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Temperate growth at the boundary for solutions of hypoelliptic equations

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Abstract

In our main result we shall consider an hypoelliptic linear partial differential operator $p(x, D)$ with $\mathcal{C}^\infty$ coefficients defined on an open set U and consider a solution $u \in \mathcal{C}^\infty(U)$ of the equation $p(x, D)u = 0$ which extends to a distribution defined in a neighborhood of some point x^0 in the boundary ∂U of U. If hypoellipticity is a consequence of the existence of a suitable right parametrix of pseudodifferential type, then we shall show that u must have temperate growth at the boundary near x^0.

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1 Boundary values in distributions and temperate growth at the boundary

Let U be open in $\mathbb{R}^n$ and consider a point x^0 in the boundary ∂U of U. We say that a measurable function $f : U \to \mathbb{C}$ is of temperate growth at ∂U near x^0 if we can find $c > 0, k \geq 0$ and a neighborhood W of x^0 such that

$$|f(x)| \leq c \operatorname{dist}(x, \partial U)^{-k} \text{ for almost all } x \in U \cap W. \tag{1.1}$$

Also consider (with standard multi-index notation and conventions; see e.g., [12]), a linear partial differential operator $p(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha$ with $\mathcal{C}^\infty$ coefficients a_α defined in a neighborhood V of x^0. We recall that an operator p is called hypoelliptic if for every distribution u it follows from $p(x, D)u \in \mathcal{C}^\infty(W)$ for some open set $W \subset V$ that also u must be $\mathcal{C}^\infty$ on W. We will assume in this note that p admits a right parametrix given by a pseudodifferential operator associated with a symbol in a symbol class of type $S^\mu_{\rho,\rho}$ and that this parametrix is defined in a full neighborhood of x^0. (We will recall the definition of the symbol classes $S^\mu_{\rho,\rho}$ in section 2.) Many conditions on the symbol $p(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x)\xi^\alpha = \exp[-i\langle x, \xi\rangle] p(x, D) \exp[i\langle x, \xi\rangle]$ of $p(x, D)$ are known to imply the existence of such parametrices, foremost the cases considered by

Hörmander, [11], or L.Boutet de Monvel in [3]. Actually, somewhat less than a parametrix in a good symbol class is needed in the argument: the main result will remain valid whenever $p(x, D)$ admits a parametrix which satisfies the properties (2.9), (2.10), of section 2 below. Indeed, we could have worked with symbols in the classes of R.Beals, [1], as well, and our preference for the symbol classes $S^{\mu}_{\rho,\rho}$ is purely opportunistic: the definition of the symbol classes $S^{\mu}_{\rho,\rho}$ is directly accessible also to non-specialists, whereas the symbol classes in [1] have rather complicated definitions and the meaning of the choices in the definitions is perhaps somewhat intricate. Anyway, we will make some comments on more general symbol classes in section 2.

The main result of this note is

Theorem 1.1. *Assume, under the above assumptions, that $u \in \mathcal{D}'(V)$ is a solution of $p(x, D)u = 0$ on $U \cap V$. Then u is of temperate growth at the boundary of U near x^0.*

Note that u and $p(x, D)$ are defined in a full neighborhood of x^0, but the equation $p(x, D)u = 0$ is only assumed to hold in $U \cap V$. An alternative way to state this could be to start from a solution u of $p(x, D)u = 0$ on $V \cap U$ and to assume that u extends, as a distribution (and not necessarily as a solution of $p(x, D)u = 0$) to a full neighborhood of x^0. We will say for short in such a situation that u extends near x^0 across the boundary of U. We should further perhaps point out that for general linear partial differential operators $p(x, D)$ with $\mathcal{C}^\infty$ coefficients, distributions defined in a neighborhood W of x^0 and which satisfy $p(x, D)u = 0$ on $U \cap W$ need not always have temperate growth at ∂U near x^0: see for example remark 3.1 in section 3 below. Even simpler is the remark that extendibility in distributions across the boundary of u is not immediately related to the fact that u be temperate at the boundary: see remark 3.9. In the other direction, measurable functions on U which are of temperate growth at the boundary near some point x^0 are in quite general situations often locally extendible across the boundary, even if they do not satisfy some specific equation. That this is so even in interesting situations is an elementary consequence of classical results on the division of distributions by real analytic functions: for completeness, an example will be given in lemma 3.3 later on. In section 3 we also make further comments which relate theorem 1.1 to some results on the boundary behavior of solutions to homogeneous linear partial differential equations.

All these remarks and also the proof of theorem 1.1 are based on classical results and arguments. One reason why they may have some interest lies in their generality and in the fact that they are related to some results in complex function theory. We comment on this in section 3.

2 Parametrices and proof of theorem 1.1

We start by recalling some terminology related to “parametrices”. A right parametrix for $p(x, D)$ on an open set V is a linear continuous operator $T : \mathcal{E}'(V) \to \mathcal{D}'(V)$ such that $T \circ p(x, D) = I + K$ on $\mathcal{E}'(V)$ where I is the identity operator and $K : \mathcal{E}'(V) \to \mathcal{C}^\infty(V)$ is an integral operator with a $\mathcal{C}^\infty$ kernel. If the operator $p(x, D)$ is hypoelliptic, then from $u \in \mathcal{E}'(V)$ and from the fact that u is $\mathcal{C}^\infty$ near some point $\tilde{x}$ it must follow that also Tu is $\mathcal{C}^\infty$ near $\tilde{x}$. It is then easy to see that the kernel $F \in \mathcal{D}'(V \times V)$ of T (given, if not already known explicitly, by the Schwartz kernel theorem) must be $\mathcal{C}^\infty$ outside the diagonal $\{(x, x); x \in V\}$ of $V \times V$.

(We say that a distribution $F \in \mathcal{D}'(V \times V)$ is the kernel of an operator $T : \mathcal{E}'(V) \to \mathcal{D}'(V)$ if we have that $F(\varphi \otimes \psi) = T(\varphi)(\psi)$ for every $\varphi, \psi \in \mathcal{C}_0^\infty(V)$.) Actually, this can be seen by abstract functional analysis, and there is no need to know by what kind of argument the hypoellipticity of $p(x, D)$ was established, but if the parametrix is given by a pseudodifferential operator, then the fact that F must be $\mathcal{C}^\infty$ outside the diagonal of $V \times V$ is trivial. It does not seem automatic however that by assuming hypoellipticy alone, we can also deduce immediately that F has to be temperate near the diagonal, in the sense that for every W open in V with $\bar{W}$ compact in V, and every $k \in \mathbb{N}$, it will follow that for some k' we have $|x-y|^{k'} E(x, y) \in \mathcal{C}^k(W \times W)$. Why this is not a consequence of abstract results (i.e., which do not involve the operators under consideration) is clear from the example which we give in remark 3.1. This is the reason why in theorem 1.1 we shall assume that the parametrix of p is associated with a pseudodifferential operator, which, moreover, we will actually assume to belong to some specific class of such operators. In fact, if q is the symbol of some pseudodifferential operator, then its kernel is formally given by

$$F(x, y) = (2\pi)^{-n} \int_{\mathbb{R}^n} \exp\left[i\langle x - y, \xi\rangle\right] q(x, y, \xi) d\xi, \tag{2.1}$$

where (2.1) is to be regarded as an oscillatory integral. In our specific case we will assume that q is a symbol in the symbol class $S^\mu_{\rho,\rho}(V \times V)$ for some $\rho \leq 1$. We recall that for given constants $0 < \delta \leq \rho \leq 1$ and $\mu \in \mathbb{R}$, $S^\mu_{\rho,\delta}(V \times V)$, V open in $\mathbb{R}^n$, denotes the class of $\mathcal{C}^\infty$ functions $q : V \times V \times \mathbb{R}^n \to \mathbb{C}$ such that for every compact $K \subset V$ and for every two multi-indices α, β we can find constants $c_{\alpha,\beta}$ such that

$$|\partial_\xi^\alpha \partial_{x,y}^\beta q(x, y, \xi)| \leq c_{\alpha,\beta}(1 + |\xi|)^{\mu - \rho|\alpha| + \delta|\beta|}, \text{ if } (x, y) \in K \times K, \xi \in \mathbb{R}^n. \tag{2.2}$$

We also recall that when $\delta < \rho$, then the calculus of pseudodifferential operators associated with symbols in the class $S^\mu_{\rho,\delta}(V \times V)$ is rather simple. Modulo smoothing operators, one may then always assume that the symbol $q(x, y, \xi)$ depends only on the variables (x, ξ), but in our theorem we are only interested in the final form of the parametrix and then the variable y brings no additional complications also for general symbol classes. Anyway in the case of the classes with $\delta < \rho$, a parametrix will exist if p can be inverted in the symbol algebra $S^\mu_{\rho,\delta}(V)$. This is the situation in L.Hörmander, [11] and the assumption on p boils down to the following: there are $\mu \in \mathbb{R}$, $q \in S^\mu_{\rho,\delta}(V)$ (note that here we assume q to be independent of the variable y) and a sequence $\mu_j \in \mathbb{R}$ which tends to $-\infty$, such that for every k we have

$$1 - \sum_{|\alpha| < k} \frac{i^{|\alpha|}}{\alpha!} \partial_\xi^\alpha q(x, \xi) \partial_x^\alpha p(x, \xi) \in S^{\mu_k}_{\rho,\delta}(V). \tag{2.3}$$

While this case already covers many classes of pseudodifferential operators (including all hypoelliptic operators with constant coefficients), L. Boutet de Monvel in [3] was the first to consider operators with parametrices in the classes $S^\mu_{\delta,\rho}$, for the case $\rho = \delta = 1/2$. In a parallel development, with many intersections with the work of L.Boutet de Monvel, R.Beals, [1], considered symbols in very general classes $S^\lambda_{\varphi,\Phi}$, for suitable pairs of weight functions (φ, Φ). The symbol classes in [1] in particular contain parametrices for many examples of hypoelliptic operators (foremost perhaps the examples of V. Grushin, [7]), which were not in the classical symbol classes of [11]. In this paper we will for simplicity work with symbols in the classes $S^m_{\rho,\delta}$, $\delta \leq \rho$. In the case $\delta = \rho$ the symbolic calculus will work efficiently only if some additional information on the symbols involved is available, as was in fact the case in [3], and in a large number of papers written about the same time or later. Since the number of papers in which parametrices associated with symbols close to the classes $S^{\mu}_{1/2,1/2}$ is huge, we only mention a number of papers written immediately after 1974: we cite L.Boutet de Monvel-F.Treves [5], L.Boutet de Monvel-A.Grigis-B.Helffer, [4], B.Helffer, [8], A. Grigis,[6], J.Sjöstrand, [18]. Also see L.Hörmander, [12]. On the other hand, the number of papers where parametrices are constructed in $S^{\mu}_{\rho,\rho}$, $\rho < 1/2$, is apparently much smaller and we mention here M.Mughetti, [16], L.Maniccia-M.Mughetti, [15] and M.Mughetti-F.Nicola, [17], together with the references in these papers. We also mention that the parametrices associated with symbols in the classes of [1] are of the type needed in this paper, provided that the pair (φ, Φ) is "localizable" in the sense of [1], page 5. Due to the complicated nature of the symbols in [1], we will not give details.

The assumption in theorem 1.1 is now the following: we assume that a right parametrix for q is given with $q \in S^{\mu}_{\rho,\rho}(V \times V)$, V a (full) neighborhood of x^0. We recall here that u is also defined on a full neighborhood of x^0, but that x^0 is assumed to be a boundary point of U and that u is assumed to satisfy $p(x,D)u = 0$ possibly only on $U \cap V$.

We now turn effectively to the proof of theorem 1.1. We fix a point $x \in U \setminus \partial U$ in a neighborhood of x^0. If $\operatorname{dist}(y,x) \leq (2/3)\operatorname{dist}(x,\partial U)$, we have $y \in U$, $\operatorname{dist}(y,\partial U) \geq (1/3)\operatorname{dist}(x,\partial U)$.

We now consider two $\mathcal{C}_0^\infty(\mathbb{R}^n)$ functions φ and ψ such that:

- $\varphi(y) = 1$ for $|y| < 1/3$, $\varphi(y) = 0$ for $|y| \geq 2/3$,
- $\psi(y) = 1$ for $1/3 \leq |y| \leq 2/3$,
- $\psi(y) = 0$ for $|y| \leq 1/6$ or $|y| \geq 5/6$.

Also consider the functions $\varphi_x(y) = \varphi((y-x)/\operatorname{dist}(x,\partial U))$, $\psi_x(y) = \psi((y-x)/\operatorname{dist}(x,\partial U))$. Thus we will have

- $\varphi_x(y) = 1$ for $|y-x| < (1/3)\operatorname{dist}(x,\partial U)$, and $\varphi_x(y) = 0$ for $|y-x| \geq (2/3)\operatorname{dist}(x,\partial U)$,
- $\psi_x(y) = 1$ for $(1/3)\operatorname{dist}(x,\partial U) \leq |y-x| \leq (2/3)\operatorname{dist}(x,\partial U)$,
- $\psi_x(y) = 0$ for $|y-x| \leq (1/6)\operatorname{dist}(x,\partial U)$ or $|y-x| \geq (5/6)\operatorname{dist}(x,\partial U)$.

In particular, by the last property of $\psi_x(y)$

$$\varphi_x(y)\psi_x(y) = 0, \text{ for } |y| \leq (1/6)\operatorname{dist}(x,\partial U) \text{ or } |y| \geq (2/3)\operatorname{dist}(x,\partial U). \tag{2.4}$$

We also have

$$\psi_x(y)p(y,D_y)(\varphi_x(y)u(y)) = p(y,D_y)(\varphi_x(y)u(y)) \text{ for all } y. \tag{2.5}$$

Indeed, this is trivial at points y for which we have either $\psi_x(y) \equiv 1$ or else $\varphi_x(y) \equiv 0$ in a neighborhood of y. For $\psi_x(y) \neq 1$ we have either $|y-x| \leq (1/3)\operatorname{dist}(x,\partial U)$, or $|y-x| \geq$

$(2/3)\,\mathrm{dist}(x,\partial U)$, but in the second case we have $\varphi_x(y) \equiv 0$. It thus remains to consider the case when y lies in the set $\{y; |y-x| \le (1/3)\,\mathrm{dist}(x,\partial U)\}$. In this case $\varphi_x(y) = 1$ near y, so on both sides of (2.5), $p(y,D_y)(\varphi_x(y)\ u(y)) = 0$ since u is a solution of $p(y,D_y)u(y) = 0$.

Also observe that both (2.4) and (2.5) can be derivated in y as many times as we wish.

Finally we mention that for every multi-index α there is a constant c_α which does not depend on x such that

$$|\partial_y^\alpha \varphi_x(y)| + |\partial_y^\alpha \psi_x(y)| \le c_\alpha^{|\alpha|+1}\,\mathrm{dist}(x,\partial U)^{-|\alpha|}. \tag{2.6}$$

We observe here that the fact that in the sequel we only need to estimate derivatives of the functions $\varphi_x(y)$, $\psi_x(y)$ in y, and not also in x, simplifies the situation a little bit.

As a preparation for things to come, we observe that since u is a distribution defined on V it has a finite order near x^0. For suitable constants c, c', k, we will therefore have

$$|u(g)| \le c|g|_k, \text{ if } g(y) = 0 \text{ for } |y-x^0| \ge c', \tag{2.7}$$

where for a given $\mathcal{C}_0^\infty(\mathbb{R}^n)$ function g which vanishes for $|y-x^0| \ge c'$, $|g|_k$ is the norm $|g|_k = \sup_{y\in\mathbb{R}^n, |\alpha|\le k} |D_y^\alpha g(y)|$. We will assume that x is close enough to x^0 in order to have the inclusion $\{y; |y-x| \le (1/2)\,\mathrm{dist}(x,\partial U)\} \subset \{y; |y-x^0| \le c'\}$. In particular then, $\varphi_x(y)\psi_{\tilde{x}}(y) \ne 0$ will imply $|y-x^0| \le c'$.

We now consider the (or rather "a") pseudodifferential operator $q(x,D)$ associated with the symbol $q(x,\xi)$. We may assume that $w \to q(x,D)w$, $w \in \mathcal{E}'(V)$, is formally given by the oscillatory integral

$$(2\pi)^{-n} \int \exp\,[i\langle x,\xi\rangle] q(x,y,\xi) w(x,y)(\xi)\, dy d\xi\,,$$

(For the exact meaning of this, see later on. We apply the preceding anyway only for $w \in \mathcal{C}_0^\infty(V)$.) We denote the kernel of $q(x,D)$, as an operator $\mathcal{E}'(V) \to \mathcal{D}'(V)$, by F and will then have for some integral operator $\mathcal{K}$ with $\mathcal{C}^\infty$ kernel that $q(x,D)\circ p(x,D) = I + \mathcal{K}$ as operators on $\mathcal{E}'(V)$. It will follow that we have

$$v(x) = q(x,D)(p(x,D)v) + \int_V K(x,y)v(y)dy,\ \forall v \in \mathcal{C}_0^\infty(V), \text{ and } \forall x \in V, \tag{2.8}$$

where we have denoted the kernel of $\mathcal{K}$ by K. We could in principle improve this representation if we replaced F and K by properly supported kernels; see in particular (2.17) later on. We will however also make a comment there on why we do not do so.

We also need some elementary information on the singularity of $(x,y) \to F(x,y)$ near the diagonal $\{(y,y); y \in V\}$ of $V \times V$. Since F is given by the oscillatory integral (2.1) (we recall that the exact meaning of $F(x,y)$ as an oscillatory integral is that $F(x,y)(w) = (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{V\times V} \exp\,[i\langle x-y,\xi\rangle] q(x,\xi) w(x,y)\,dx\,dy\,d\xi$ for $w \in \mathcal{C}_0^\infty(V\times V)$) it is immediate from the theory of oscillatory integrals that the following proposition holds:

Proposition 2.1. *Consider $q \in S^\mu_{\rho,\rho}(V\times V)$ for some $\rho \le 1$ and define F by (2.1). Let further k be given by (2.7) and let m be the order of $p(x,D)$. Then there is k' such that $|x-y|^{k'}F(x,y)$*

is $\mathcal{C}^{k+m}(V \times V)$. *In particular, we have when* $\overline{W} \subset V$, W *a bounded neighborhood of* x^0 *and* $\overline{W}$ *its closure, the following estimate:*

$$\sup_{x \in W, y \in W} |\partial_y^\alpha (|x-y|^{k'} F(x,y))| < \infty, \ \textit{for } |\alpha| \leq k+m. \tag{2.9}$$

Moreover, since F *is* $\mathcal{C}^\infty$ *away from the diagonal, it is also clear from this that*

$$\sup_{x \in W, y \in W} |x-y|^{k'} |\partial_y^\alpha F(x,y)| < \infty, \ \textit{for } |\alpha| \leq k+m. \tag{2.10}$$

(In fact, the formal part of the argument is

$$\begin{aligned}(x-y)^\beta \partial_x^\alpha F(x,y) &= \int_{\mathbb{R}^n} (x-y)^\beta \partial_x^\alpha \{\exp[i\langle x-y, \xi\rangle] q(x,y,\xi)\} d\xi \\ &= i^\beta \int_{\mathbb{R}^n} \partial_x^\alpha \{\exp[i\langle x-y,\xi\rangle] \partial_\xi^\beta q(x,y,\xi)\} d\xi,\end{aligned} \tag{2.11}$$

and thus $(x-y)^\beta \partial_x^\alpha F(x,y)$ is associated with a symbol which lies in $S_{\rho,\rho}^{\mu+|\alpha|-\rho|\beta|}$. By the definition of the symbol class $S_{\rho,\rho}^\mu$ we have here since $\bar{W}$ is compact in V that $|\partial_x^\alpha \{\exp[i\langle x-y,\xi\rangle] \partial_\xi^\beta q(x,y,\xi)\}| \leq c_{\alpha,\beta}(1+|\xi|)^{\mu+|\alpha|-\rho|\beta|}$, if $(x,y) \in W \times W$ for some constants $c_{\alpha,\beta}$. The integrand in (2.11) is integrable, if $\mu + |\alpha| - \rho|\beta| < -n$, and this will happen, once α is fixed, for large β, etc.)

When we apply (2.7), the role of g will be played by the function

$$\chi(y) = \varphi_x(y)\, {}^t p(y, D_y)(\psi_x(y) F(x,y))$$

with $\psi_x(y), \varphi_x(y)$, the functions introduced above. Here x will be considered as a parameter and ${}^t p(y, D_y)$ is the operator adjoint to $p(y, D_y)$. Note that initially, χ is not defined when $y = x$. However, when y is a point with $|y - x| < \operatorname{dist}(x, \partial U)/6$, then ψ_x is identically zero in a neighborhood of y, so we can extend $\chi(y)$ by 0 at $y = x$ and the extended function will be $\mathcal{C}^\infty$ in y everywhere since F was $\mathcal{C}^\infty$ away from the diagonal. We will denote the extended function also by χ and since the support of χ lies in the support of ψ_x, χ will have support in $|y - x^0| \leq c'$, which fact is one of the conditions we need in order to apply (2.7). The fact now that $|y - x| \geq \operatorname{dist}(x, \partial U)/6$ when $y \in \operatorname{supp} \chi$ will help us with the estimates in that we will have that there is a constant $c_1 > 0$ for which

$$|\partial_y^\alpha F(x,y)| \leq c_1 \operatorname{dist}(x, \partial U)^{-k'}, \text{ if } |\alpha| \leq k+m \text{ on } \operatorname{supp} \varphi_x \cap \operatorname{supp} \psi_x. \tag{2.12}$$

We furthermore will have to estimate the norm $|\chi|_k$ of χ. This is just done by using the Leibniz rule. If $|\alpha| \leq k$, then for $y \neq x$ we have that $\partial_y^\alpha \chi(y)$ is a finite sum of terms of form $a_{\alpha,\beta,\beta',\gamma}(x,y)(\partial_y^\beta \varphi_x(y))(\partial_y^{\beta'} \varphi_x(y))\partial_y^\gamma F(x,y)$ where $|\beta| + |\beta'| + |\gamma| \leq |\alpha| + m$ and where the $a_{\alpha,\beta,\beta',\gamma}(x,y)$ are some $\mathcal{C}^\infty$ functions in y, which can be calculated in terms of the coefficients of ${}^t p(y, D_y)$. It is then easy to see that there is a constant c_2 which does not depend on x as long as x remains close to x^0 such that

$$|\chi|_k \leq c_2 (\operatorname{dist}(x, \partial U)^{-k'-k-m}. \tag{2.13}$$

In the last part of the argument of the proof of theorem 1.1, we now observe that $u(x) = \varphi_x(x)u(x)$ and write down (2.8) for $v(y) = \varphi_{\tilde{x}}(y)u(y)$. To do so, we observe that by (2.5) $q(x,D)(p(x,D)v)(x) = q(x,D)(\psi_{\tilde{x}}(x)p(x,D)v)$, which when written in terms of the kernel F, comes to

$$q(x,D)(p(x,D)v)(x) = \int F(x,y)\psi_{\tilde{x}}(y)[p(y,D_y)\varphi_{\tilde{x}}(y)u(y)]dy, \tag{2.14}$$

where the integral is to be read in distribution sense. More precisely, we have

$$\int F(x,y)[\psi_{\tilde{x}}(y)p(y,D_y)\varphi_{\tilde{x}}(y)u(y)]dy = [\,{}^t p(y,D_y)\psi_{\tilde{x}}(y)F(x,y)](\varphi_{\tilde{x}}u), \tag{2.15}$$

i.e., it is the action (in the y variable) of the distribution $[{}^t p(y,D_y)\psi_{\tilde{x}}(y)\, F(x,y)] \in \mathcal{D}'(V)$ on the $\mathcal{C}_0^\infty$-function $y \to \varphi_{\tilde{x}}(y)u(y)$. We have thus obtained so far

$$u(x) = [{}^t p(y,D_y)\psi_{\tilde{x}}(y)\, F(x,y)](\varphi_{\tilde{x}}(y)u(y)) - \int_V K(x,y)\varphi_{\tilde{x}}(y)u(y)dy. \tag{2.16}$$

We further write the first term on the right hand side of (2.16) as $u[\varphi_{\tilde{x}}(y){}^t p(x,D_y)\;\psi_{\tilde{x}}(y)\,F(x,y)] = u(\chi)$. We can therefore apply (2.7) and obtain from (2.13) that $|u(\chi)| \le c_3 \operatorname{dist}(x,\partial U)^{-k'}$. This shows that the function $x \to \int_V F(x,y)[p(y,D_y)\;\varphi_x(y)u(y)]dy$ is temperate at the boundary, concluding the argument for the first term in (2.16).

To conclude the proof of theorem 1.1, we need to show that the same is true for the function $x \to \int_V K(x,y)\varphi_{\tilde{x}}(y)u(y)dy$. This is however obvious, in that it is a $\mathcal{C}^\infty$ function on all of V.

Remark 2.2. In the above argument we could also have worked with properly supported pseudodifferential operators which live completely in the region $\tilde{V} = V \cap U$. This would have led to a formally simpler representation formula for functions $v \in \mathcal{C}^\infty(\tilde{V})$, namely

$$v(x) = \int_{\tilde{V}} \tilde{F}(x,y)[p(y,D_y)v(y)]dy - \int_{\tilde{V}} \tilde{K}(x,y)v(y)dy,\ \forall v \in \mathcal{C}^\infty(\tilde{V}), \tag{2.17}$$

where we have changed the notation from F, K, to $\tilde{F}, \tilde{K}$, in order to make clear that we replaced our original kernels with properly supported ones.

When we then apply (2.17) for $v = u$, assuming that $p(x,D)u = 0$, we obtain the representation formula $u(x) = -\int_{\tilde{V}} K(x,y)v(y)dy$, which is of course much simpler than (2.16). However, it is not immediate how to estimate $u(x)$ for x near the boundary with this formula, since the (new) kernel $\tilde{K}(x,y)$ is not immediately known well enough when $x-y$ is small and both x and y are close to the boundary.

3 Comments, remarks and examples

Remark 3.1. *(Folklore)* We observe at first that it does not follow from $E \in \mathcal{D}'(\mathbb{R}^n)$ and $\operatorname{singsupp} E = \{0\}$ that the restriction of E to $\mathbb{R}^n \setminus \{0\}$ is temperate at 0. Indeed, e.g., for $n = 1$, $\sin(\exp[1/x])$, is bounded on $\mathbb{R} \setminus \{0\}$ and defines therefore a distribution on all of $\mathbb{R}$. Therefore also $E = (d/dx)\sin(\exp[1/x])$ defines a distribution on all of $\mathbb{R}$, but the restriction of E to $\mathbb{R} \setminus \{0\}$ is not temperate at 0. Now denote for a given $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ which is identically one in a neighborhood of $0 \in \mathbb{R}$ by $S = \varphi E$ and by $T : \mathcal{E}'(\mathbb{R}) \to \mathcal{D}'(\mathbb{R})$ the operator $Tu = S * u$. Then T has the kernel $F(x, y) = S(x - y)$. F has ($\mathcal{C}^\infty$) singular support on the diagonal $\{(x, x); x \in \mathbb{R}\}$ of $\mathbb{R} \times \mathbb{R}$ and therefore shrinks singular supports (as the parametrices of hypoelliptic operators would do). T is thus an example of an operator which shrinks singular supports, but which has a kernel which is not temperate at the diagonal of $\mathbb{R} \times \mathbb{R}$.

Remark 3.2. a) We now want to make some comments on the main assumption in theorem 1.1, viz. the type of hypoellipticity of $p(x, D)$. We have assumed above for commodity that hypoellipticity comes from the existence of a parametrix in the symbol class $S^\mu_{\rho,\rho}$. Actually, all we needed was a parametrix with a kernel F which satisfied (2.10). In this sense the conclusion in the theorem is more of a statement on the parametrices involved in the argument than on the operator $p(x, D)$ itself.

b) We also mention that in some cases of hypoellipticity, parametrices are constructed in a very explicit way, e.g., in R.Beals-B.Gaveau-P.Greiner-Y.Kannai, [2] and that in other cases, the solutions of the homogeneous equation are known well enough to study them by direct introspection. (This is the case e.g., for solutions to ordinary differential equations with regular singular points.) See e.g., Y.Kannai, [13], B.Helffer-Y.Kannai, [9].) We omit further comments on this question.

There is of course also a vast literature on hypoellipticity where the latter is established by estimates. We have not tried to understand what kind of estimates so obtained are suited to prove results as in theorem 1.1.

We continue this section with a discussion of the relation between temperate growth at a boundary and extendibility across that boundary.

Lemma 3.3. *Consider U open in $\mathbb{R}^n$ and let $x^0 \in \partial U$. Assume that after a $\mathcal{C}^\infty$ change of variables in a neighborhood of x^0, $\partial U \cap V$ is of form $\{x \in V; q(x) = 0\}$ where V is an open neighborhood of x^0 and q is a real analytic function defined on V. Also let $f : U \to \mathbb{C}$ be a measurable function which is of temperate growth at the boundary of U near x^0. Then f extends to a distribution defined on a neighborhood of x^0.*

Proof: Since extendibility at x^0 does not depend on the (smooth) coordinate system which we use near x^0, we may assume from the very beginning that $\partial U \cap V$ has the form $\{x \in V; q(x) = 0\}$ with q real analytic. Moreover, since near x^0, $|q(x)|$ can be estimated by $c_1 \operatorname{dist}(x, \partial U)$, we have, with k a constant for which we have (1.1) that $|q^k(x)f(x)| \le c_2$ for $x \in U \cap W$ for some constant c_2. For notational simplicity we have assumed here that $W \subset V$. The function $q^k f$ is therefore bounded near x^0, and, being measurable, is in particular a distribution. We can then define a distribution extension of $q^k f$ to $\mathbb{R}^n$ by setting $q^k f = 0$ outside W. Denote this distribution by u. A distributional extension of f is now any solution v of $u = q^k v$. Such a distribution v exists in view of a result on the division of distributions by real analytic functions due to S.Lojasiewicz, respectively by polynomials, due to L.Hörmander. (See [14], [10].) ∎

Particular cases of situations as in lemma 3.3 are if locally near x^0, $\partial U = \{x^0\}$ or when, possibly after a local $\mathcal{C}^\infty$ change of variables, ∂U is of form $x_n = 0$ and $U = \{x; x_n > 0\}$. Actually, the case when $\partial U = \{x^0\}$ is rather simple. We may assume without loss of generality that $x^0 = 0$. If then $n = 1$ and $|f(x)| \leq |x|^{-k}$ for $x \neq 0$ and $k \in \mathbb{N}$, then we can take a $(k+1)$-order primitive G of f in $x > 0$, respectively $x < 0$ and will have that G is bounded in a neighborhood of 0. Since the initial f is thus near 0 a $(k+1)$ order derivative of a bounded function, it follows that f can be given a meaning on all of $\mathbb{R}$ as a distribution of order $k+1$. An example when this is in some sense optimal in Sobolev spaces is $\psi(x) = \underline{x}^{-k} = ((x+i0)^{-k} + (x-i0)^{-k})/2$. The Fourier transform of this is $\hat{\psi}(\xi) = \pi i^{-k}(\operatorname{sgn}\xi)\xi^{k+1}/(k+1)!$ (see e.g., [12], example 7.1.17, vol. I), which shows that convolution by ψ maps $H_0^s(\mathbb{R})$ to $H_{loc}^{s-k-1}(\mathbb{R})$. ($H_0^s(\mathbb{R})$ denotes the compactly supported distributions in the Sobolev space or order s, whereas $H_{loc}^{s-k-1}(\mathbb{R})$ are the distributions which are locally in the Sobolev space of order $s-k-1$.)

Remark 3.4. The case $\partial U = \{0\}$ is easy to understand also when $n > 1$. Assume again that $|f(x)| \leq |x|^{-k}$ near 0 for $k \in \mathbb{N}$. Also assume now that f has compact support and consider, assuming for simplicity, $n \geq 3$, $f * |x|^{n-2}$. Recall that $(\sigma_n(n-2))^{-1}|x|^{-n+2}$ is (with σ_n the surface area of the sphere in n dimensions) the fundamental solution of the Laplace operator Δ in $\mathbb{R}^n$. Moreover, the convolution of $|x|^{-k}$ with $|x|^{-n+2}$ is $c|x|^{-k+2}$ for some c. It follows estimating $|\int_{\mathbb{R}^n} f(x-y)|y|^{-n+2}dy|$ by $|\int_{\mathbb{R}^n} |x-y|^{-k}|y|^{-n+2}dy$ that $|(f*|x|^{-n+2})(x)| \leq c'|x|^{-k+2}$. Iterating this we see that we can find a bounded $\mathcal{C}^{k'}(\mathbb{R}^n \setminus 0)$ function g such that $\Delta^{k'} g = f$ for $x \neq 0$, $k' = [k/2] + 1$, $[k/2]$ the integer part of $k/2$. Again, f is extendible to $\mathbb{R}^n$ as a distribution of order $[k/2] + 1$.}

Remark 3.5. We make a remark on the mapping properties of an operator

$$Tu(x) = \int_U \tilde{E}(x,y)u(y)dy$$

where $\tilde{E}$ is a distribution on $U \times U$ which is $\mathcal{C}^\infty$ outside the diagonal $\{(x,x); x \in U\}$ of $U \times U$ and satisfies an estimate of form $|x-y|^k \tilde{E}(x,y)| \leq c$ for some k for $x \neq y$. The integral is here understood in distribution sense, i.e., the meaning of $Tu(x)$ is that if $v \in \mathcal{C}_0^\infty(U)$, then $(Tu)(v) = \tilde{E}(v(x) \otimes u(y))$. We will assume for simplicity at first that $n = 1$, and since we are only interested in local properties, that $U = \mathbb{R}$. At the end of this remark we will then remove the assumption $n = 1$. What follows is somehow folklore.

If we add to $\tilde{E}$ a distribution F concentrated on the diagonal Y of $\mathbb{R} \times \mathbb{R}$ (i.e., $Y = \{(x,x); x \in \mathbb{R}\}$), then the restriction of $\tilde{E} + F$ to $\mathbb{R}^2 \setminus Y$ is not changed, so we can not study the mapping properties of T by studying only the restriction of the initial $\tilde{E}$ to $\mathbb{R}^2 \setminus Y$. It seems therefore reasonable to reformulate the problem in the following way: given a $\mathcal{C}^\infty$ function E on $\mathbb{R}^2 \setminus Y$ which satisfies $|x-y|^k|E(x,y)| \leq c$ on $x \neq y$, extend this E in some natural way to a distribution $\tilde{E}$ in $\mathbb{R}^2$ and study the mapping properties of the integral operator associated with the kernel $\tilde{E}$. It is this what we want to do briefly in what follows.

As mentioned before, we now consider a "natural extension" of the restriction of E to $\mathbb{R}^2 \setminus Y$. To do so we will proceed as follows. As a distribution on $\mathbb{R}^2 \setminus Y$, E acts by $\int_{\mathbb{R}^2} E(x,y)w(x,y)dxdy$, $w \in \mathcal{C}_0^\infty(\mathbb{R}^2 \setminus Y)$, and we can rewrite this as $\int_{\mathbb{R}^2} E(x,t-x)u(x,t-x)dxdt$, and denote $E(x,t-x)$ by $G(x,t)$. The set $\mathbb{R}^2 \setminus Y$ has transformed to the set $\{(x,t); t \neq 0\}$ and

G is $\mathcal{C}^\infty$ on this set and satisfies $|t^k G(x,t)| \le c$ there. Now we take a primitive $H(x,t)$ of order $(k+1)$ in t for $G(x,t)$. The main property of H is thus that $G(x,t) = (d/dt)^{k+1} H(x,t)$. We can take this primitive in such a way that H remains $\mathcal{C}^\infty$ for $t \neq 0$ and becomes bounded with all derivatives in x. For $w \in \mathcal{C}_0^\infty(\mathbb{R}^2 \setminus Y)$ we then have

$$\begin{aligned}\int_{\mathbb{R}^2} E(x,y)w(x,y)dy &= \int_{t\neq 0} (d/dt)^{k+1} G(x,t) w(x,t-x)dxdt \\ &= \int_{t\neq 0} (d/dt)^{k+1} G(x,t) w(x,t-x)dxdt \\ &= (-1)^{k+1} \int_{t\neq 0} G(x,t)(d/dt)^{k+1} w(x,t-x)dxdt.\end{aligned}$$

A natural extension of E from $\mathbb{R}^2 \setminus Y$ to a distribution $\tilde{E}$ defined on $\mathbb{R}^2$ is then

$$w \to \int_{t\neq 0} G(x,t)(d/dt)^{k+1} w(x,t-x)dt, \quad \text{for } w \in \mathcal{C}_0^\infty(\mathbb{R}^2).$$

We now regard the distribution so extended as a kernel of an operator $\mathcal{C}_0^\infty(\mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R})$. The resulting operator is of course

$$u \to (-1)^{k+1} \int_{t\neq 0} G(x,t)(d/dt)^{k+1} u(t-x)dt.$$

Clearly, this operator maps $\mathcal{C}^\ell$ functions u to $\mathcal{C}^{\ell-k-1}$ functions. This is then the searched for relation between the singularity type "k" at the diagonal and the "loss" of derivatives of the "natural" integral operator associated with E defined on $\mathbb{R}^2 \setminus Y$. That this "loss" cannot be improved in general, is clear from explicit examples.

Actually, the preceding arguments can be applied with only small changes also in the case of several variables if we argue as above replacing primitives by taking suitable solutions of $\Delta_t^{k'} g(x,t) = f(x,t)$ (as in remark 3.4). We omit details.

Remark 3.6. Our main assumption on the operator $p(x,D)$ was that it had to be hypoelliptic. While this is possibly not the best result under which one can prove results of the type of theorem 1.1, some condition on the operator $p(x,D)$ is certainly needed if we want the conclusion in the theorem to hold. We give a simple example in one variable where extendible solutions of some differential equation will not have temperate growth at the boundary. We put $U = \{x \in \mathbb{R};\, x > 0\}$. The operator will be defined on $V = \mathbb{R}$ and it will be of form $u \to A(x)(d/dx)u(x) + B(x)u(x)$, with $A, B \in \mathcal{C}^\infty(\mathbb{R})$.

It follows that the solutions of $p(x,D)u = 0$ on U are given, up to an additive constant, by

$$u(x) = \exp\,[\textstyle\int_x^1 B(t)/A(t)dt]. \tag{3.1}$$

If we want this u to tend to infinity at $t = 0+$, we must assume that $A(0) = 0$. We will define some suitable A in a moment, but will have to assume a rather strong vanishing at 0. Let us at first denote by $\tilde{a}, \tilde{b} : U \to \mathbb{R}$ the functions $\tilde{a}(x) = u(x), \tilde{b}(x) = -(d/dx)u(x)$, where

$u = (d/dx)\sin(1/x)$ is the function in remark 3.1. Then we have trivially $\tilde{a}(x)(d/dx)u(x) + \tilde{b}(x)u(x) = 0$ on $(0, \infty)$. Next we consider a function $h : U \to \mathbb{R}$ in $\mathcal{C}^\infty(U)$, which tends sufficiently rapidly to 0 for $x \to 0$ in order to imply that $a(x) = h(x)\tilde{a}(x)$ and $b(x) = h(x)\tilde{b}(x)$ extend to $\mathcal{C}^\infty$ functions defined on $[0, \infty)$, and such that all (one-sided) derivatives of $a(x), b(x)$ vanish at 0. We may also assume that $h(x) > 0$ for $x \neq 0$. Finally extend a, b, to $\mathcal{C}^\infty$ functions A, B defined this time on $\mathbb{R}$, for example by setting $A(x) = a(-x), B(x) = b(-x)$ for $x < 0$ and $A(x) = a(x), B(x) = b(x)$ for $x \geq 0$. The operator $p(x, D)f = A(x)(d/dx)f(x) + B(x)f(x)$ has then $\mathcal{C}^\infty$ coefficients on $\mathbb{R}$ and we have $p(x, D)u(x) = 0$ for $x \in U$.

Theorem 1.1 also has a bearing on the structure of the singularities of solutions of $p(D)u = 0$, $p(D)$ a hypoelliptic constant coefficients operator. We assume that u is defined outside 0, say on $\{x \in \mathbb{R}^n; 0 < |x| < 1\}$ and are interested in the structure of u near 0 when u has temperate growth at 0. We next fix in an arbitrary way a fundamental solution E of $p(D)$. We then have

Proposition 3.7. *Under the above assumptions, we can find a constant coefficient linear partial differential operator $R(D)$ and a $\mathcal{C}^\infty$ function G which satisfies $p(D)G = 0$ on $U = \{x; 0 < |x| < 1\}$ such that $u = G + R(D)E$ (still on $0 < |x| < 1$). Conversely, every function u on U of this form is a solution of $p(D)u = 0$ with temperate growth at 0.*

Proof: We denote in this argument by δ the Dirac distribution at 0 and consider an extension w of u to the set $\{x; |x| < 1\}$. Such an extension exists since u has temperate growth at 0. Then $p(D)w$ is a distribution concentrated at 0. It follows that w has the form $w = R(D)\delta$, for some constant coefficient partial differential operator $R(D)$. Then $G = w - R(D)E$ will satisfy $p(D)G = 0$, which shows in particular that G is a $\mathcal{C}^\infty$ function on $|x| < 1$. This gives the first half of the proposition. The other half follows from the fact that E restricted to $x \neq 0$, is a solution of $p(D)E = 0$ which has temperate growth at 0. ∎

We have not tried to find reasonable cases for variable coefficient operators where results similar to proposition 3.7 hold.

Remark 3.8. Another remark is that in general solutions of $p(D)u = 0$ defined for $x \neq 0$, $p(D)$ constant coefficient and hypoelliptic, will not all have temperate growth at 0. To give an example, we recall that for the constant coefficient case all solutions of a homogeneous hypoelliptic equation will have Gevrey-s regularity for some $s > 0$. More precisely, given a constant coefficient hypoelliptic operator $p(D)$, there is some $s > 0$ with the following property: if $u \in \mathcal{C}^\infty(U)$ satisfies $p(D)u = 0$ in a neighborhood of some point $\tilde{x} \in U$, and if we fix $c > 0$ such that $\{x; |x - \tilde{x}| \leq c\} \subset U$, then we have for some $d > 0$ that

$$\sup_{|x - \tilde{x}| \leq c} |\partial_x^\alpha u(x)| / (d|\alpha|)^{|\alpha|/s} < \infty. \tag{3.2}$$

When p is elliptic, we obtain this for $s = 1$. Since the case $s = 1$ would be a little bit special in what follows, we will also in this case use (3.2) with $s < 1$, since if (3.2) holds for $s = 1$, it also holds for every $s < 1$.

Now consider an infra-exponential infinite order partial differential operator with constant coefficients of Gevrey type s. More precisely, we consider operators $G(D)$ which are constructed as follows. We start from an entire function $\zeta \to R(\zeta)$, defined on $\mathbb{C}^n$, for which $\sup_{\zeta\in\mathbb{C}^n} |R(\zeta)| \exp[-\varepsilon|\zeta|^s] < \infty$ for every $\varepsilon > 0$, and then we associate with it an operator on functions φ of Gevrey type s with compact support (it is here that it pays to assume $s < 1$, since otherwise φ would have to be real analytic) by setting

$$R(D)\varphi = \mathcal{F}^{-1}(R(\xi)\hat{\varphi}(\xi)). \tag{3.3}$$

(the "hat" denotes again the Fourier transform and $\mathcal{F}^{-1}(R(\xi)\hat{\varphi}(\xi))$ denotes the Fourier inverse transform of $R(\xi)\hat{\varphi}(\xi)$.) The fact that this is well-defined is easy to check. Actually, $R(D)$ is a local operator, so $R(D)$ immediately extends to Gevrey -s functions which do not necessarily have compact support. Now consider a fundamental solution E of $p(D)$ and define $u = R(D)E$ for $x \neq 0$. We then also have that on $\mathbb{R}^n$, $R(D)E = R(D)\delta$, where the last expression is calculated in ultra-distributions. When $\zeta \to R(\zeta)$ is not a polynomial, it follows from proposition 3.7 that the function $u = R(D)E$, defined for $x \neq 0$, is a solution of $p(D)u = 0$ which cannot have temperate growth at 0.

Remark 3.9. A slightly more interesting example of the same type, but for a much bigger boundary, is the following. We consider the Laplace operator $\Delta = (\partial/\partial x)^2 + (\partial/\partial y)^2$ as an operator on $U = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 < 1\}$ and consider an analytic functional v on $x^2 + y^2 = 1$ which is not a distribution. Let u be a solution on U of the Dirichlet problem: $\Delta u = 0$ in $\{(x, y) \in \mathbb{R}^2; x^2 + y^2 < 1\}$, $u_{|x^2+y^2=1} = v$. Note that a solution for this problem is given (e.g.) by Poisson's formula. It is then classical, and it also follows from results proved above, that u cannot have temperate growth at the boundary. (Otherwise, it were extendible across the boundary, and the boundary value would have to be a distribution, e.g., by theorem B.2.9 in Hörmander, [12], vol. III.)

Remark 3.10. In another context and with another motivation, temperate growth at the boundary is interesting in the theory of several complex variables. More precisely, if $U \subset \mathbb{R}^n$ is open, if $\Gamma \subset \mathbb{R}^n \setminus \{0\}$ is an open cone in $\mathbb{R}^n \setminus \{0\}$, and if finally, f is a holomorphic function on $\Omega = \{z \in (U + i\Gamma), |z| < c_1\}$, then f admits distributional boundary values at the edge $\{z; \mathrm{Re}\, z \in U, \mathrm{Im}\, z = 0\}$ of Ω precisely if f is of temperate growth at $\mathrm{Im}\, x = 0$. By this we mean that for every compact $K \subset U$ there are constants c, k, such that $|f(z)| \leq c|\,\mathrm{Im}\, z|^{-k}$, if $\mathrm{Re}\, z \in K, \mathrm{Im}\, z \in \Gamma, |z| < d$.

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