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A Note On S^1-Equivariant Maps Between 3-Spheres

by
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Abstract

In this paper, we study the non-existence of equivariant maps of 2-connected compact manifold with effective circle actions. It was given a necessary condition for a map between 3-spheres to be equivariant for an effective circle action.

Key Words: Inverse limit, Hilbert-Smith conjecture, effective group action, aquivariant map.
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1 Introduction

This short note is motivated by the following statement known as the generalized Hilbert-Smith conjecture

Conjecture: If G is a compact group which acts effectively on a connected finite dimensional manifold, then G is a Lie group. A well known fact ([7]) states Hilbert-Smith conjecture is equivalent the following conjecture:

Conjecture: A p-adic group cannot act effectively on a connected finite dimensional manifold.

Therefore constructing effective p-adic space plays an important role in the study of the Hilbert-Smith conjecture.

We show that one way to obtain a compact space whith an effective p-adic group action, is to take inverse limit of inverse systems of effective S^1-spaces with bonding maps that satisfy certain equivariant properties.

2 Preliminaries and Background

In this section we give necessary definitions and facts that will be used in the note. Suppose that to every α in a set I directed by the relation $\leq$ corresponds a topological space X_α, and that for any $\alpha, \beta \in I$ satisfying $\alpha \leq \beta$ a continuous mapping $f_\alpha^\beta : X_\beta \to X_\alpha$ is defined; suppose further that $f_\gamma^\alpha f_\alpha^\beta = f_\gamma^\beta$ for any $\alpha, \beta, \gamma \in I$ satisfying $\gamma \leq \alpha \leq \beta$ and that $f_\alpha^\alpha : X_\alpha \to X_\alpha$ is the identitiy map for all $\alpha \in I$. In this case we say that the family $S = \{X_\alpha, f_\alpha^\beta, I\}$ is an inverse system of the spaces X_α; the mappings f_α^β are called bonding mappings of the inverse system S.

Let $S = \{X_\alpha, f_\alpha^\beta, I\}$ be an inverse system; an element (x_α) of the cartesian product $\prod_{\alpha \in I} X_\alpha$ is called a *thread* of S if $f_\alpha^\beta(x_\beta) = x_\alpha$ for any $\alpha, \beta \in I$ satisfying $\alpha \leq \beta$, and the subspace of $\prod_{\alpha \in I} X_\alpha$ consisting of all threads of S is called the limit of the inverse system $S = \{X_\alpha, f_\alpha^\beta, I\}$ and is denoted by $\varprojlim S$ or by $\varprojlim X_\alpha$.

We denote by $\pi_\beta : \varprojlim X_\alpha \to X_\beta$ the restriction of the canonical projection $\prod_{\alpha \in I} X_\alpha \to X_\beta$. Clearly, for any $\alpha, \beta \in I$ such that $\alpha \leq \beta$, the projections π_α and π_β satisfy the equality $\pi_\alpha = f_\alpha^\beta \pi_\beta$. The following results are well known and we refer to [10] for more details.

Theorem 1. [10, *pg.* 99] *The limit of inverse system of T_i-spaces is a T_i-space for $i \leq 3\frac{1}{2}$.*

Theorem 2. [10, *pg.* 355] *The inverse limit of inverse system $S = \{X_\alpha, f_\alpha^\beta, I\}$ of continua is a continuum.*

If $S = \{G_\alpha, f_\alpha^\beta, I\}$ is an inverse system of topological groups (f_α^β are continuous group homomorphisms), then $\varprojlim G_\alpha$ is also a topological group with the subspace topology from $\prod_{\alpha \in I} G_\alpha$ and with the group operation $(g_\alpha)(h_\alpha) \mapsto (g_\alpha h_\alpha)$. Further, if all groups G_α in the inverse system are compact then inverse limit is compact group.

Given a prime number p. Set $G_n = S^1 = \{z \in \mathbb{C} : |z| = 1\}$ (the usual torus group) for all $n \in \mathbb{N}$ and define $f_n^{n+1} : G_{n+1} \to G_n$, $f_n^{n+1}(z) = z^p$ for all $n \in \mathbb{N}$ and $z \in S^1$. The inverse limit of this inverse system is called the p-adic solenoid T_p. This inverse limit has the p-adic integers as a totally disconnected closed normal subgroup. Solenoids are one of the prototypes of compact abelian groups that are connected but not arc-wise connected.

Let G be a group and X be a topological space. Then G acts on X if there is a continuous function $G \times X \to X$ denoted by $(g, x) \mapsto gx$, such that

$$1x = x$$

$$(gh)x = g(hx)$$

for all $x \in X$ and $g, h \in G$ (here 1 is the identity element of G). Call X a G-space if G acts on X. If X is a G-space and $x \in X$, then the subspace $G(x) = \{gx \in X : g \in G\}$ is called the orbit of x. We denote the set whose elements are the orbits by X/G. An action of G on X is said to be transitive if there is precisely one orbit, X itself and it is said to be effective if for any $g \neq 1$ in G there exists an x in X such that $gx \neq x$.

3 Main Result

Let X be a G-space, Y be an H-space and $\varphi : G \to H$ be continuous group homomorphism. A continuous map $f : X \to Y$ is called φ-equivariant if

$$f(gx) = \varphi(g) f(x)$$

for all $g \in G$ and $x \in X$.

If $\{X_\alpha, f_\alpha^\beta, I\}$ is an inverse system of topological spaces and $\{G_\alpha, \varphi_\alpha^\beta, I\}$ is an inverse system of topological groups, where each X_α is a G_α-space and each bonding map f_α^β is φ_α^β-equivariant, then we get another inverse system of topological spaces $\{X_\alpha/G_\alpha, f_\alpha^\beta, I\}$ by passing to orbit spaces. Also, under above conditions $\varprojlim X_\alpha$ is a $\varprojlim G_\alpha$-space with the action given by

$$(g_\alpha)(x_\alpha) = (g_\alpha x_\alpha)$$

for $(g_\alpha) \in \varprojlim G_\alpha$ and $(x_\alpha) \in \varprojlim X_\alpha$. Singh [9] has proved, for the inverse systems of spaces with bonding maps satisfy certain properties, that $(\varprojlim X_\alpha)/(\varprojlim G_\alpha)$ is homeomorphic to $\varprojlim(X_\alpha/G_\alpha)$. As a consequence of this, under the above conditions, if each X_α is transitive G_α-space then $\varprojlim X_\alpha$ is a transitive $\varprojlim G_\alpha$-space. Here we consider the following question.

Problem: In the above discussion, is it possible to replace "effective" by "transitive".

Before answer the problem, we recall some facts about ordered sets. If I is a set and $\leq$ is an order relation on X, and if $a < b$ (i.e. $a \leq b$ and $a \neq b$), we use the notation (a, b) to denote the set $\{x : a < x < b\}$; is called an open interval in X. If this set is empty, we call a the immediate predecessor of b, and we call b the immediate successor of a.

Lemma 1. *Let $\{X_\alpha, f_\alpha^\beta, I\}$ be an inverse system of non-empty Hausdorff topological spaces with an linear order ($\leq$) on I and let $\{G_\alpha, \varphi_\alpha^\beta, I\}$ be an inverse system of topological groups, where each X_α is an effective G_α-space and each bonding map f_α^β is φ_α^β-equivariant and onto. Further, assume that each element of I has an immediate predecessor and immediate successor. Then $X = \varprojlim X_\alpha$is an effective $G = \varprojlim G_\alpha$-space.*

Proof: Suppose that the induced G action on X is not effective. Then, there exists an element $g = (g_\alpha) \in G \setminus \{1\}$ such that $gx = x$ for each $x = (x_\alpha) \in X$. Since $g = (g_\alpha) \in G \setminus \{1\}$, there exists at least an element $\alpha' \in I$ such that $p_{\alpha'}(g) = g_{\alpha'} \in G_{\alpha'} \setminus \{1\}$ where $p_{\alpha'}$ denotes the canonical projection $p_{\alpha'} : \varprojlim G_\alpha \to G_{\alpha'}$.

Let y be an arbitrary point of $X_{\alpha'}$. Since each bonding map f_α^β is onto, we have an element $z = (x_\alpha) \in \pi_{\alpha'}^{-1}(y)$ ($\pi_{\alpha'}$ denotes the canonical projection $\pi_{\alpha'} : \varprojlim X_\alpha \to X_{\alpha'}$) such that

$$(i) \qquad f_\lambda^\gamma(\pi_\gamma(z)) = \pi_\lambda(z)$$

for $\alpha' \leq \lambda$ and immediate successor γ of λ

$$(ii) \qquad f_\beta^\theta(\pi_\theta(z)) = \pi_\beta(z)$$

for $\theta \leq \alpha'$ and immediate predecessor β of θ

Since $gz = z$, we have $g_{\alpha'} y = y$. This implies that $X_{\alpha'}$ is not effective $G_{\alpha'}$-space which is a contradiction. □

Example 1. *Let p be a prime number. Set $G_n = S^1 = \{z \in \mathbb{C} : |z| = 1\}$ and define $\varphi_n^{n+1} : G_{n+1} \to G_n, \varphi_n^{n+1}(z) = z^p$. Similarly set $X_n = \mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ and define $f_n^{n+1} : X_{n+1} \to X_n, f_n^{n+1}(w) = w^p$ for all $n \in \mathbb{N}$. We consider the effective action of G_n on X_n given by multiplication. It is clear that each bonding map f_n^{n+1} is φ_n^{n+1}-equivariant. So inverse limit of the inverse system $\{X_n, f_n^{n+1}, \mathbb{N}\}$ is an efective compact $\varprojlim G_n = T_p$-space. Since the p-adic group, Z_p, is a subgroup of T_p, $\varprojlim X_n$ is effective Z_p-space.*

Let X be a compact and 2-connected (i.e. $\pi_i(X) = \{0\}$ for $i \leq 2$) manifold and $q : X \to Y$ be a finite sheeted covering map over a compact manifold Y. Consider a continuous map $f : X \to X$ such that $qf = q$. Let p be a prime number and $\varphi : S^1 \to S^1, \varphi(z) = z^p$.

Theorem 3. *If there is an effective S^1 action on X which admits f as a φ-equivariant map, then the $p-$adic integers acts effectively on X.*

Proof: Suppose there exists an effective S^1-action on X such that f is φ-equivariant map (i.e. $f(zx) = z^p f(x)$ for all $z \in S^1$ and $x \in X$). Let $X_f = \varprojlim(X, f) = \{(x_i)_{i\in\mathbb{N}} : f(x_{i+1}) = x_i\}$ and $T_p = \varprojlim(S^1, \varphi) = \{(z_i)_{i\in\mathbb{N}} : \varphi(z_{i+1}) = z_{i+1}^p = z_i\}$ (i.e. p-adic solenoid). On the other hand Cohen [6] defined $\varprojlim^1$ functor and proved that there is a short exact sequence for the homotopy classes of maps from a space Z into the inverse limit of spaces X_α, namely

$$0 \to \varprojlim{}^1[SZ, X_\alpha] \to [Z, \varprojlim X_\alpha] \to \varprojlim[Z, X_\alpha] \to 0$$

where SZ be the suspension of Z and $[Z, W]$ denotes the homotopy classes of maps from the space Z to W. It is clear that $[SZ, W] = [Z, \Omega W]$, where ΩW loop space of W. In our case, we consider the homotopy classes of maps from S^1 into the inverse limit, X_f, of the inverse system $\{X, f\}$. Since $\pi_1(X) = \{0\}$, we have $\pi_1(X_f) = \varprojlim^1[SS^1, X]$.

In our case $\varprojlim^1[SS^1, X] = coker\psi$ where $\psi : [SS^1, X]^{\mathbb{N}} \to [SS^1, X]^{\mathbb{N}}$, $\psi((x_n)) = (x_n - Sf(x_{n+1})$ ($Sf : [SS^1, X] \to [SS^1, X]$ induced by f). Since $[SS^1, X] = [S^1, \Omega X] = \pi_2(X) = \{0\}$, we have $\pi_1(X_f) = 0$.

Now we identify Y with inverse limit of inverse system $\{Y, 1\}$. Since $qf = q$, it induces a continuous map $g : X_f \to Y$ given by $g((x_n)) = (q(x_n))$. Then, by [11] g will also be a finite sheeted (in fact has the same number of sheet as q) covering map over Y. However since X_f is simply connected, it is the universal cover of Y. Therefore $X_f = X$ by the uniqueness of the universal cover. According to Lemma 1. there is an effective T_p (therefore Z_p)-action on X. □

We shift now our attention to 3-manifolds. It is well known fact that fundamental group is the most important invariant to distinguish 3-manifolds. In particular, if X is simply connected compact 3-manifold, then X has the homology of a 3-sphere by Poincare duality. In fact X is homotopy equivalent to S^3 by the Hurewicz theorem. But Poincare conjecture ([2,3,4]) asserts that S^3 is the only such manifold. If X is orientable compact manifold of odd dimension then the Euler characteristic is 0 by Poincare duality. If X is not orientable compact manifold then it has a two fold covering by a compact orientable manifold. And it is obvious that the Euler characteristic of this cover is twice the Euler characteristic of X. Therefore if X is a non-orientable 3-manifold, then the Euler characteristic of X is 0. This implies that $H_1(X, Z)$ is infinite. Therefore S^3 is the only 2-connected compact 3-manifold.

Remark 1. *A 3-manifold X is called spherical if there exists a finite subgroup ,Γ, of $SO(4)$ acting freely by rotations on S^3 such that $X = S^3/\Gamma$. On the other hand if $q : S^3 \to Y$ is a finite covering, then the covering transformation group and hence $\pi_1(Y)$ is finite. Thurston's elliptization conjecture, which was proved in 2003 by G. Perelman ([3,4]), states that a closed*

3-manifold with finite fundamental group is spherical. So Y must be a spherical 3-manifold. On the other hand Γ is either cyclic, or is a central extension of a dihedral, tetrahedral, octahedral, or icosahedral group by a cyclic group of even order. The most basic example for spherical 3-manifolds are Lens spaces(cyclic case)and links of quotient (alias simple) surface singularities. (see[1] or [8], pp 59-60.)

Recently J. Pardon [5] has proved that there is no effective action of a p-adic group on any connected 3-manifold. So we have the following theorem.

Theorem 4. *If Γ is one of the those classes of groups and acts freely by rotations on S^3 and θ is an effective S^1 action on S^3 such that it admits a φ-equivariant map $f : S^3 \to S^3$, then there exists $x \in S^3$ such that $f(x) \notin \Gamma(x)$.*

Proof: Assume $f(x) \in \Gamma(x)$ for each $x \in S^3$. Then we have $qf = q$ where $q : S^3 \to S^3/\Gamma$ is the canonical, finite sheeted covering map. In accordance with Theorem 3. there is an effective p-adic group action on S^3 which is a contradiction. □

Corollary 1. *There is no effective S^1-action on S^3 which admits antipodal map as a φ-equivariant map.*

Proof: The proof is trivial from Theorem 4. □

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