + 4(t^3 + \dots$
$\tau = 4,\ \ ct = 3,\ \ st = 3,\ \ mdr = 2,\ \ def = 3,\ \ lc = 3.$
$2A_1 + A_2 : f = wxz + y^2(x+y+z) = 0$
$S(t) = 1 + 4t + 6t^2 + 4(t^3 + \dots$
$\tau = 4,\ \ ct = 3,\ \ st = 3,\ \ mdr = 2,\ \ def = 3,\ \ lc = 3.$
$A_1 + A_3 : f = wxz + (x+z)(y^2 - x^2) = 0$
$S(t) = 1 + 4t + 6t^2 + 4(t^3 + \dots$
$\tau = 4,\ \ ct = 3,\ \ st = 3,\ \ mdr = 2,\ \ def = 3,\ \ lc = 3.$
$2A_2 : f = x^3 + y^3 + x^2y + xy^2 + xzw = 0$
$S(t) = 1 + 4t + 6t^2 + 5t^3 + 4(t^4 + \dots$
$\tau = 4,\ \ ct = 2,\ \ st = 4,\ \ mdr = 1,\ \ def = 1,\ \ lc = 4.$
$A_4 : f = y^2z + yx^2 - z^3 + xzw = 0$
$S(t) = 1 + 4t + 6t^2 + 4(t^3 + \dots$
$\tau = 4,\ \ ct = 3,\ \ st = 3,\ \ mdr = 2,\ \ def = 3,\ \ lc = 3.$
$D_4 : f = w(x+y+z)^2 + xyz = 0$
$S(t) = 1 + 4t + 6t^2 + 4(t^3 + \dots$
$\tau = 4,\ \ ct = 3,\ \ st = 3,\ \ mdr = 2,\ \ def = 3,\ \ lc = 3.$
$2A_1 + A_3 : f = wxz + (x+z)y^2 = 0$
$S(t) = 1 + 4t + 6t^2 + 5(t^3 + \dots$
$\tau = 5,\ \ ct = 2,\ \ st = 3,\ \ mdr = 1,\ \ def = 1,\ \ lc = 3.$
$A_1 + 2A_2 : f = wxz + xy^2 + y^3 = 0$
$S(t) = 1 + 4t + 6t^2 + 5(t^3 + \dots$
$\tau = 5,\ \ ct = 2,\ \ st = 3,\ \ mdr = 1,\ \ def = 1,\ \ lc = 3.$
$A_1 + A_4 : f = wxz + y^2z + yx^2 = 0$
$S(t) = 1 + 4t + 6t^2 + 5(t^3 + \dots$
$\tau = 5,\ \ ct = 2,\ \ st = 3,\ \ mdr = 1,\ \ def = 1,\ \ lc = 3.$
$A_5 : f = wxz + y^2z + x^3 - z^3 = 0$
$S(t) = 1 + 4t + 6t^2 + 5(t^3 + \dots$
$\tau = 5,\ \ ct = 2,\ \ st = 3,\ \ mdr = 1,\ \ def = 1,\ \ lc = 3.$
$D_5 : f = wx^2 + xz^2 + y^2z; = 0$
$S(t) = 1 + 4t + 6t^2 + 5(t^3 + \dots$
$\tau = 5,\ \ ct = 2,\ \ st = 3,\ \ mdr = 1,\ \ def = 1,\ \ lc = 3.$
$A_1 + A_5 : f = wxz + y^2z + x^3 = 0$
$S(t) = 1 + 4t + 6(t^2 + \dots$

$\tau = 6,\quad ct = 2,\quad st = 2,\quad mdr = 1,\quad def = 2,\quad lc = \infty.$

$3A_2 : f = x^3 + yzw = 0$
$S(t) = 1 + 4t + 6(t^2 + \dots$
$\tau = 6,\quad ct = 2,\quad st = 2,\quad mdr = 1,\quad def = 2,\quad lc = \infty.$

$E_6 : f = wx^2 + xz^2 + y^3 = 0$
$S(t) = 1 + 4t + 6(t^2 + \dots$
$\tau = 6,\quad ct = 2,\quad st = 2,\quad mdr = 1,\quad def = 2,\quad lc = \infty.$

- **Kummer quartic surface with 16 nodes**

$f = x^4 + y^4 + z^4 - y^2z^2 - z^2x^2 - x^2y^2 - x^2w^2 - y^2w^2 - z^2w^2 + w^4 = 0$
$S(t) = 1 + 4t + 10t^2 + 16t^3 + 19t^4 + 16(t^5 + \dots$
$\tau = 16,\quad ct = 5,\quad st = 5,\quad mdr = 3,\quad def = 6,\quad lc = 5.$

- **Octic nodal surface with 144 nodes**

$f = 16(x^8 + y^8 + z^8 + w^8) + 224(x^4y^4 + x^4z^4 + x^4w^4 + y^4z^4 + y^4w^4 + z^4w^4) + 2688x^2y^2z^2w^2 - 9(x^2 + y^2 + z^2 + w^2)^4.$
$S(t) = 1 + 4t + 10t^2 + 20t^3 + 35t^4 + 56t^5 + 84t^6 + 116t^7 + 149t^8 + 180t^9 + 206t^{10} + 224t^{11} + 231t^{12} + 224t^{13} + 206t^{14} + 180t^{15} + 158t^{16} + 148t^{17} + 145t^{18} + 144(t^{19} + \dots$
$\tau = 144,\quad ct = 15,\quad st = 19,\quad mdr = 9,\quad def = 9,\quad lc = 15.$

- **A non nodal octic surface (van Straten)**

$f = x^6y^2 - 2x^4y^4 + x^2y^6 - x^6z^2 - 3x^4y^2z^2 - 3x^2y^4z^2 - y^6z^2 + 5x^4z^4 + 10x^2y^2z^4 + 5y^4z^4 - 8x^2z^6 - 8y^2z^6 - 508z^8 + 1024z^6w^2 - 640z^4w^4 + 128z^2w^6 - 8w^8.$
$\tau(S/J) = 124, \tau(S/I) = 100$ so this surface is not nodal
S(t)$=1 + 4t + 10t^2 + 20t^3 + 35t^4 + 56t^5 + 84t^6 + 116t^7 + 149t^8 + 180t^9 + 206t^{10} + 224t^{11} + 231t^{12} + 224t^{13} + 206t^{14} + 180t^{15} + 157t^{16} + 139t^{17} + 128t^{18} + 125t^{19} + 124(t^{20} + \dots$
$\tau = 124,\quad ct = 15,\quad st = 20,\quad mdr = 9,\quad def = 8,\quad lc = 16.$

- **Singularities in P^4**

Cubic 3-fold with 10 **nodes**
$f = x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_0x_1x_2 - x_0x_3x_4 = 0$
$S(t) = 1 + 5t + 10(t^2 + \dots$
$\tau = 10,\quad ct = 3,\quad st = 2,\quad mdr = 2,\quad def = 5,\quad lc = \infty.$

Burkhardt quartic 3-fold with 45 **nodes**
$f = x_0^4 - x_0(x_1^3 + x_2^3 + x_3^3 + x_4^3) + 3x_1x_2x_3x_4$
$S(t) = 1 + 5t + 15t^2 + 30t^3 + 45t^4 + 51t^5 + 45(t^6 + \dots$
$\tau = 45,\quad ct = 6,\quad st = 6,\quad mdr = 4,\quad def = 15,\quad lc = 6.$

Quintic with 125 **nodes**
$f = x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5x_0x_1x_2x_3x_4 = 0$
$S(t) = 1 + 5t + 15t^2 + 35t^3 + 65t^4 + 101t^5 + 135t^6 + 155t^7 + 155t^8 + 135t^9 + 125(t^{10} + \dots$
$\tau = 125,\quad ct = 9,\quad st = 10,\quad mdr = 6,\quad def = 24,\quad lc = 9.$

- **Nodal cubic with** 15 **nodes in** $\mathbb{P}^5$

$f = 4(x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3) - (x_0 + x_1 + x_2 + x_3 + x_4 + x_5)^3 = 0$
$S(t) = 1 + 6t + 15t^2 + 20t^3 + 15(t^4 + \dots$
$\tau = 15,\quad ct = 4,\quad st = 4,\quad mdr = 3,\quad def = 9,\quad lc = 4.$

3 Some properties for invariants

If the hypersurface is nodal, Tjurina number τ is the number of nodes.

To decide if the hypersurface f is nodal, we consider the following ideals: $J = J_f$ and $I = I_f$ the radical of J and the invariants $\tau(S/J) = \dim S_m/J_m$ for $m >> 0$ and $\tau(S/I) = \dim S_m/I_m$ for $m >> 0$. The invariant $\tau(S/J)$ represents the total Tjurina number and for a nodal hypersurfaces represents the number of nodes. The invariant $\tau(S/I)$ represent the number of singularities and if we have $\tau(S/J) = \tau(S/I)$, then the hypersurface $f = 0$ is nodal.

Recall that Hilbert-Poincaré series of a graded S-module M of finite type is defined by

$$HP(M)(t) = \sum_{k \geq 0} (\dim M_k) t^k \tag{3.1}$$

and that for smooth hypersurface we have

$$F(t) = HP(M(f_s)) = \frac{(1-t^{d-1})^{n+1}}{(1-t)^{n+1}} = (1 + t + t^2 + \ldots + t^{d-2})^{n+1}, \tag{3.2}$$

see for instance [4], p. 112. In particular, if we set $T = T(n, d) = (n + 1)(d - 2)$, it follows that $M(f_s)_j = 0$ for $j > T$ and $\dim M(f_s)_j = \dim M(f_s)_{T-j}$ for $0 \leq j \leq T$.

Smooth hypersurfaces have the same Hilbert-Poincaré series associated to Fermat type

$$F(t) = (1 + t + t^2 + \ldots + t^{d-2})^{n+1} = \sum_{k=0}^{k=T} a_k t^k$$

.

In a very important recent paper [10], J. Huh has constructed for any homogeneous polynomial $h \in S_d$ a log-concave sequence $\mu^i(h)$, $i = 0, ..., n$ via the mixed multiplicities of the pair of ideals J_h and $m = (x_0, ..., x_n)$ in S and showed surprising relation to the topology of the corresponding hypersurface $V(h) : h = 0$ in $\mathbb{P}^n$.

It is natural therefore to ask whether our sequences $\dim M(f)_i$ give rise to some log-concave sequences. Here is the result.

Proposition 1. *Let $f \in S_d$ and set as usual $T = (n + 1)(d - 2)$.*

- *If the hypersurface $D : f = 0$ is smooth, then the sequence $a_k = \dim M(f)_k$, $k = 0, ..., T$ is log-concave.*
- *If the hypersurface $D : f = 0$ is singular, then*

  the sequence $a_k = \dim M(f)_k$, $k = 0, ..., ct(D)$ is log-concave, but the extended sequence $a_k = \dim M(f)_k$, $k = 0, ..., ct(D), ct(D) + 1$ may be no longer log-concave even for nodal plane curves.

Proof. For the smooth case, the coefficients a_k appear as the coefficients of the polynomial

$$(1 + t + t^2 + ... + t^{d-2})^{n+1}.$$

In other words, they are obtained by applying $n+1$ convolution products of sequences, starting with the constant sequence $1, 1, ..., 1$. Since this constant sequence is clearly log-concave and the convolution product preserves the log-concavity, see [10], the first claim is proved.

The first part of the second claim is obvious by the definition of $ct(D)$. To justify the second part, consider the case where the curve is Lemniscate of Bernoulli . □

Remark 1. *If the hypersurface $D' : h = 0$ has only isolated singularities, Huh's invariants $\mu^i(h)$ are very easy to compute: in fact $\mu^i(h) = (d-1)^i$ for $0 \leq i < n$ and*

$$\mu^n(h) = (d-1)^n - \mu(D'),$$

see Example 12 in [10]. It follows that these invariants are not sensitive to the position of the singularities of D' as our invariants $a_k = \dim M(h)_k$ are.

It is easy to see that one has

$$ct(D) = mdr(D) + d - 2 \text{ and } S(t) - F(t) = \operatorname{def}(D) t^{ct(D)+1} + \dots \tag{3.3}$$

with

$$def(D) = \operatorname{def} S_{(n+1)(d-2)-ct(D)-1}(\Sigma_f) \tag{3.4}$$

where Σ_f denotes the singular subscheme of the projective hypersurface $f = 0$ as in [5].

Note that computing the Hilbert-Poincaré series of the Milnor algebra $M(f)$ using an appropriate software like Singular, is much easier than computing the defects $\operatorname{def} S_k(\Sigma_f)$, because the Jacobian ideal comes with a given set of $(n+1)$ generators $f_0, ..., f_n$, while the ideal I of polynomials vanishing on Σ_f has not such a given generating set.

4 Projectively rigid hypersurfaces

Recall that the automorphism group of the projective space $\mathbb{P}^n$ is the linear group $G = PGL(n+1)$. It follows that any hypersurface D in $\mathbb{P}^n$ can be moved around by using translations by elements of G. A natural question is whether there are any other equisingular deformations of D inside $\mathbb{P}^n$. A recent answer to this question was given by E. Sernesi, who proved the following result, see formula (5) and Corollary 2.2 in [12] and refer to [11] for the general theory of deformations.

Proposition 2. *Let $D : f = 0$ be a degree d reduced hypersurface in $\mathbb{P}^n$. Let $\hat{J}$ be the saturation of the Jacobian ideal J of f. Then the vector space $\hat{J}_d/J_d$ is naturally identified with the space of first order locally trivial deformations of D in $\mathbb{P}^n$ modulo those arising from the above $PGL(n+1)$-action.*

In view of this result we introduce the following definition.

Definition 2. *We say that a degree d hypersurface $D : f = 0$ is projectively rigid if $\hat{J}_d/J_d = 0$.*

Example 2. *Let D be a Cayley cubic surface in $\mathbb{P}^3$, i.e. a cubic surface having 4 A_1 singularities. Since any two such surfaces differ by an element of the group $G = PGL(3)$, see for instance [1], it follows that D is projectively rigid.*

In this section we show that many other classical surfaces and 3-folds are rigid in this sense. To do this, we first first explain how to compute the dimension of the vector space $\hat{J}_d/J_d$. The formula (2.3) in [5] gives

$$\dim S_d/\hat{J}_d = \tau(D) - \operatorname{def}_d \Sigma_f.$$

Next we apply Theorem 1 in [5] and get

$$\operatorname{def}_d \Sigma_f = \dim M(f)_{T-d} - \dim M(f_s)_d.$$

If we put everything together we get the following.

Proposition 3.

$$\dim \hat{J}_d/J_d = \dim M(f)_d - \dim M(f_s)_d + \dim M(f)_{T-d} - \tau(D).$$

Corollary 1. *A hypersurface d degree $D : f = 0$ in $\mathbb{P}^n$ with isolated singularities and $d \leq ct(D)$ is projectively rigid if and only if $\dim M(f)_{T-d} = \tau(D)$.*

By a direct inspection of the results listed in section 2.2 we get the following.

Corollary 2.

- *A cubic surface in $\mathbb{P}^3$ with isolated singularities and defined by an equation listed in section (2.2) is projectively rigid if and only if it is one of the following types: A_4, $A_1 + A_3$, A_5, D_4, $2A_1 + A_2$, $A_1 + A_4$, D_5, $4A_1$, $A_1 + 2A_2$, $2A_1 + A_3$, $A_1 + A_5$, E_6, $3A_2$.*

- *In $\mathbb{P}^4$, the cubic of type $10\ A_1$, the Burkhardt quartic of type $45A_1$ and the quintic of type $125A_1$ defined by an equation listed in section (2.2) are projectively rigid.*

- *In $\mathbb{P}^3$, the Kummer surface and the octic of type $144A_1$ defined by an equation listed in section (2.2) are not projectively rigid.*

- *The nodal cubic with 15 nodes in $\mathbb{P}^5$ defined by an equation listed in section (2.2) is not projectively rigid.*

5 Singular program to compute our invariants

For mathematical computations, we can use any CAS (Computer Algebra Systems) software like Mathematica, Matlab or Maple, but for Algebraic Geometry, the best are Singular, Macaulay2 or CoCoA.

Singular is a computer algebra system for polynomial computations, with special emphasis on commutative and non-commutative algebra, algebraic geometry, and singularity theory, developed at the University of Kaiserslautern (see http://www.singular.uni-kl.de/ and also [9]). The following Singular program compute the Hilbert-Poincaré series and our invariants and decide if the hypersurface is nodal and projectively rigid.

The program is build for any number of variables and degree of the hypersurface.

```
// run with command:  inv(n), where n+1 = number of variables
proc inv(int n){
LIB "primdec.lib"; //library for  radical(Ideal);
ring R=0,(x(0..n)),dp;
poly f;
// you can change here the polynomial f
f=(x(0)^2+x(1)^2)^2-2*(x(0)^2-x(1)^2)*x(2)^2;  // Lemniscata
int d=deg(f);
int T=(n+1)*(d-2); // T+1=maximal index of stabilization
ideal J=jacob(f);
ideal G=std(J);
ideal G2=std(radical(G));
int n1=mult(G); int n2=mult(G2);
print("Total Tjurina= "+string(n1));
print("Number of singularities= "+string(n2));
if (n1==n2)
{ print("The hypersurface is nodal ! ");}
else
{print("The Hypersurface  is not nodal ! ");}
intvec v;
int i;
for (i=1; i<=T+2;i++) {v[i]=size(kbase(G,i-1));}
ring R=0,t,ds;
poly S;
S=0;
for(i=1;i<=T+2;i++) {S=S+v[i]*t^(i-1); }
poly F=((1-t^(d-1))/(1-t))^(n+1); // Fermat series
poly D=S-F;
print("F(t)=Hilbert Poincare series for smooth hypersurface ");
print(F);
print("S(t)=Hilbert Poincare series hypersurface");
print(S);
number def1=leadcoef(D);
string stau1, sct1, sst1, smdr1, sdef1, slc1;
i=1;
while (coeffs(D,t)[i,1]==0) {i=i+1;}
int ct1=i-2;
int mdr1=ct1-(d-2);
i=T+2;
while (coeffs(S,t)[i,1]==coeffs(S,t)[i-1,1]) {i=i-1;}
int st1=i-1;
i=1;
while ((coeffs(S,t)[i,1]*coeffs(S,t)[i+2,1]
<=coeffs(S,t)[i+1,1]*coeffs(S,t)[i+1,1]) && (i<=1+st1))
{i=i+1;}
if (i==2+st1)
{ slc1=  "  lc="+"infinity ";}
else
{slc1=  "  lc="+string(i);}
stau1="   \\tau= "+ string(coeffs(S,t)[T+2,1]);
sct1 = "  ct="+string(ct1);
sst1 = "  st="+string(st1);
smdr1= "  mdr="+string(mdr1);
sdef1= "  def="+string(def1);
string ss1="invariants=  "+stau1+sct1+sst1+smdr1+sdef1+slc1;
```

```
print(ss1);
if ( coeffs(S,t)[d+1,1] - coeffs(F,t)[d+1,1] + coeffs(S,t)[T-d+1,1]
- coeffs(S,t)[T+2,1] == 0)
{ print("The hypersurface is projectively rigid");}
else
{ print("The hypersurface is not projectively rigid");}
};
```

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