

Algebraic models, Alexander-type invariants, and Green–Lazarsfeld sets

by

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To Bazil Brînzănescu on the occasion of his 70th birthday

Abstract

We relate the geometry of the resonance varieties associated to a commutative differential graded algebra model of a space to the finiteness properties of the completions of its Alexander-type invariants. We also describe in simple algebraic terms the non-translated components of the degree-one characteristic varieties for a class of non-proper complex manifolds.

Key Words: Resonance varieties, characteristic varieties, Alexander invariants, completion, Gysin models, intersection form.

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1 Introduction

Throughout, X will denote a connected CW-complex. Without loss of generality, we may assume that X has a single 0-cell. Let $\pi = \pi_1(X, x_0)$ be the fundamental group of our space based at this point, and let $\pi_{\text{ab}} = H_1(X, \mathbb{Z})$ be the abelianization of π .

An old idea, going back to J.W. Alexander’s definition of his eponymous knot polynomial, is to consider the homology groups of the universal abelian cover X^{ab} , viewed as modules over the group-algebra $\mathbb{C}[\pi_{\text{ab}}]$. The resulting Alexander invariants, $H_i(X^{\text{ab}}, \mathbb{C})$, play an important role in low-dimensional topology and geometry.

Let I be the augmentation ideal of $\mathbb{C}[\pi_{\text{ab}}]$, and let $\widehat{H}_i(X^{\text{ab}}, \mathbb{C})$ be the completion of the i -th Alexander invariant with respect to the I -adic filtration. The first main result of this note is about the finiteness properties of these modules (as well as their natural generalizations), when viewed as complex vector spaces.

Our approach is via commutative differential graded algebras (for short, **cdga**’s) A^* , and their resonance varieties $\mathcal{R}_r^i(A)$, which sit inside $H^1(A)$. We say that A is a q -model for X if A has the same Sullivan q -minimal model as the de Rham **cdga** of X [18]. When the space X has a q -model A with good finiteness properties (i.e., when A is q -finite in the sense explained

in Section 3), we relate the finite-dimensionality of the completions of the Alexander invariants of X to the geometry of the resonance varieties of A , as follows.

Theorem 1. *Let X be a connected CW-complex with finite q -skeleton. Suppose X admits a q -finite q -model A . Then the complex vector space $\bigoplus_{i \leq q} \widehat{H}_i(X^{\text{ab}}, \mathbb{C})$ is finite-dimensional if and only if 0 is an isolated point in the variety $\bigcup_{i \leq q} \mathcal{R}_1^i(A)$.*

A particularly interesting class of spaces to which Theorem 1 applies consists of smooth, connected, quasi-projective varieties, for which we may take as a suitable **cdga** model the Gysin model constructed by Morgan in [14]. Our theorem extends results from both [6] and [17], where only the formal case was considered, i.e., the case when A may be taken to be the cohomology ring $H^*(X, \mathbb{C})$ with zero differential, and $\mathcal{R}_r^i(X) := \mathcal{R}_r^i(A)$ are the usual resonance varieties of X .

The classical theory of Alexander polynomials of knots and links has a vast and fruitful generalization. Given a connected, finite-type CW-complex X , the characteristic varieties $\mathcal{V}_r^i(X)$ are certain algebraic subsets of the character torus $H^1(X, \mathbb{C}^*)$, defined as the jump loci for cohomology with coefficients in rank 1 local systems on X . (In the context of Kähler geometry, these varieties are also known as the Green–Lazarsfeld sets [10, 11].) In degree $i = 1$, the characteristic varieties depend only on the fundamental group $\pi = \pi_1(X)$, and provide powerful invariants for this group. When X is a quasi-projective manifold as above, the set $\mathcal{V}_1^1(\pi)$ controls fibrations of X onto curves of general type, cf. Arapura [1].

Let X be a connected, quasi-projective manifold which is not a curve; that is, $X = \overline{X} \setminus D$, where \overline{X} is a projective manifold of dimension at least 2, and D is a union of smooth divisors with normal crossings. Let $D = \bigcup_{k=1}^m D^k$ be the decomposition of D into *connected* components. For each index k , let I_k be the intersection matrix associated to the divisor D^k ; more specifically, $I_k = (D_i^k \cdot D_j^k)$, where D_i^k and D_j^k run through the irreducible components of D^k . Denoting by $U_{(u)}$ the germ of the analytic set U at a point u , we may state our second main result, as follows.

Theorem 2. *Assume that the intersection matrix I_k is either negative or positive definite, for each $k = 1, \dots, m$. Then there is an analytic germ isomorphism $H^1(X, \mathbb{C}^*)_{(1)} \cong H^1(\overline{X}, \mathbb{C})_{(0)}$ which induces analytic germ isomorphisms $\mathcal{V}_r^1(X)_{(1)} \cong \mathcal{R}_r^1(\overline{X})_{(0)}$, for all $r \geq 0$.*

In [5], the germs $\mathcal{V}_r^i(X)_{(1)}$ were identified with the germs $\mathcal{R}_r^i(A)_{(0)}$, with no additional assumptions on X , by choosing A to be the Gysin model associated to the above compactification of X . The main point in Theorem 2 is the replacement of the Gysin model by the smaller, much simpler sub-**cdga** $(H^*(\overline{X}, \mathbb{C}), d = 0)$. This theorem extends similar results from [7, 8].

The unifying idea behind both Theorem 1 and Theorem 2 is the description of germs of characteristic varieties of spaces in terms of suitable **cdga** models, established in [5] and further elaborated in [3, 13].

Each of these theorems considers some rather intricate objects, namely, the Alexander invariants and the characteristic varieties of a reasonably nice space, and manages to relate these objects (upon completion, or by passing to germs at the origin) to some simpler objects, namely, the resonance varieties of an appropriate **cdga**.

2 Some commutative algebra

Let R be a Noetherian ring, $\mathfrak{m} \subset R$ a maximal ideal, and M a finitely generated R -module. We will denote by \widehat{M} the \mathfrak{m} -adic completion of M , and view it as a module over \widehat{R} , the \mathfrak{m} -adic completion of R .

Let $\mathfrak{m}_0 = \mathfrak{m} \cdot R_{\mathfrak{m}}$ be the maximal ideal of the localized ring $R_{\mathfrak{m}} = S^{-1}R$, where $S = R \setminus \mathfrak{m}$. The following result says that completion with respect to \mathfrak{m} corresponds to ‘analytic localization’, and hence is stronger than algebraic localization.

Proposition 1. *The natural morphism $M \rightarrow M_{\mathfrak{m}} = S^{-1}M$ induces an isomorphism $\widehat{M} \rightarrow \widehat{M}_{\mathfrak{m}}$, where the first completion is with respect to \mathfrak{m} , and the second completion is with respect to \mathfrak{m}_0 .*

Proof: Applying Theorem 7.2 from [9], we see that $\widehat{M} = M \otimes_R \widehat{R}$ and $\widehat{M}_{\mathfrak{m}} = M_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} \widehat{R}_{\mathfrak{m}}$. Since $M_{\mathfrak{m}} = M \otimes_R R_{\mathfrak{m}}$, we also have $\widehat{M}_{\mathfrak{m}} = M \otimes_R \widehat{R}_{\mathfrak{m}}$. Hence, it is enough to consider only the case $M = R$.

To prove that $\widehat{R} \cong \widehat{R}_{\mathfrak{m}}$, it is enough to show that there are isomorphisms

$$R/\mathfrak{m}^k \rightarrow R_{\mathfrak{m}}/\mathfrak{m}_0^k$$

for all $k \in \mathbb{N}$. Using Proposition 2.5 from [9], we have that

$$R_{\mathfrak{m}}/\mathfrak{m}_0^k \cong S^{-1}R/S^{-1}\mathfrak{m}^k \cong S^{-1}(R/\mathfrak{m}^k).$$

We claim that any element $s \in S$ becomes invertible in R/\mathfrak{m}^k . To see this, let $I = \mathfrak{m}^k + (s)$. This ideal cannot be contained in a maximal ideal \mathfrak{n} , since $\mathfrak{m}^k \subset I \subset \mathfrak{n}$ implies, by taking radicals, that $\mathfrak{m} \subset \mathfrak{n}$, and hence $\mathfrak{m} = \mathfrak{n}$. This is a contradiction, since $s \in \mathfrak{n} \setminus \mathfrak{m}$. It follows that $I = R$ and hence there are elements $x \in \mathfrak{m}^k$ and $t \in R$ such that $x + st = 1$.

The claim shows that $S^{-1}(R/\mathfrak{m}^k) \cong R/\mathfrak{m}^k$, thereby concluding the proof. \square

Assume now in addition that R is a finitely generated \mathbb{C} -algebra, and let $\text{Spec}(R)$ be its maximal spectrum. The *support* of the R -module M , denoted $\text{supp}(M)$ is the subvariety of $\text{Spec}(R)$ defined by the annihilator ideal of M in R , denoted $\text{ann}(M)$.

Proposition 2. *Let M be a finitely-generated R -module, let \mathfrak{m} be a maximal ideal in R , and let \widehat{M} be the \mathfrak{m} -adic completion of M . Then, the following conditions are equivalent.*

1. *The \mathbb{C} -vector space \widehat{M} is finite-dimensional.*
2. *The point \mathfrak{m} is isolated with respect to $\text{supp}(M)$.*

Proof: Suppose first that $\widehat{M} = 0$. This condition is equivalent to $M_{\mathfrak{m}} = 0$, that is, $\mathfrak{m} \notin \text{supp}(M)$.

Suppose now that $\widehat{M} \neq 0$. By Proposition 1, $\dim_{\mathbb{C}} \widehat{M} < \infty$ if and only if $\dim_{\mathbb{C}} \widehat{M}_{\mathfrak{m}} < \infty$. In turn, the second condition is equivalent to saying that the module $\widehat{M}_{\mathfrak{m}}$ has finite length; in other words, its Krull dimension is zero, see [9, Proposition 10.8]. By [9, Corollary 12.5], this is equivalent to the Krull dimension of $M_{\mathfrak{m}}$ being equal to zero. By [9, Corollary 2.18], the last condition is equivalent to the fact that $\{\mathfrak{m}\}$ is an irreducible component of $\text{supp}(M)$. \square

3 Resonance varieties of a cdga

Let $A = (A^*, d)$ be a commutative, differential graded algebra (cdga) over the field of complex numbers. That is to say, $A = \bigoplus_{i \geq 0} A^i$ is a graded \mathbb{C} -vector space, endowed with a graded-commutative multiplication map $\cdot : A^i \otimes A^j \rightarrow A^{i+j}$ and a differential $d : A^i \rightarrow A^{i+1}$ satisfying the graded Leibnitz rule.

We say that A is q -finite, for some $q \geq 1$, if $A^0 = \mathbb{C}$ and A^i is finite-dimensional, for each $i \leq q$. Furthermore, we say that two cdga's A and B have the same q -type if there is a zig-zag of cdga maps from one to the other, inducing isomorphisms in cohomology in degree up to q , and a monomorphism in cohomology in degree $q + 1$.

For each cohomology class $\omega \in H^1(A)$, we make A into a cochain complex,

$$(A, d_\omega): A^0 \xrightarrow{d_\omega} A^1 \xrightarrow{d_\omega} A^2 \xrightarrow{d_\omega} \dots, \quad (3.1)$$

using as differentials the maps given by

$$d_\omega(\alpha) = d\alpha + \omega \cdot \alpha, \quad (3.2)$$

for all $\alpha \in A$. Computing the homology of these chain complexes for various values of the parameter ω , and keeping track of the resulting Betti numbers yields some interesting subsets of the affine space $H^1(A)$. More precisely, for each i and r , define

$$\mathcal{R}_r^i(A) = \{\omega \in H^1(A) \mid \dim_{\mathbb{C}} H^i(A, d_\omega) \geq r\} \quad (3.3)$$

to be the i -th *resonance variety of depth r* of the cdga (A, d) . It is readily seen that these sets are Zariski closed subsets of the affine space $H^1(A)$, for all $i \leq q$. The set $\mathcal{R}_1^i(A)$ is simply denoted by $\mathcal{R}^i(A)$.

When the differential d is zero, the resonance varieties of the algebra A are homogeneous subsets of $H^1(A) = A^1$. In general, though, the resonance varieties of a cdga are not homogeneous. Here is a simple example, extracted from [13].

Example 3. Let A be the exterior algebra on generators x, y in degree 1, endowed with the differential given by $dx = 0$ and $dy = y \wedge x$. Then $H^1(A) = \mathbb{C}$, generated by x , and $\mathcal{R}^1(A) = \{0, 1\}$.

4 Algebraic models and resonance

We now return to the topological setting from the Introduction. Throughout, X will be a connected CW-complex with finite q -skeleton, for some $q \geq 1$ (for short, a q -finite space). There are two kinds of resonance varieties that one can associate to the space X , depending on which cdga one uses to approximate it.

The most direct approach is to take the cohomology algebra $H^*(X, \mathbb{C})$, endowed with the zero differential. Let $\mathcal{R}_r^i(X)$ be the resonance varieties of this cdga. By the above discussion, these sets are Zariski closed, homogeneous subsets of the affine space $H^1(X, \mathbb{C})$, for all $i \leq q$. As before, we will denote $\mathcal{R}_1^i(X)$ by $\mathcal{R}^i(X)$.

While relatively easy to compute, these varieties may not provide accurate enough information about our space, since the cohomology algebra $H^\bullet(X, \mathbb{C})$ may not be a (rational homotopy) model for X .

We thus turn to Sullivan’s model of polynomial forms on X (see [18]). This model, which we denote by $(\Omega^\bullet(X), d)$, is a *cdga* defined over \mathbb{Q} which imitates the de Rham algebra of differential forms on a smooth manifold; in particular, $H^\bullet(\Omega(X)) \cong H^\bullet(X, \mathbb{C})$, as graded rings.

The difficulty is that, in general, Sullivan’s model does not have good finiteness properties. So let us assume $\Omega(X)$ has the same q -type as a q -finite *cdga* (A, d) . (As pointed out in [5, 13], this assumption is satisfied in many situations of geometric interest.) In this case, the resonance varieties $\mathcal{R}_r^i(A)$ are identified with Zariski-closed subsets of $H^1(X, \mathbb{C})$, for all $i \leq q$, since

$$H^1(A) \cong H^1(\Omega(X)) \cong H^1(X, \mathbb{C}). \tag{4.1}$$

The next result provides a comparison between the two types of resonance varieties associated to our space X , under the above assumptions.

Theorem 4 ([13]). *Suppose X is q -finite and $\Omega^\bullet(X)$ has the same q -type as a q -finite *cdga* A . Then, for all $i \leq q$, the tangent cone at 0 to the resonance variety $\mathcal{R}_r^i(A)$ is contained in $\mathcal{R}_r^i(X)$.*

As the next example shows, the inclusion from Theorem 4 can well be strict.

Example 5. Let X be the 3-dimensional Heisenberg nilmanifold. Then X is a circle bundle over the torus, with Euler number 1; thus, $H^1(X, \mathbb{C}) = \mathbb{C}^2$ and all cup products of degree 1 classes vanish. It follows that $\mathcal{R}^1(X)$ coincides with $H^1(X, \mathbb{C})$.

On the other hand, X admits as a model the exterior algebra A on generators x, y, z in degree 1, with differential $dx = dy = 0$ and $dz = x \wedge y$. It is now a simple matter to check that $\mathcal{R}^1(A) = \{0\}$, thereby proving the claim.

5 Characteristic varieties

We now turn to another type of homological jump loci associated to our space X . Let $\pi = \pi_1(X)$, and let $\text{Hom}(\pi, \mathbb{C}^*)$ be the algebraic group of complex-valued, multiplicative characters on π , with identity 1 corresponding to the trivial representation. For each character $\rho: \pi \rightarrow \mathbb{C}^*$, let \mathbb{C}_ρ be the corresponding rank 1 local system on X .

The *characteristic varieties* of X are the jump loci for homology with coefficients in such local systems,

$$\mathcal{V}_r^i(X) = \{\rho \in \text{Hom}(\pi, \mathbb{C}^*) \mid \dim_{\mathbb{C}} H_i(X, \mathbb{C}_\rho) \geq r\}. \tag{5.1}$$

For each $i \leq q$, the sets $\mathcal{V}_r^i(X)$ are Zariski-closed subsets of the character group $\text{Hom}(\pi, \mathbb{C}^*) = H^1(X, \mathbb{C}^*)$. The set $\mathcal{V}_1^i(X)$ is simply denoted by $\mathcal{V}^i(X)$.

When the space X has an algebraic model A with good finiteness properties, the characteristic varieties of X may be identified around the identity with the resonance varieties of A . More precisely, we have the following basic result.

Theorem 6 ([5]). *Assume X is q -finite and $\Omega^\bullet(X)$ has the same q -type as a q -finite *cdga* A . Then, for all $i \leq q$ and all $r \geq 0$, the germ at 1 of $\mathcal{V}_r^i(X)$ is isomorphic to the germ*

at 0 of $\mathcal{R}_r^i(A)$. Furthermore, all these isomorphisms are induced by an analytic isomorphism $H^1(X, \mathbb{C}^*)_{(1)} \cong H^1(A)_{(0)}$ obtained from the map $\exp_*: \text{Hom}(\pi, \mathbb{C}) \rightarrow \text{Hom}(\pi, \mathbb{C}^*)$.

A precursor to this theorem can be found in the pioneering work of Green and Lazarsfeld [10, 11] on the cohomology jump loci of compact Kähler manifolds. An important particular case (for $i = 1$ and under a 1-formality assumption) was first established in [7].

6 Alexander-type invariants

As before, let X be a connected CW-complex. Let $\nu: \pi \twoheadrightarrow G$ be an epimorphism from the fundamental group of X to an abelian group G , and let X^ν be the corresponding Galois cover. The action of the group of deck-transformations on this cover puts a $\mathbb{C}[G]$ -module structure on the homology groups $H_i(X^\nu, \mathbb{C})$. We shall call these modules the *Alexander-type invariants* of the cover $X^\nu \rightarrow X$.

Assume now that the CW-complex X has finite q -skeleton, for some $q \geq 1$. Since X has finitely many 1-cells, its fundamental group is finitely generated. Thus, the quotient G is a finitely-generated abelian group, and the group-algebra $R = \mathbb{C}[G]$ is a commutative, finitely generated \mathbb{C} -algebra. Likewise, the Alexander-type invariants $H_i(X^\nu, \mathbb{C})$ are finitely-generated R -modules, for all $i \leq q$.

Theorem 7 ([16]). *With notation as above, the following equality holds:*

$$\nu^* \left(\bigcup_{i \leq q} \text{supp}(H_i(X^\nu, \mathbb{C})) \right) = \text{im}(\nu^*) \cap \left(\bigcup_{i \leq q} \mathcal{V}^i(X) \right),$$

where $\nu^*: \text{Hom}(G, \mathbb{C}^*) \rightarrow \text{Hom}(\pi, \mathbb{C}^*)$ is the monomorphism induced by ν .

In particular, taking $\nu = \text{ab}$, we see that the variety $\bigcup_{i \leq q} \mathcal{V}^i(X)$ coincides with the union up to degree q of the support varieties of the Alexander invariants $H_i(X^{\text{ab}}, \mathbb{C})$.

7 Completion and resonance

As above, let X be a q -finite CW-complex. Set $\pi = \pi_1(X)$, and let $\nu: \pi \twoheadrightarrow G$ be an epimorphism to an abelian group G . Let $\nu^*: H^1(G, \mathbb{C}) \rightarrow H^1(\pi, \mathbb{C})$ be the induced homomorphism. Identifying $H^1(\pi, \mathbb{C})$ with $H^1(X, \mathbb{C})$, we may view the image of ν^* as a linear subspace of $H^1(X, \mathbb{C})$.

Theorem 8 ([17]). *With notation as above, the following implication holds:*

$$\text{im}(\nu^*) \cap \left(\bigcup_{i \leq q} \mathcal{R}^i(X) \right) = \{0\} \implies \dim \bigoplus_{i \leq q} \widehat{H}_i(X^\nu, \mathbb{C}) < \infty.$$

We now sharpen this result, under the assumption that X has a q -finite q -model A , i.e., $\Omega^*(X)$ has the same q -type as A .

Theorem 9. *Suppose that the q -finite CW-complex X has a q -finite q -model A . Then the following conditions are equivalent.*

1. The complex vector space $\bigoplus_{i \leq q} \widehat{H}_i(X^\nu, \mathbb{C})$ is finite-dimensional.
2. The point 0 is an isolated point in the variety $\text{im}(\nu^*) \cap (\bigcup_{i \leq q} \mathcal{R}^i(A))$.

Proof: By Proposition 2, the \mathbb{C} -vector space $\bigoplus_{i \leq q} \widehat{H}_i(X^\nu, \mathbb{C})$ is finite-dimensional if and only if the identity character $1 \in \text{Hom}(G, \mathbb{C}^*)$ is an isolated point in the variety $\bigcup_{i \leq q} \text{supp}(H_i(X^\nu, \mathbb{C}))$.

By Theorem 7, this variety may be identified with the intersection of the algebraic subgroup $\text{im}(\nu^*: \text{Hom}(G, \mathbb{C}^*) \rightarrow \text{Hom}(\pi, \mathbb{C}^*))$ with the corresponding union of characteristic varieties, $\bigcup_{i \leq q} \mathcal{V}^i(X)$.

Finally, by Theorem 6, the germ at 1 of the above intersection may be identified with the germ at 0 of the trace on the linear subspace $\text{im}(\nu^*: H^1(G, \mathbb{C}) \rightarrow H^1(\pi, \mathbb{C}))$ of the corresponding union of resonance varieties, $\bigcup_{i \leq q} \mathcal{R}^i(A)$.

Putting these facts together completes the proof. □

Taking $\nu = \text{ab}$ in the above result proves Theorem 1 from the Introduction.

8 Positive weights and formal spaces

Let A be a rationally defined **cdga**. We say that A has *positive weights* if $A^i = \bigoplus_{j \in \mathbb{Z}} A_j^i$ for each $i \geq 0$, and, moreover, these vector space decompositions are compatible with the **cdga** structure and satisfy the condition $A_j^1 = 0$, for all $j \leq 0$. As we shall see in Section 9, the Gysin models of a connected quasi-projective manifold have positive weights.

The existence of positive weights on a **cdga** model A for a space X leads to a strong connection between the resonance varieties of A and X .

Theorem 10 ([13]). *Assume X is q -finite, $\Omega^*(X)$ has the same q -type as a q -finite **cdga** A with positive weights, and the identification (4.1) preserves \mathbb{Q} -structures. Then $\mathcal{R}_r^i(A) \subseteq \mathcal{R}_r^i(X)$, for all $i \leq q$ and $r \geq 0$.*

Under the above (more restrictive) assumptions, Theorem 8 also follows from Theorems 9 and 10. The positive weight property also enters into the following result, applicable to Gysin models of quasi-projective manifolds.

Corollary 1. *Under the assumptions from the above theorem, the complex vector space $\bigoplus_{i \leq q} \widehat{H}_i(X^{\text{ab}}, \mathbb{C})$ is finite-dimensional if and only if $\bigcup_{i \leq q} \mathcal{R}^i(A) = \{0\}$.*

Proof: By Theorem 9, the dimension of $\bigoplus_{i \leq q} \widehat{H}_i(X^{\text{ab}}, \mathbb{C})$ is finite if and only if 0 is an isolated point in $\bigcup_{i \leq q} \mathcal{R}^i(A)$. Note that the \mathbb{C}^* -action on A^1 associated to the positive weight decomposition leaves both $H^1(A)$ and the resonance varieties $\mathcal{R}_r^i(A)$ invariant. Since plainly the orbit $\mathbb{C}^* \cdot \alpha$ is positive-dimensional for $0 \neq \alpha \in A^1$ and 0 belongs to the closure of this orbit, our claim follows. □

A connected space X is said to be q -formal if Sullivan’s model $\Omega^\bullet(X)$ has the same q -type as the cohomology algebra $H^\bullet(X, \mathbb{C})$ endowed with the 0 differential. By [4], compact Kähler manifolds are ∞ -formal. For a detailed discussion of formality (especially 1-formality) in our context, we refer to [15].

Notice that, if X is q -finite and q -formal, we may take $A = (H^\bullet(X, \mathbb{C}), d = 0)$ as an appropriate cdga model in Theorem 6.

Theorem 11 ([6]). *Suppose X is 1-finite and 1-formal. Then the following holds:*

$$\dim \bigoplus_{i \leq 1} \widehat{H}_i(X^{\text{ab}}, \mathbb{C}) < \infty \iff \bigcup_{i \leq 1} \mathcal{R}^i(X) = \{0\}.$$

We may now generalize this result, as follows.

Theorem 12. *Let X be a q -formal CW-complex with finite q -skeleton. Then*

$$\dim \bigoplus_{i \leq q} \widehat{H}_i(X^\nu, \mathbb{C}) < \infty \iff \text{im}(\nu^*) \cap \left(\bigcup_{i \leq q} \mathcal{R}^i(X) \right) = \{0\},$$

for any abelian Galois cover X^ν of X .

Proof: By assumption, the cohomology algebra $A = H^\bullet(X, \mathbb{C})$ endowed with the 0 differential is a q -finite q -model for X . Clearly, this cdga has positive weights: simply set $A_j^i = A^i$ if $j = i$, and zero otherwise. Furthermore, the \mathbb{C}^* -action on A^1 associated to this positive weight decomposition is just \mathbb{C}^* -multiplication. The desired conclusion follows in the same way as in Corollary 1, using the fact that each resonance variety $\mathcal{R}^i(X)$ is a homogeneous variety. \square

Example 13. Let X be the Heisenberg manifold from Example 5. This space is not 1-formal, and $\mathcal{R}^1(X) = \mathbb{C}^2$; thus, neither Theorem 11 nor Theorem 8 apply in this case.

On the other hand, as we saw previously, X admits a finite model (A, d) for which $\mathcal{R}^1(A) = \{0\}$. Thus, we may apply Theorem 9 to conclude that $\widehat{H}_1(X^{\text{ab}}, \mathbb{C})$ is finite-dimensional. (In fact, direct computation shows that $H_1(X^{\text{ab}}, \mathbb{C}) = \mathbb{C}$.)

9 Gysin models

Let X be a connected quasi-projective manifold. Choose a smooth compactification, $X = \overline{X} \setminus D$, where $D = \bigcup_{j \in J} D_j$ is a finite union of smooth divisors with normal crossings. There is then an associated rational cdga, $(A^\bullet, d) = A^\bullet(\overline{X}, D)$, called the *Gysin model* of the compactification, constructed as follows.

As a vector space, $A^k = \bigoplus_{p+l=k} A^{p,l}$, with

$$A^{p,l} = \bigoplus_{|S|=l} H^p \left(\bigcap_{i \in S} D_i, \mathbb{Q} \right) (-l), \tag{9.1}$$

where S runs through the l -element subsets of J and $(-l)$ denotes the Tate twist. The multiplication in A is induced by the cup-product, and has the property that $A^{p,l} \cdot A^{p',l'} \subseteq A^{p+p',l+l'}$.

The differential, $d: A^{p,l} \rightarrow A^{p+2,l-1}$, is defined by using the various Gysin maps coming from intersections of divisors.

Morgan proved in [14] that $\Omega_{\mathbb{Q}}^{\bullet}(X)$ has the same ∞ -type as (A^{\bullet}, d) , and thus $\Omega^{\bullet}(X)$ has the same ∞ -type as $(A^{\bullet}, d) \otimes \mathbb{C}$; moreover, the induced homology isomorphisms preserve \mathbb{Q} -structures. The weight of $A^{p,l}$ is by definition $p + 2l$, and this clearly gives a positive-weight decomposition of (A^{\bullet}, d) .

Note that $(H^{\bullet}(\overline{X}, \mathbb{C}), d = 0)$ is a sub-cdga of $A^{\bullet}(\overline{X}, D) \otimes \mathbb{C}$, much simpler than the whole Gysin model. Unfortunately, this subalgebra does not give a model for X , in general.

We will need a more detailed description of $(A^{\bullet}, d) = A^{\bullet}(\overline{X}, D)$ in low degrees. Omitting the coefficients for cohomology (they will be assumed to be either \mathbb{Q} or \mathbb{C} for the rest of this section), we have:

$$\begin{aligned} A^0 &= A^{0,0} = H^0(\overline{X}) \\ A^1 &= A^{1,0} \oplus A^{0,1} = H^1(\overline{X}) \oplus \bigoplus_{j \in J} H^0(D_j) \\ A^2 &= A^{2,0} \oplus A^{1,1} \oplus A^{0,2} = H^2(\overline{X}) \oplus \bigoplus_{j \in J} H^1(D_j) \oplus \bigoplus_{\{j,j'\} \subset J} H^0(D_j \cap D_{j'}), \end{aligned} \tag{9.2}$$

with differential $d: A^0 \rightarrow A^1$ the zero map, and differential $d: A^1 \rightarrow A^2$ given by

$$d(\eta, (b_j)_{j \in J}) = \left(\sum_{j \in J} \iota_{j!}(b_j), 0, 0 \right) \tag{9.3}$$

for $\eta \in H^1(\overline{X})$ and $b_j \in H^0(D_j)$. Here $\iota_j: D_j \rightarrow \overline{X}$ denotes the inclusion and $\iota_{j!}: H^0(D_j) \rightarrow H^2(\overline{X})$ the corresponding Gysin map. Note that $\iota_{j!}(1) \in H^2(\overline{X})$ is the Poincaré dual of the fundamental class $[D_j]$. In analytic terms, $\iota_{j!}(1) = c_1(\mathcal{O}_{\overline{X}}(D_j))$; see e.g. [12, p. 141].

To conclude this section, we provide the following cohomological criterion, as a warm-up exercise with Gysin models.

Lemma 1. *Let $\iota: X \rightarrow \overline{X}$ be the inclusion map, and assume that each divisor $D_j \subset \overline{X}$ is irreducible. Then the induced map $\iota^*: H^1(\overline{X}) \rightarrow H^1(X)$ is an isomorphism if and only if the classes $\{\iota_{j!}(1)\}_{j \in J}$ are linearly independent.*

Proof: By functoriality of Gysin models [14], the map ι^* may be identified with the homomorphism induced on H^1 by the inclusion $(H^{\bullet}(\overline{X}), d = 0) \hookrightarrow A^{\bullet}(\overline{X}, D)$. By inspecting the definition (9.3) of the differential $d: A^1 \rightarrow A^2$, we infer that the induced map on H^1 is an isomorphism if and only if the restriction of d to $A^{0,1}$ is injective. \square

10 Intersection forms and resonance

Let \overline{X} be a connected projective manifold of dimension $n \geq 2$, and let D be a union of smooth divisors in \overline{X} . Let $\{D_j\}_{j \in J}$ be the irreducible components of D . We may define the intersection multiplicity $D_i \cdot D_j$ of two such components by the usual formula,

$$D_i \cdot D_j = \langle \eta_i \eta_j \alpha^{n-2}, [\overline{X}] \rangle, \tag{10.1}$$

where α is a Kähler form on \overline{X} and $\eta_i \in H^2(\overline{X}, \mathbf{Z})$ is the cohomology class dual to $[D_i] \in H_{2n-2}(\overline{X}, \mathbf{Z})$. Alternatively, using the projection formula from [2, p. 11], we have that

$$D_i \cdot D_j = \langle \iota_j^*(\eta_i \alpha^{n-2}), [D_j] \rangle, \tag{10.2}$$

where $\iota_j: D_j \rightarrow \overline{X}$ is the inclusion. When $n = 2$ the choice of α is not necessary, since $D_i \cdot D_j$ coincides then with the usual intersection number of curves on a smooth surface.

Now let $(A^*, d) = A^*(\overline{X}, D)$ be the Gysin model (over \mathbb{C}) associated to the divisor $D \subset \overline{X}$, as described in low degrees in the previous section. Also let $X = \overline{X} \setminus D$, and let $\iota: X \rightarrow \overline{X}$ be the inclusion map. The decomposition of D into connected components leads to a partition $J = J_1 \cup \dots \cup J_m$. It follows from (10.1) that the intersection matrix of D splits into blocks I_1, \dots, I_m , given by the intersection matrices of the divisors $D^k = \bigcup_{j \in J_k} D_j$.

Lemma 2. *Suppose each of the intersection matrices I_1, \dots, I_m is invertible. Then the map $\iota^*: H^1(\overline{X}, \mathbb{C}) \rightarrow H^1(X, \mathbb{C})$ is an isomorphism.*

Proof: For each $j \in J$, set $\eta_j := \iota_{j!}(1) \in H^2(\overline{X}, \mathbb{C})$. By Lemma 1, we only need to check that the classes $\{\eta_j\}_{j \in J}$ are independent.

Suppose $\sum_{j \in J} b_j \eta_j = 0$. Taking the cup product of $\sum_{j \in J} b_j \eta_j$ with the classes $\eta_i \alpha^{n-2}$ for each $i \in J$ and evaluating on the fundamental class $[\overline{X}]$, we see using formula (10.1) that the vector $(b_j)_{j \in J}$ is in the kernel of the intersection matrix of D . By our hypothesis, this matrix is invertible. Hence, $b_j = 0$ for all $j \in J$. \square

Consider now a 1-form $\omega \in H^1(\overline{X}, \mathbb{C}) = H^1(A)$ and the associated covariant derivative d_ω from (3.2), given by $d_\omega(a) = a\omega$ for $a \in A^0$ and

$$d_\omega(\eta, (b_j)_{j \in J}) = \left(\sum_{j \in J} \iota_{j!}(b_j) + \omega \wedge \eta, (b_j \iota_j^*(\omega))_{j \in J}, 0 \right) \tag{10.3}$$

for $(\eta, (b_j)_{j \in J}) \in A^1$. With a stronger hypothesis on the divisor D , we obtain the following lemma.

Lemma 3. *Suppose each of the intersection matrices I_1, \dots, I_m is definite. If $d_\omega(\eta, (b_j)_{j \in J}) = 0$, then $b_j = 0$ for all $j \in J$.*

Proof: Let D^1, \dots, D^m be the connected components of D , and let $\{D_j\}_{j \in J_k}$ be the set of irreducible components of D^k . Let J'_k be the set of indices $i \in J_k$ for which $\iota_i^*(\omega) = 0$, and set $J''_k = J_k \setminus J'_k$. The intersection matrix I_k contains the diagonal block I'_k . Since, by assumption, I_k is a definite matrix, each of these blocks is an invertible matrix.

If $i \in J''_k$, then the condition $d_\omega(\eta, (b_j)_{j \in J}) = 0$ implies that $b_i = 0$. If $i \in J'_k$, let us apply ι_i^* to the equality $\sum_{j \in J} \iota_{j!}(b_j) + \omega \wedge \eta = 0$. Using formula (10.2), we find that the vector $(b_j)_{j \in \bigcup_{k=1}^m J'_k}$ belongs to the kernel of the invertible matrix I' with blocks I'_1, \dots, I'_m . \square

Observe now that the hypothesis of Theorem 2 coincides with that of Lemma 3, and implies that of Lemma 2. Thus, we may invoke these two lemmas in order to finish the proof of the theorem, as shown next.

Proof: Recall that $X = \overline{X} \setminus D$, and (A^\bullet, d) is the Gysin model (over \mathbb{C}) associated to the divisor $D \subset \overline{X}$. By Theorem 6, then, there is an analytic isomorphism $H^1(X, \mathbb{C}^*)_{(1)} \cong H^1(A)_{(0)}$ which identifies $\mathcal{V}_r^1(X)_{(1)}$ with $\mathcal{R}_r^1(A)_{(0)}$, for all $r \geq 0$.

On the other hand, the inclusion $(H^1(\overline{X}, \mathbb{C}), d = 0) \hookrightarrow (A^\bullet, d)$ identifies $H^1(\overline{X}, \mathbb{C})$ with $H^1(A)$, by Lemma 2, and $\mathcal{R}_r^1(\overline{X})$ with $\mathcal{R}_r^1(A)$ for all $r \geq 0$, by Lemma 3, and we are done. \square

Corollary 2. *Suppose that the intersection matrices associated to the divisor $D = \overline{X} \setminus X$ are definite. Then for any fixed projective curve C_g of genus $g > 1$, the inclusion $\iota: X \rightarrow \overline{X}$ induces a bijection between the equivalence classes of fibrations $\overline{X} \rightarrow C_g$ (i.e., surjective morphisms with connected general fiber) and the equivalence classes of fibrations defined on X .*

Proof: In view of the relationship between fibrations $X \rightarrow C_g$ and irreducible components of $\mathcal{V}_1^1(X)_{(1)}$, and likewise for \overline{X} , established by Arapura in [1], the desired conclusion follows from Theorem 2, in conjunction with the identification $\mathcal{V}_1^1(\overline{X})_{(1)} \cong \mathcal{R}_1^1(\overline{X})_{(0)}$ from [7]. \square

11 Examples and discussion

In this last section, we make a few additional remarks and we examine several classes of examples related to Theorem 2.

Remark 14. There is an alternate way to prove Corollary 2, one which is both more direct, and also covers the cases $g = 0$ and $g = 1$. If $\dim \overline{X} = 2$, Zariski’s Lemma (see Lemma (8.2) from [2, p. 90]), when coupled with our assumption on D prevents any of the connected components D^k of D to be a fiber of a fibration $\overline{X} \rightarrow C_g$. The general case can be reduced to the surface case by taking a general linear section.

Remark 15. It may happen that the map $\iota^*: H^1(\overline{X}, \mathbb{C}) \rightarrow H^1(X, \mathbb{C})$ is an isomorphism, yet the conclusion of Theorem 2 does not hold.

For instance, take $\overline{X} = C_1 \times C_1$ to be the product of two elliptic curves, and D to be the diagonal in the product. In view of Lemma 1 (see also [12, p. 64]), the map ι^* is an isomorphism. On the other hand, $\mathcal{V}_1^1(X)$ is 2-dimensional at 1, by [7, Example 10.2], yet $\mathcal{R}_1^1(\overline{X}) = \{0\}$, by an easy computation.

Example 16. Let S be a normal, projective, connected complex surface. Then the singular locus of S is a finite set, say $\{a_1, \dots, a_m\}$. Take X to be the regular locus, $S_{\text{reg}} = S \setminus \{a_1, \dots, a_m\}$. By resolving each of these singularities, we obtain a surface \overline{X} as in Theorem 2: indeed, the corresponding matrices I_k are all negative definite in this case, in view of the Mumford–Grauert criterion (see Theorem 2.1 from [2, p. 72]). The case $S = S_{\text{reg}}$ was treated in [7].

Example 17. Fix a set of strictly positive weights $w = (w_1, w_2, w_3)$ and let $f \in \mathbb{C}[x, y, z]$ be a polynomial such that the affine surface Y given by the equation $f = 0$ in \mathbb{C}^3 has only isolated singularities, and the top degree part f_d of f with respect to the weights w defines an isolated singularity at the origin of \mathbb{C}^3 .

Consider the closure Z of Y in the weighted projective space $\mathbb{P}(w_1, w_2, w_3, 1)$. Then $Z = Y \cup D$, where D is a smooth curve such that $D \cdot D > 0$. It follows as above that the surface $X = Y_{\text{reg}}$, the smooth part of Y , satisfies the assumption of our Theorem 2. The case of a weighted homogeneous polynomial (i.e., when $f = f_d$) was considered in [8], where it was shown that such surfaces X may not be 1-formal.

Example 18. Let M be the 3-dimensional Heisenberg nilmanifold discussed in Example 5, as well as in [7, Example 6.17] and [8, Example 8.6]. Since M is an S^1 -bundle with Euler number 1, it has the same homotopy type as a \mathbb{C}^* -bundle X over an elliptic curve $C = C_1$, associated to the line bundle $L = \mathcal{O}_C(s)$. We can construct a compactification \overline{X} by taking the projective bundle $\mathbb{P}(E)$ associated to the rank two vector bundle $E = L \oplus \mathcal{O}_C$. Indeed, if D_0 and D_∞ denote the divisors $\mathbb{P}(0 \oplus \mathcal{O}_C) \subset \mathbb{P}(E)$ and $\mathbb{P}(L \oplus 0) \subset \mathbb{P}(E)$, then clearly $X = \overline{X} \setminus (D_0 \cup D_\infty)$.

It is readily seen that $D_0^2 = 1$ and $D_\infty^2 = -1$, which shows that we can apply Theorem 2. Using the description of the cohomology of a projective bundle, it follows that $H^1(C, \mathbb{C}) = H^1(\overline{X}, \mathbb{C})$ and $H^2(C, \mathbb{C})$ is a subspace of $H^2(\overline{X}, \mathbb{C})$, whence $\mathcal{R}_1^1(\overline{X})_{(0)} = \{0\}$. Theorem 2 now shows that there are no positive-dimensional components of $\mathcal{V}_1^1(X)$ passing through the origin.

This behavior was predicted in Example 5, where it was pointed out that $\mathcal{R}_1^1(A) = \{0\}$, for an explicit model A of M . The Gysin model $A^*(\overline{X}, D)$ for X can also be computed explicitly, but it is slightly more complicated than the model A^* . For instance, the Hilbert series of A^* is $(1+t)^3$, whereas the Hilbert series of $A^*(\overline{X}, D)$ is $(1+t)^4$.

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