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# On dilation, scattering and spectral theory for two-interval singular differential operators

by

[(1)]Bilender P. Allahverdiev and [(2)]Ekin Uğurlu

**Abstract**

This paper aims to construct a space of boundary values for minimal symmetric singular impulsive-like Sturm-Liouville (SL) operator in limit-circle case at singular end points $a, b$ and regular inner point $c$. For this purpose all maximal dissipative, accumulative and self-adjoint extensions of the symmetric operator are described in terms of boundary conditions. We construct a self-adjoint dilation of maximal dissipative operator, a functional model and we determine its characteristic function in terms of the scattering matrix of the dilation. The theorem verifying the completeness of the eigenfunctions and the associated functions of the dissipative SL operator is proved.

**Key Words**: Impulsive-like Sturm-Liouville operator, extensions of the symmetric operator, dissipative operator, self-adjoint dilation, completeness of the eigenfunctions and the associated functions.
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## 1 Introduction

An important class of non-self-adjoint operators is the class of dissipative operators. In recent years, a lot of papers have been published about dissipative operators (for example, see [2-6, 8, 9, 24]). It is known that all eigenvalues of a dissipative operators lie in the closed upper half-plane. One of the most important problems for non-self-adjoint dissipative operators is the completeness of the system of all eigenfunctions and associated functions of these operators. So, it is important to analyse the spectral properties of dissipative operators. There are some methods to investigate the spectral theory of dissipative operators. One of the methods is the functional model which is associated with the equivalence of Lax-Phillps scattering matrix [17] and Sz.-Nagy-Foiaş characteristic function [18].

In 1967, Lax and Phillips defined the abstract scattering matrix, also called the Lax-Phillips scattering matrix, which exactly coincides with the scattering matrix [17]. This scattering matrix acts in the subspaces $D_-$ and $D_+$ called the incoming and outgoing subspaces, respectively, of the Hilbert space $H$. Further an unitary group $V(s)$ and the subspaces $D_-$ and $D_+$ have the following properties:

$$
\begin{aligned}
&\text{(i) } V(s)D_- \subset D_-,\ s \leq 0; V(s)D_+ \subset D_+,\ s \geq 0,\\
&\text{(ii) } \bigcap_{s\leq 0} V(s)D_- = \bigcap_{s\geq 0} V(s)D_+ = \{0\},\\
&\text{(iii) } \overline{\bigcup_{s\geq 0} V(s)D_+} = \overline{\bigcup_{s\leq 0} V(s)D_-} = H,\\
&\text{(iv) } D_- \perp D_+.
\end{aligned}
$$

On the other hand, to investigate the spectral properties of contractive operators, Sz.-Nagy and Foiaş studied the characteristic functions of contractions [18]. Moreover they proved a theorem on completeness of eigenvectors and associated vectors of contractions. It is fortunate that there is an equivalence between Lax-Phillips scattering matrix and Sz.-Nagy-Foiaş characteristic function. This fact may allow us to know that whether all eigenvectors and associated vectors of a dissipative operator are complete or not in the Hilbert space. In the literature, there are some works containing the spectral theory of non-self-adjoint (dissipative) operators. For example the spectral analysis of dissipative operators defined on a single interval was investigated in detail in [1, 2, 4, 5, 20, 21]. Other non-self-adjoint problems were investigated in [3, 6, 8, 9, 24].

In this paper, we consider the minimal symmetric singular impulsive-like Sturm-Liouville (SL) operator acting in the Hilbert space $L_r^2(\Omega)$, where $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 = (a, c)$, $\Omega_2 = (c, b)$ with deficiency indices $(4, 4)$, i.e., limit-circle case holds at singular end points $a, b$ and inner point $c$ is regular. We construct a space of boundary values and describe all maximal dissipative accumulative and self-adjoint extensions in terms of the boundary conditions. We construct a self-adjoint dilation of maximal dissipative operator and its incoming and outgoing spectral representations, which makes it possible to determine the scattering matrix of dilation according to the scheme of Lax and Phillips [17]. We also construct a functional model of dissipative operator and its characteristic function. Finally, on the basis of the results obtained regarding the theory of the characteristic function, we prove a theorem on completeness of the system of eigenfunctions and associated functions of dissipative SL operators.

## 2 Space of the boundary values and extensions of the symmetric operator

In this paper we consider the following differential expression

$$\tau(y) := \frac{1}{r(t)}\left[-(p(t)y')' + q(t)y\right],\ t \in (a,c) \cup (c,b).$$

We set $\Omega_1 := (a,c)$, $\Omega_2 := (c,b)$, $-\infty \le a < c < b \le +\infty$ and $\Omega := \Omega_1 \cup \Omega_2$. We assume that the points $a$ and $b$ are singular and $c$ is regular for the differential expression $\tau$. $r, p$ and $q$ are real-valued, Lebesgue measurable functions on $\Omega$ and $r, \frac{1}{p}, q \in L^1_{loc}(\Omega_k)$, $k = 1,2$, $r(t) > 0$ for almost all $t \in \Omega$ and $r(t) = \left\{\begin{smallmatrix} r_1(t), & t\in\Omega_1 \\ r_2(t), & t\in\Omega_2 \end{smallmatrix}\right.$. The point $c$ is regular if $r, \frac{1}{p}, q \in L^1[c-\epsilon, c+\epsilon]$ for some $\epsilon > 0$.

Denote by $\mathcal{D}$ the linear set of all function $y \in \mathfrak{H}$ such that $y$, $py'$ are locally absolutely continuous functions on $\Omega$, one-sided limits $y(c\pm), (py')(c\pm)$ exist and are finite and $\tau(y) \in \mathfrak{H}$, where $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$, $\mathfrak{H}_m := L^2_r(\Omega_m)$, $m = 1,2$, denotes the Hilbert space containing all complex-valued functions $y$ such that $\int_\Omega r(t)\,|y(t)|^2\,dt < +\infty$ and equipped with the inner product $(y,z) = (y,z)_{\mathfrak{H}_1} + (y,z)_{\mathfrak{H}_2}$ and

$$(y,z)_{\mathfrak{H}_m} = \int_{\Omega_m} r_m(t) y_m(t) \overline{z_m(t)} dt, m = 1,2.$$

The operator $T$ defined by $Ty = \tau(y)$ is called the maximal operator $T$ on $\mathcal{D}$. Let us adopt the notation $[y,z](t) := p(t)(y(t)\overline{z'(t)} - y'(t)\overline{z(t)})$. Then the values $[y,z](a) := \lim_{t\to a^+}[y,z](t)$ and $[y,z](b) := \lim_{t\to b^-}[y,z](t)$ exist and are finite. In fact one gets the Green's formula: for arbitrary $y,z \in \mathcal{D}$, Green's formula is

$$\int_\Omega r(t)\tau(y)\overline{z(t)}dt - \int_\Omega r(t)y(t)\overline{\tau(z)}dt = [y,z](c-) - [y,z](a) + [y,z](b) - [y,z](c+).$$

Let us consider the set $\mathcal{D}_0$ consisting all functions $y$ from $\mathcal{D}$ satisfying the following conditions $[y,z](a) = y(c-) = (py')(c-) = y(c+) = (py')(c+) = [y,z](b) = 0$, where $z \in \mathcal{D}$. The operator $T_0$ which is the restriction of the operator $T$ to $\mathcal{D}_0$ is called the minimal operator generated by $\tau$ and it is closed symmetric operator with deficiency indices $(s,s)$, $s = 2,3,4$. Moreover $T_0^* = T$ [10, 11, 19, 25, 26]. In this paper we assume that the deficiency indices of the minimal symmetric operator $T_0$ are $(4,4)$ [7, 10, 11, 14, 15, 19, 23, 25, 26].

Let us set $u = \left\{\begin{smallmatrix} u_1, & t\in\Omega_1 \\ u_2, & t\in\Omega_2 \end{smallmatrix}\right.$ and $v = \left\{\begin{smallmatrix} v_1, & t\in\Omega_1 \\ v_2, & t\in\Omega_2 \end{smallmatrix}\right.$ satisfying

$$\left\{\begin{array}{ll} u_1(c-) = 1, & (pu_1')(c-) = 0, \\ v_1(c-) = 0, & (pv_1')(c-) = 1 \end{array}\right., \quad \left\{\begin{array}{ll} u_2(c+) = 1, & (pu_2')(c+) = 0, \\ v_2(c+) = 0, & (pv_2')(c+) = 1. \end{array}\right.$$

Then $\{u,v\}$ is the fundamental system of the equation $\tau(y) = 0$ $(t \in \Omega)$.

Consider the following linear mappings from $\mathcal{D}$ into $\mathcal{E} := \mathbb{C}^4$ defined by $G_1 y = (\left(G^1\right)_1 y, \left(G^2\right)_1 y)^T$ and $G_2 y = (\left(G^1\right)_2 y, \left(G^2\right)_2 y)^T$, where $\left(G^1\right)_i$ and $\left(G^2\right)_i$ $(i = 1,2)$ are linear mappings of $\mathcal{D}\mid_{\Omega_1}$ and $\mathcal{D}\mid_{\Omega_2}$, respectively, into

$\mathbb{C}^2$ with $\left(G^1\right)_1 y := (-[y,u](a), y(c-))^T$, $\left(G^2\right)_1 y := (-y(c+), [y,u](b))^T$ and $\left(G^1\right)_2 y := ([y,v](a), (py')(c-))^T$, $\left(G^2\right)_2 y := ((py')(c+), [y,v](b))^T$, where superscript $T$ denotes the transpose of the vector.

The proof of the following Lemma can be obtained by using Naimark's Patching Theorem [19].

**Lemma 2.1.** *For any complex numbers* $\alpha_i, \beta_i, \theta_i, \gamma_i \in \mathbb{C}$ $(i = 0, 1)$, *there is a function* $y \in \mathcal{D}$ *satisfying*

$$\begin{aligned} &[y,u](a) = \gamma_0, \quad [y,v](a) = \theta_0, \quad y(c-) = \alpha_0, \quad (py')(c-) = \alpha_1, \\ &y(c+) = \beta_0, \quad (py')(c+) = \beta_1, \quad [y,u](b) = \gamma_1, \quad [y,v](b) = \theta_1. \end{aligned} \tag{2.1}$$

Using Lemma 2.1 and the theory of space of boundary values (see [13]) we have the following theorem.

**Theorem 2.2.** *The triplet* $(\mathcal{E}, G_1, G_2)$ *is a space of boundary values of the operator* $T_0$.

**Proof**: Let $y$ and $z$ be two functions from $\mathcal{D}$. Then one can get the equality

$$(T_0^* y, z) - (y, T_0^* z) = [y,z](c-) - [y,z](a) + [y,z](b) - [y,z](c+).$$

On the other side, the equality $(T_0^* y, z) - (y, T_0^* z) = (G_1 y, G_2 z)_{\mathcal{E}} - (G_2 y, G_1 z)_{\mathcal{E}}$ holds. So the theorem is proved. □

Using Theorem 2.2, [13, Theorem 1.6, p.156] and linear mappings $G_1$ and $G_2$ we can state the following theorem.

**Theorem 2.3.** *For any contraction* $\mathfrak{K}$ *in* $\mathcal{E}$, *i.e.,* $\|\mathfrak{K}\|_{\mathcal{E}} \leq 1$, *the restriction of the operator* $T$ *to the set of functions* $f \in \mathcal{D}$ *satisfying the boundary condition*

$$(\mathfrak{K} - I)\, G_1 f + i\, (\mathfrak{K} + I)\, G_2 f = 0 \tag{2.2}$$

*or*

$$(\mathfrak{K} - I)\, G_1 f - i\, (\mathfrak{K} + I)\, G_2 f = 0 \tag{2.3}$$

*is respectively, a maximal dissipative or a maximal accumulative extension of the operator* $T_0$. *Conversely, every maximal dissipative (maximal accumulative) extension of* $T_0$ *is the restriction of* $T$ *to the set of vectors* $f \in \mathcal{D}$ *satisfying (2.2) ((2.3)), and the contraction* $\mathfrak{K}$ *is uniquely determined by the extensions. These conditions give self-adjoint extension if* $\mathfrak{K}$ *is unitary.*

In this paper, we consider the maximal dissipative operator $A_{\mathfrak{K}}$, where $\mathfrak{K}$ is a strict contraction in $\mathcal{E}$, i.e., $\|\mathfrak{K}\|_{\mathcal{E}} < 1$, generated by the differential expression $\tau$ and the boundary condition (2.2). The boundary condition (2.2) is equivalent to the condition

$$G_2 f + \mathfrak{T} G_1 f = 0, \; f \in \mathcal{D}, \tag{2.4}$$

where $\mathfrak{T} = -i\,(\mathfrak{K} + I)^{-1}(\mathfrak{K} - I)$, $\Im \mathfrak{T} > 0$, and $-\mathfrak{K}$ is the Cayley transform of the dissipative operator $\mathfrak{T}$. We denote by $L_{\mathfrak{T}}$ $(= A_{\mathfrak{K}})$ the maximal dissipative operator generated by the expression $\tau$ and the boundary condition (2.4). Let $\mathfrak{T}$ be an $4 \times 4$ matrix as $\mathfrak{T} = diag\,(h_1, h_2, h_3, h_4)$. Then the boundary condition (2.4) coincides with the separated boundary conditions.

The proof of the following lemma follows from the fact that $\tau$ is in limit-circle case at $a$ and $b$ and $\Im\mathfrak{T} > 0$.

**Lemma 2.4.** *The operator $L_{\mathfrak{T}}$ is completely non-self-adjoint (simple).*

## 3 Self-adjoint dilation, scattering matrix of dilation and spectral analysis of the dissipative operator

To analyse the operator $L_{\mathfrak{T}}$, we shall construct the scattering matrix. For this purpose, we form the main Hilbert space of the dilation $\mathbf{H} = L^2(\mathbb{R}_-;\mathcal{E}) \oplus \mathfrak{H} \oplus L^2(\mathbb{R}_+;\mathcal{E})$, where $\mathbb{R}_- := (-\infty, 0]$ and $\mathbb{R}_+ := [0,\infty)$. Consider the mappings $P : \mathbf{H} \to \mathfrak{H}$ and $P_1 : \mathfrak{H} \to \mathbf{H}$ the mappings with the rules $P : \langle \sigma_-, y, \sigma_+\rangle \to y$ and $P_1 : y \to \langle 0, y, 0\rangle$.

Construct the operator $\mathbf{L}_{\mathfrak{T}}$ in $\mathbf{H}$ generated by

$$\mathbf{L}\,\langle \sigma_-, y, \sigma_+\rangle = \left\langle i\frac{d\sigma_-}{d\xi}, \tau(y), i\frac{d\sigma_+}{d\zeta}\right\rangle, \tag{3.1}$$

on the set of vectors $D(\mathbf{L}_{\mathfrak{T}})$ satisfying the conditions: $\sigma_- \in W_2^1(\mathbb{R}_-;\mathcal{E})$, $\sigma_+ \in W_2^1(\mathbb{R}_+;\mathcal{E})$, $y \in \mathcal{D}$,

$$G_2 y + \mathfrak{T} G_1 y = \mathfrak{C}\sigma_-(0), \quad G_2 y + \mathfrak{T}^* G_1 y = \mathfrak{C}\sigma_+(0). \tag{3.2}$$

Here $\mathfrak{C}^2 := 2\Im\mathfrak{T}$, $\mathfrak{C} > 0$ and $W_2^1$ denotes the Sobolev space.

**Theorem 3.1.** *The operator $\mathbf{L}_{\mathfrak{T}}$ is self-adjoint in $\mathbf{H}$.*

**Proof**: Let $Y = \langle \sigma_-, y, \sigma_+\rangle, Z = \langle \omega_-, z, \omega_+\rangle \in D(\mathbf{L}_{\mathfrak{T}})$. Then we have

$$\begin{aligned}(\mathbf{L}_{\mathfrak{T}}Y, Z)_{\mathbf{H}} - (Y, \mathbf{L}_{\mathfrak{T}}Z)_{\mathbf{H}} \;=\; & [y,z](c-) - [y,z](a) + [y,z](b) - [y,z](c+)\\ & + i\,(\sigma_-(0), \omega_-(0))_{\mathcal{E}} - i\,(\sigma_+(0), \omega_+(0))_{\mathcal{E}}\,.\end{aligned} \tag{3.3}$$

Using the boundary conditions (3.2) in (3.3) we obtain that $\mathbf{L}_{\mathfrak{T}}$ is symmetric in $\mathbf{H}$.

Consider the function $Y = \langle \sigma_-, 0, \sigma_+\rangle$, $\sigma_\mp \in W_2^1(\mathbb{R}_\mp;\mathcal{E})$, $\sigma_\mp(0) = 0$. Then for $Z = \langle \omega_-, z, \omega_+\rangle \in D(\mathbf{L}_{\mathfrak{T}}^*)$, we obtain

$$(\mathbf{L}_{\mathfrak{T}}Y, Z)_{\mathbf{H}} = (\langle \sigma_-, 0, \sigma_+\rangle, \langle id\omega_-/d\xi, z^*, id\omega_+/d\zeta\rangle)_{\mathbf{H}},$$

and therefore $\mathbf{L}_{\mathfrak{T}}^* Z = \left\langle i\frac{d\omega_-}{d\xi}, z^*, i\frac{d\omega_+}{d\zeta}\right\rangle$, where $\omega_\mp \in W_2^1(\mathbb{R}_\mp)$, $z^* \in \mathfrak{H}$.

Now, let $Y = \langle 0, y, 0\rangle \in D(\mathbf{L}_{\mathfrak{T}})$ and put $Y$ in the Eq. (3.3). Then we arrive at $\mathbf{L}_{\mathfrak{T}}^* Z = \langle id\omega_-/d\xi, \tau(z), id\omega_+/d\zeta\rangle$, $z \in \mathcal{D}$. Consequently, we obtain

$$[y,z](c-) - [y,z](a) + [y,z](b) - [y,z](c+) + i\,(\sigma_-(0), \omega_-(0))_{\mathcal{E}} - i\,(\sigma_+(0), \omega_+(0))_{\mathcal{E}} = 0. \tag{3.4}$$

Further, solving the boundary conditions (3.2), we find that

$$G_1 y = -i\mathfrak{C}^{-1}\left(\sigma_-(0) - \sigma_+(0)\right), \ G_2 y = \mathfrak{C}\sigma_-(0) + i\mathfrak{T}\mathfrak{C}^{-1}\left(\sigma_-(0) - \sigma_+(0)\right). \quad (3.5)$$

Using the equalities (3.4) and (3.5) we obtain

$$\begin{aligned} i\left(\sigma_-(0), \omega_-(0)\right)_{\mathcal{E}} - i\left(\sigma_+(0), \omega_+(0)\right)_{\mathcal{E}} &= (G_1 y, G_2 z)_{\mathcal{E}} - (G_2 y, G_1 z)_{\mathcal{E}} \\ = -i\left(\mathfrak{C}^{-1}\left(\sigma_-(0) - \sigma_+(0)\right), G_2 z\right)_{\mathcal{E}} &- (\mathfrak{C}\sigma_-(0), G_1 z)_{\mathcal{E}} \\ -i\left(\mathfrak{T}\mathfrak{C}^{-1}\left(\sigma_-(0) - \sigma_+(0)\right), G_1 z\right)_{\mathcal{E}}. \end{aligned}$$

Comparing the coefficients of $\sigma_\mp(0)$, on the left and right of the last equality, it is proved that the vector $Z = \langle \omega_-, z, \omega_+ \rangle$ satisfies the boundary conditions $G_2 z + \mathfrak{T} G_1 z = \mathfrak{C}\sigma_-(0)$ and $G_2 z + \mathfrak{T}^* G_1 z = \mathfrak{C}\sigma_+(0)$. Therefore, $D(\mathbf{L}_{\mathfrak{T}}^*) \subseteq D(\mathbf{L}_{\mathfrak{T}})$, and this completes the proof. □

Let us consider the unitary group $V(s) = \exp(iL_{\mathfrak{T}} s)$ $(s \in \mathbb{R} := (-\infty, \infty))$ on $\mathbf{H}$, strongly continuous semi-group of completely non-unitary contractions $Z(s) := PV(s)P_1$ $(s \geq 0)$ on $\mathbf{H}$ and the generator $By = \lim_{s\to+0} \frac{1}{is}(Z(s)y - y)$ of $Z(s)$ [16,18]. Note that $B$ is a maximal dissipative operator and the operator $\mathbf{L}_{\mathfrak{T}}$ is called the self-adjoint dilation of $B$.

**Theorem 3.2.** *The operator $\mathbf{L}_{\mathfrak{T}}$ is a self-adjoint dilation of the operator $L_{\mathfrak{T}}$.*

**Proof**: Construct the equality $(\mathbf{L}_{\mathfrak{T}} - \lambda I)^{-1} P_1 y = g = \langle \omega_-, z, \omega_+ \rangle$. Hence $\tau(z) - \lambda z = y$, $\omega_-(\xi) = \omega_-(0) e^{-i\lambda\xi}$ and $\omega_+(\zeta) = \omega_+(0) e^{-i\lambda\zeta}$. Since $g \in D(\mathbf{L}_{\mathfrak{T}})$, we have $\omega_-(0) = 0$. This implies that $z$ satisfies the boundary condition $G_2 z + \mathfrak{T} G_1 z = 0$ and $z \in D(\mathbf{L}_{\mathfrak{T}})$. Moreover for $\Im\lambda < 0$ one can consider that $z = (L_{\mathfrak{T}} - \lambda I)^{-1} y$. Hence, for $y \in H$ and $\Im\lambda < 0$ we have $(\mathbf{L}_{\mathfrak{T}} - \lambda I)^{-1} P_1 y = \langle 0, (L_{\mathfrak{T}} - \lambda I)^{-1} y, \mathfrak{C}^{-1}\left(G_2 y + \mathfrak{T}^* G_1 y\right) e^{-i\lambda\zeta} \rangle$. Applying the mapping $P$ to the last equality, we have $P(\mathbf{L}_{\mathfrak{T}} - \lambda I)^{-1} P_1 y = (L_{\mathfrak{T}} - \lambda I)^{-1} y$, where $y \in \mathfrak{H}$ and $\Im\lambda < 0$. Therefore we obtain for $\Im\lambda < 0$ that $(L_{\mathfrak{T}} - \lambda I)^{-1} = P(\mathbf{L}_{\mathfrak{T}} - \lambda I)^{-1} P_1 = (B - \lambda I)^{-1}$ and from which we have $L_{\mathfrak{T}} = B$. □

We set $H_- = \overline{\underset{s\geq 0}{\cup} V(s) D_-}$ and $H_+ = \overline{\underset{s\leq 0}{\cup} V(s) D_+}$, where $D_- = \langle L^2(\mathbb{R}_-; \mathcal{E}), 0, 0 \rangle$ and $D_- = \langle 0, 0, L^2(\mathbb{R}_+; \mathcal{E}) \rangle$. Using Lemma 2.4 we get that $H_- + H_+ = \mathbf{H}$.

Let us consider the functions $\varphi(t,\lambda) = \begin{cases} \varphi_1(t,\lambda), & t\in\Omega_1 \\ \varphi_2(t,\lambda), & t\in\Omega_2 \end{cases}$, $\psi(t,\lambda) = \begin{cases} \psi_1(t,\lambda), & t\in\Omega_1 \\ \psi_2(t,\lambda), & t\in\Omega_2 \end{cases}$ satisfying the initial conditions

$$\begin{cases} [\varphi_1, v](a) = 0, & [\varphi_1, u](a) = -1, \\ [\psi_1, v](a) = 1, & [\psi_1, u](a) = 0, \end{cases} \qquad \begin{cases} \varphi_1(c+,\lambda) = 0, & (p\varphi')(c+,\lambda) = -1, \\ \psi_1(c+,\lambda) = 1, & (p\psi')(c+,\lambda) = 0. \end{cases}$$

We denote by $M(\lambda)$ as the form

$$M(\lambda) = \begin{pmatrix} M_1(\lambda) & 0 \\ 0 & M_2(\lambda) \end{pmatrix},$$

where $M_1(\lambda)$ and $M_2(\lambda)$ the matrix-valued functions satisfying the conditions: $M_1(\lambda)\left(G^1\right)_1\varphi_1 = \left(G^1\right)_2\varphi_1$, $M_1(\lambda)\left(G^1\right)_1\psi_1 = \left(G^1\right)_2\psi_1$ and $M_2(\lambda)\left(G^2\right)_1\varphi_2 = \left(G^2\right)_2\varphi_2$, $M_1(\lambda)\left(G^2\right)_1\psi_2 = \left(G^2\right)_2\psi_2$. Then we have $M(\lambda)G_1\varphi = G_2\varphi$ and $M(\lambda)G_1\psi = G_2\psi$. Note that $M(\lambda)$ is meromorphic in $\mathbb{C}$ with all its poles on real axis $\mathbb{R}$, and it has the following properties

*i)* $\Im M(\lambda) \leq 0$ if $\Im\lambda > 0$ and $\Im M(\lambda) \geq 0$ if $\Im\lambda < 0$;

*ii)* $M^*(\lambda) = M(\lambda)$ for all $\lambda \in \mathbb{R}$, except for the poles of $M(\lambda)$.

We denote by $\chi_j(t) = \left\{\begin{smallmatrix}(\chi_j)_1(t), & t\in\Omega_1\\ (\chi_j)_2(t), & t\in\Omega_2\end{smallmatrix}\right.$ and $\theta_j(t) = \left\{\begin{smallmatrix}(\theta_j)_1(t), & t\in\Omega_1\\ (\theta_j)_2(t), & t\in\Omega_2\end{smallmatrix}\right.$ being the solutions of the equations $\tau(y) = \lambda y$, $t \in \Omega$, which satisfy the conditions $G_1\chi_j = \left(M(\lambda) + \mathfrak{T}\right)^{-1}\mathfrak{C}\phi_j$ and $G_1\theta_j = \left(M(\lambda) + \mathfrak{T}^*\right)^{-1}\mathfrak{C}\phi_j$ $(j = 1, ...4)$, where $\phi_j$ are the orthonormal basis for $\mathcal{E}$.

Let us define the vector $\Psi^-_{\lambda j}$, $j = 1, ..., 4$,

$$\Psi^-_{\lambda j}(t,\xi,\zeta) = \left\langle e^{-i\lambda\xi}\phi_j, \chi_j(t), \mathfrak{C}^{-1}\left(M(\lambda) + \mathfrak{T}^*\right)\left(M(\lambda) + \mathfrak{T}\right)^{-1}\mathfrak{C}e^{-i\lambda\zeta}\phi_j\right\rangle.$$

With the help of the vector $\Psi^-_{\lambda j}(t,\xi,\zeta)$, we define the transformation $F_- : \mathbf{f} \to \widetilde{f}_-(\lambda)$ by $(F_-\mathbf{f})(\lambda) := \widetilde{f}_-(\lambda) := \sum\limits_{j=1}^{4} f^-_j(\lambda)\phi_j$, where $f^-_j(\lambda) = \frac{1}{\sqrt{2\pi}}(\mathbf{f}, \Psi^-_{\lambda j})$, $j = 1, ..., 4$, on the vectors $\mathbf{f} = \langle\sigma_-, f, \sigma_+\rangle$ in which $\sigma_-$, $\sigma_+$ and $f$ are smooth, compactly supported functions.

The transformation $F_-$ isometrically maps $H_-$ onto $L^2(\mathbb{R};\mathcal{E})$. For all vectors $\mathbf{f}, \mathbf{g} \in H_-$ the Parseval equality and the inverse formula hold

$$(\mathbf{f},\mathbf{g})_{\mathbf{H}} = (\widetilde{f}_-, \widetilde{g}_-)_{L^2} = \int\limits_{-\infty}^{\infty}\sum_{j=1}^{4} f^-_j(\lambda)\overline{g^-_j(\lambda)}d\lambda, \quad \mathbf{f} = \tfrac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\sum_{j=1}^{4}\Psi^-_{\lambda j}f^-_j(\lambda)d\lambda.$$

We set $\Psi^+_{\lambda j}(t,\xi,\zeta) = \left\langle\Theta(\lambda)e^{-i\lambda\xi}\phi_j, \theta_j(t), e^{-i\lambda\zeta}\phi_j\right\rangle$, $j = 1, ..., 4$, where

$$\Theta(\lambda) = \mathfrak{C}^{-1}\left(M(\lambda) + \mathfrak{T}\right)\left(M(\lambda) + \mathfrak{T}^*\right)^{-1}\mathfrak{C}. \tag{3.6}$$

With the help of the vector $\Psi^+_{\lambda j}(t,\xi,\zeta)$, we define the transformation $F_+ : \mathbf{f} \to \widetilde{f}_+(\lambda)$ by $(F_+\mathbf{f})(\lambda) := \widetilde{f}_+(\lambda) := \sum\limits_{j=1}^{4} f^+_j(\lambda)\phi_j$, where $f^+_j(\lambda) = \frac{1}{\sqrt{2\pi}}(\mathbf{f}, \Psi^+_{\lambda j})_{\mathbf{H}}$, $j = 1, ..., 4$, on the vectors $\mathbf{f} = \langle\sigma_-, f, \sigma_+\rangle$ in which $\sigma_-$, $\sigma_+$ and $f$ are smooth, compactly supported functions.

The transformation $F_+$ isometrically maps $H_+$ onto $L^2(\mathbb{R};\mathcal{E})$. For all vectors $\mathbf{f}, \mathbf{g} \in H_+$ the Parseval equality and the inverse formula hold

$$(\mathbf{f},\mathbf{g})_{\mathbf{H}} = (\widetilde{f}_+, \widetilde{g}_+)_{L^2} = \int\limits_{-\infty}^{\infty}\sum_{j=1}^{4} f^+_j(\lambda)\overline{g^+_j(\lambda)}d\lambda, \quad \mathbf{f} = \tfrac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\sum_{j=1}^{4}\Psi^+_{\lambda j}f^+_j(\lambda)d\lambda.$$

It is clear that the matrix-valued function $\Theta(\lambda)$ is meromorphic in $\mathbb{C}$ and all poles are in the lower half-plane. It is easy to obtain that $\|\Theta(\lambda)\| \leq 1$ for $\Im\lambda > 0$ and $\Theta(\lambda)$ is the unitary matrix for all $\lambda \in \mathbb{R}$.

Since $\Theta(\lambda)$ is the unitary matrix for $\lambda \in \mathbb{R}$, then, it follows from the definitions of the vectors $\Psi^{+}_{\lambda j}$ and $\Psi^{-}_{\lambda j}$ that

$$\Psi^{+}_{\lambda j} = \sum_{k=1}^{4} \Theta_{jk}(\lambda)\Psi^{-}_{\lambda j},\ j = 1, ..., 4, \tag{3.7}$$

where $\Theta_{jk}(\lambda)$, $j, k = 1, ..., 4$, are entries of the matrix $\Theta(\lambda)$.

$F_{-}$ is the incoming spectral representation for the group $\{V(s)\}$. Similarly, $F_{+}$ is the outgoing spectral representation for $\{V(s)\}$. It follows from (3.7) that $\Theta_{\mathfrak{T}}(\lambda) : \widetilde{f}_{-} = \Theta(\lambda)\widetilde{f}_{+}$. According to [17], we have the following theorem.

**Theorem 3.4.** *The matrix $\Theta^{-1}(\lambda)$ is the scattering matrix of the group $\{V(s)\}$ (of the self-adjoint operator $\mathbf{L}_{\mathfrak{T}}$).*

It follows from the explicit form of the unitary transformation $F_{-}$ that under the mapping $F_{-}$,

$$\begin{aligned} &\mathbf{H} \to L^2(\mathbb{R}),\ \mathbf{f} \to \widetilde{f}_{-}(\lambda),\ D_{-} \to H^2_{-},\ D_{+} \to \Theta H^2_{+}, \\ &\langle 0, \mathfrak{H}, 0\rangle \to H^2_{+} \ominus \Theta H^2_{+},\ V(s)\mathbf{f} \to (F_{-}V(s)F_{-}^{-1}\widetilde{f}_{-})(\lambda) = e^{i\lambda s}\widetilde{f}_{-}(\lambda), \end{aligned} \tag{3.8}$$

where $H^2_{+}$ denotes Hardy class in $L^2(\mathbb{R}; \mathcal{E})$ consisting of the vector valued functions analytically extendible to the upper half-plane.

**Theorem 3.5.** *The characteristic function of the maximal dissipative operator $L_{\mathfrak{T}}$ coincides with the matrix-valued function $\Theta_{\mathfrak{T}}(\lambda)$ defined by (3.6).*

**Proof**: The formulas (3.8) show that our operator $L_{\mathfrak{T}}$ is a unitary equivalent to the model dissipative operator with the characteristic function $\Theta(\lambda)$ [16, 18]. Since the characteristic functions of unitary equivalent dissipative operators coincide with each other the proof is completed. □

Let us introduce the inner product $\langle T, S\rangle = trS^*T$ for $T, S \in [\mathcal{E}]$ ($trS^*T$ is the trace of the operator $S^*T$). Hence, we may introduce the $\Gamma$-capacity of a set of $[\mathcal{E}]$ (see [12, 22]).

It is known [12] that the inner matrix-valued function $\Theta_{\mathfrak{K}}(\lambda)$ is a Blaschke-Potapov product if and only if $\det \Theta_{\mathfrak{K}}(\lambda)$ is a Blaschke product. We can infer that the characteristic function $\Theta_{\mathfrak{K}}(\lambda)$ is a Blaschke-Potapov product if and only if the matrix-valued function

$$\mathcal{Y}_{\mathfrak{K}}(\xi) = (I - \mathfrak{K}_1\mathfrak{K}_1^*)^{-1/2}(\theta(\xi) - \mathfrak{K}_1)(I - \mathfrak{K}_1^*\theta(\xi))^{-1}(I - \mathfrak{K}_1^*\mathfrak{K}_1)^{1/2}$$

is a Blaschke-Potapov product in a unit disk. Therefore using the result of [12, 16, 18] and all the obtained results for the maximal dissipative operator $A_{\mathfrak{K}}(L_{\mathfrak{T}})$, we have proved the following theorem.

**Theorem 3.6.** *For $\Gamma$-quasi-every strictly contractive $\mathfrak{K} \in [\mathcal{E}]$ (i.e., for all strictly contractive $\mathfrak{K} \in [\mathcal{E}]$ with the possible exception of a set of $\Gamma$-capacity zero), the characteristic function $\Theta_{\mathfrak{K}}(\lambda)$ of the maximal dissipative operator $A_{\mathfrak{K}}$*

*is a Blaschke-Potapov product, and the spectrum of $A_{\mathfrak{K}}$ is purely discrete and belongs to the open upper half-plane. For $\Gamma$-quasi-every strictly contractive $\mathfrak{K} \in [\mathcal{E}]$, the operator $A_{\mathfrak{K}}$ has a countable number of isolated eigenvalues with finite multiplicity and limit point at infinity, and the system of eigenfunctions and associated functions of this operator is complete in the space $\mathfrak{H}$.*

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(1)Department of Mathematics,
Suleyman Demirel University, 32260 Isparta, Turkey
E-mail: `bilenderpasaoglu@sdu.edu.tr`

(2)Department of Mathematics and Computer Science,
Çankaya University,
06530, Ankara, Turkey
E-mail: `ekinugurlu@yahoo.com`