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Algebraic invariants of graded ideals with a given Hilbert function in an exterior algebra

by
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Abstract

Let $E = K\langle e_1, \ldots, e_n\rangle$ be the exterior algebra over an n-dimensional vector space V with basis $e_1, \ldots, e_n$ over some field K. We introduce the universal lexsegment ideals in E and we devote our attention to their Hilbert function. Hence, we analyze the depth and the graded Betti numbers of a graded ideal with a given Hilbert function in E, via such a class of monomial ideals.

Key Words: Exterior algebra, monomial ideals, lexicographic ideals, minimal resolutions, standard invariants.
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1 Introduction

Let K be a field. We denote by $E = K\langle e_1, \ldots, e_n\rangle$ the exterior algebra over an n-dimensional K-vector space V with basis $e_1, \ldots, e_n$. A monomial ideal $I \subsetneq E$ is called a *lexsegment ideal* if for all monomials $u \in I$ and all monomials $v \in E$ with $\deg u = \deg v$ and $v >_{\text{lex}} u$, then $v \in I$, where $>_{\text{lex}}$ is the *lexicographic order* on the set $\text{Mon}_d(E)$ of all monomials of degree $d \geq 1$ in E. Set $E_{[m]} = K\langle e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+m}\rangle$, where m is a positive integer. A *universal lexsegment ideal* (ULI) of E is a lexsegment ideal I of E which still remains a lexsegment ideal when we regard I as an ideal of the exterior algebra $E_{[m]}$ for all $m \geq 1$.

Let $I \subsetneq E$ be a graded ideal and $H_{E/I}$ the Hilbert function of the quotient algebra E/I. Thus, $H_{E/I}(q) = \dim_K(E/I)_q$ $(q \geq 1)$ is the dimension of the K-subspace of E/I spanned by the homogeneous elements of E/I of degree q. A result due to Kruskal-Katona [2, 11] guarantees that, given a numerical function $H : \mathbb{N} \to \mathbb{N}$, where $\mathbb{N}$ is the set of non negative integers, there exists a graded ideal $I \subsetneq E$ such that H is the Hilbert function of the quotient algebra E/I if and only if

$$H(0) = 1, \qquad H(1) \leq n, \qquad H(q+1) \leq H(q)^{(q)}, \quad \text{for } q \geq 1, \tag{1.1}$$

where the integer $H(q)^{(q)}$ is defined in [2].

Aramova, Herzog and Hibi [2, Theorem 4.1] proved that if $I \subsetneq E$ is a graded ideal, then there exists always a unique lexsegment ideal $I^{\text{lex}} \subsetneq E$ such that $H_{E/I} = H_{E/I^{\text{lex}}}$. This property justifies the next definitions. A numerical function H satisfying the properties (1.1) is said *critical* if the lexsegment ideal I of E with $H_{E/I} = H$ is a ULI, and a graded ideal $I \subsetneq E$ is said *critical* if the Hilbert function of the graded algebra E/I is critical.

In this paper, we first introduce the class of universal lexsegment ideals in E, and then we deeply study their Hilbert function. Using combinatorial arguments, we give a precise description of the Hilbert function of a ULI. Such a description allows us to obtain some relevant results on the depth and on the graded Betti numbers of a graded ideal $I \subsetneq E$ with a given Hilbert function.

The plan of the paper is as follows. In Section 2, some notions that will be used throughout the paper are recalled. In Section 3, the universal lexsegment ideals in the exterior algebra $E = K\langle e_1, \dots, e_n \rangle$ are examined, and a characterization of such graded ideals given. In Section 4, the Hilbert function of a ULI in E is analyzed; the main result is a criterion stating when a numerical function satisfying some conditions is a critical Hilbert function. This criterion allows us to obtain the main result of Section 5. In fact, we prove that for a critical graded ideal I in E, one has $\text{depth}_E E/I = \text{depth}_E E/I^{\text{lex}}$. Furthermore, we show that a critical ideal $I \subsetneq E$ and the corresponding lexsegment ideal I^{lex} have the same graded Betti numbers.

2 Preliminaries and notations

Let K be a field. We denote by $E = K\langle e_1, \dots, e_n \rangle$ the exterior algebra over an n-dimensional K-vector space V with basis $e_1, \dots, e_n$. For a subset $\sigma = \{i_1, \dots, i_d\}$ of $[n] = \{1, \dots, n\}$ with $i_1 < i_2 < \cdots < i_d$, we write $e_\sigma = e_{i_1} \wedge \ldots \wedge e_{i_d}$, and call e_σ a monomial of degree d. We set $e_\sigma = 1$, if $\sigma = \emptyset$.

In order to simplify the notation, we write fg instead of $f \wedge g$ for any two elements f and g in E. An element $f \in E$ is called *homogeneous* of degree j if $f \in E_j$, where $E_j = \bigwedge^j V$.

Let $\mathcal{M}$ be the category of finitely generated $\mathbb{Z}$-graded left and right E-modules M satisfying $am = (-1)^{\deg a \deg m} ma$ for all homogeneous elements $a \in E$, $m \in M$. Let $M \in \mathcal{M}$. The supremum of the length of a maximal M-regular sequence is called the *depth* of M over E and denoted by $\text{depth}_E M$ [1].

An important invariant related to free resolutions of M is the *Castelnuovo-Mumford regularity* $\text{reg}_E M = \max\{j \in \mathbb{Z} : \beta_{i,i+j}(M) \neq 0 \text{ for some } i \geq 0\}$ of a non-zero module M, where $\beta_{i,j}(M)$ are the *graded Betti numbers* of M. We set $\text{reg}_E 0 = -\infty$. Recall that a minimal graded free resolution of an E-module

$M \in \mathcal{M}$ has always infinite length unless the module is free. Therefore, the projective dimension is not significant. For this reason one measures the growth rate of the Betti numbers by the *complexity* [1] which is defined as follows:

$$\mathrm{cx}_E M = \inf\{c \in \mathbb{Z} : \beta_i(M) \leq \alpha i^{c-1} \text{ for some } \alpha \in \mathbb{R} \text{ and for all } i \geq 1\},$$

where $\beta_i(M) = \sum_j \beta_{i,j}(M)$ is the (*total*) *Betti number* of M.

We close this Section recalling some notions on monomial ideals that will be useful in the sequel.

For any subset S of E, we denote by $\mathrm{Mon}(S)$ the set of all monomials in S, by $\mathrm{Mon}_d(S)$ the set of all monomials of degree $d \geq 1$ in S and by $|S|$ its cardinality.

Let u be a monomial in E. We define

$$\mathrm{supp}(u) = \{i \in [n] : e_i \text{ divides } u\},$$

and we write

$$\mathrm{m}(u) = \max\{i \in [n] : i \in \mathrm{supp}(u)\}.$$

We quote the next definition from [6].

Definition 1. *Let $\mathcal{N}$ be a subset of monomials of degree $d < n$ in E. The set of monomials of degree $d + 1$*

$$\mathrm{Shad}(\mathcal{N}) = \{(-1)^{\alpha(\sigma,j)} e_j e_\sigma : e_\sigma \in \mathcal{N},\ j \notin \mathrm{supp}(e_\sigma),\quad j = 1, \ldots, n\},$$

$\alpha(\sigma, j) = |\{r \in \sigma : r < j\}|$, *is called the shadow of $\mathcal{N}$.*

We define the i-th *shadow* recursively by $\mathrm{Shad}^i(\mathcal{N}) = \mathrm{Shad}(\mathrm{Shad}^{i-1}(\mathcal{N}))$.

In order to simplify the notations, if $I = \oplus_{d\geq 0} I_d$ is a graded ideal in E, we set $\mathrm{Shad}(I_d) = \mathrm{Shad}(\mathrm{Mon}_d(I))$.

Definition 2. *Let $I \subsetneq E$ be a monomial ideal. I is called stable if for each monomial $e_\sigma \in I$ and each $j < \mathrm{m}(e_\sigma)$ one has $e_j e_{\sigma \setminus \{\mathrm{m}(e_\sigma)\}} \in I$. I is called strongly stable if for each monomial $e_\sigma \in I$ and each $j \in \sigma$ one has $e_i e_{\sigma \setminus \{j\}} \in I$, for all $i < j$.*

Finally, if $I \subsetneq E$ is a monomial ideal, we denote by $G(I)$ the unique minimal set of monomial generators of I and by $G(I)_d$ the set of all monomials of degree $d \geq 1$ of $G(I)$.

3 Universal lexsegment ideals

In this Section, we introduce the universal lexsegment ideals in the exterior algebra $E = K\langle e_1, \ldots, e_n \rangle$.

Denote by $>_{\mathrm{lex}}$ the *lexicographic order* (lex order, in short) on $\mathrm{Mon}_d(E)$, *i.e.*, if $e_\sigma = e_{i_1} e_{i_2} \cdots e_{i_d}$ and $e_\tau = e_{j_1} e_{j_2} \cdots e_{j_d}$ are monomials belonging to $\mathrm{Mon}_d(E)$ with $1 \leq i_1 < i_2 < \cdots < i_d \leq n$ and $1 \leq j_1 < j_2 < \cdots < j_d \leq n$, then $e_\sigma >_{\mathrm{lex}} e_\tau$ if $i_1 = j_1, \ldots, i_{s-1} = j_{s-1}$ and $i_s < j_s$ for some $1 \leq s \leq d$.

Definition 3. *A monomial ideal $I \subsetneq E$ is called a lexsegment ideal if for all monomials $u \in I$ and all monomials $v \in E$ with $\deg u = \deg v$ and $v >_{\text{lex}} u$, then $v \in I$.*

Every *lexsegment ideal* of E is obviously a *stable ideal.*

Set $E_{[m]} = K\langle e_1, \dots, e_n, e_{n+1}, \dots, e_{n+m}\rangle$, where m is a positive integer. Following [3], we give the following definition.

Definition 4. *A lexsegment ideal I of E is called a universal lexsegment ideal (ULI), if for any integer $m \geq 1$, the monomial ideal $IE_{[m]}$ of the exterior algebra $E_{[m]}$ is a lexsegment ideal.*

Example 1. *The lexsegment ideal $I = (e_1e_2, e_1e_3e_4)$ of $E = K\langle e_1, e_2, e_3, e_4\rangle$ is a ULI. Indeed, I is a lexsegment ideal of the exterior algebra $E_{[m]}$ for all $m \geq 1$.*

Example 2. *The lexsegment ideal $I = (e_1e_2, e_1e_3e_4, e_2e_3e_4)$ of the exterior algebra $E = K\langle e_1, e_2, e_3, e_4\rangle$ is not a ULI. Indeed, I is not a lexsegment ideal of the exterior algebra $E_{[1]} = K\langle e_1, e_2, e_3, e_4, e_5\rangle$. In fact $e_1e_3e_5 >_{\text{lex}} e_2e_3e_4$ and $e_1e_3e_5 \notin IE_{[1]}$.*

Now we discuss the combinatorics of universal lexsegment ideals.
For a sequence of non negative integers $(k_i)_{i\in\mathbb{N}}$, we define the following set:

$$\operatorname{supp}(k_i)_{i\in\mathbb{N}} = \{i \in \mathbb{N} : k_i \neq 0\}.$$

If $\operatorname{supp}(k_i)_{i\in\mathbb{N}} = \{d_1, \dots, d_t\}$, with $d_1 < d_2 < \cdots < d_t$, then we associate to $(k_i)_{i\in\mathbb{N}}$ the integers $R_j = j + \sum_{i=1}^{j} k_i$, $1 \leq j \leq d_t$. We set $R_j = 0$, for $j > d_t$.

Following [7, Characterization 2.1](see also [3, Definition 4.1]), we state the following characterization.

Characterization 1. *Assume that $I \subsetneq E$ is an ideal generated in degrees $d_1 < d_2 < \cdots < d_t$. Then I is a ULI of E if and only if*

$$G(I)_{d_i} = \left\{e_{R_1}e_{R_2}\cdots e_{R_{d_i-1}}e_\ell : R_{d_i-1} + 1 \leq \ell \leq R_{d_i} - 1\right\},$$

for $1 \leq i \leq t$, where $R_j = j + \sum_{i=1}^{j} |G(I)_i|$, for $1 \leq j \leq d_t$.

Remark 1. *Assume that $(k_i)_{i\in\mathbb{N}}$ is a sequence of non negative integers such that*

$$\operatorname{supp}(k_i)_{i\in\mathbb{N}} = \{d_1, \dots, d_t\}, \quad d_1 < d_2 < \cdots < d_t.$$

Then there exists a ULI $I \subsetneq E = K\langle e_1, \dots, e_n\rangle$ generated in degrees $d_1 < \cdots < d_t$ such that $|G(I)_{d_i}| = k_{d_i}$, $1 \leq i \leq t$, if and only if $n \geq d_t + \sum_{i=1}^{d_t} k_i - 1$, i.e, *$|G(I)| \leq n - d_t + 1$. In particular, if I is a lexsegment ideal of E generated in degree d, then I is a ULI if and only if $|G(I)| \leq n - d + 1$. Hence, if I is a ULI generated in degree d, one has:*

$$G(I) = \{e_1e_2\cdots e_{d-1}e_d, e_1e_2\cdots e_{d-1}e_{d+1}, \dots, e_1e_2\cdots e_{d-1}e_k\}, \tag{3.1}$$

with $d \leq k \leq n$.

In closing this Section we give the formula for computing the graded Betti numbers of a ULI in an exterior algebra.

Proposition 1. *Let $I \subsetneq E = K\langle e_1, \dots, e_n\rangle$ be a ULI generated in degrees $d_1 < \cdots < d_t$. Set $|G(I)_{d_i}| = k_{d_i}$, $1 \le i \le t$. Then*

$$\beta_{i,i+j}(I) = \sum_{\ell=1}^{k_j} \binom{j + \sum_{r=1}^{j-1} k_r + \ell + i - 2}{i}, \quad \textit{for all } i \ge 0.$$

Proof: Let $u \in G(I)$ a monomial of degree j. From Characterization 1, it follows that $\mathrm{m}(u) = j - 1 + \sum_{r=1}^{j-1} k_r + \ell$, for $1 \le \ell \le k_j$. From the formula on the Betti numbers for a stable ideal [2, Corollary 3.3], the assertion follows. □

4 The Hilbert function of a ULI

In this Section, we describe the Hilbert function of a ULI in the exterior algebra $E = K\langle e_1, \dots, e_n\rangle$.

In order to accomplish this task we need to introduce some notations.

For a graded ideal I we denote by $\mathrm{indeg}(I)$ the *initial degree* of I, *i.e.*, the minimum s such that $I_s \neq 0$.

Let $u \in \mathrm{Mon}_d(E)$, and define the following subset of $\mathrm{Mon}_{d+1}(E)$:

$$u\mathbf{e}_{\mathrm{m}(u)} = \{ue_{\mathrm{m}(u)+1}, \dots, ue_n\}.$$

Note that $u\mathbf{e}_{\mathrm{m}(u)} = \emptyset$ if $\mathrm{m}(u) = n$.

Example 3. *Let $u = e_1e_3e_4 \in E = K\langle e_1, \dots, e_6\rangle$, then $u\mathbf{e}_{\mathrm{m}(u)} = u\mathbf{e}_4 = \{e_1e_3e_4e_5, e_1e_3e_4e_6\}$.*

For a subset $\mathcal{N}$ of monomials of degree d of E, we define the following subset of monomials of degree $d+1$:

$$\mathrm{aShad}(\mathcal{N}) = \bigcup_{u \in \mathcal{N}} u\mathbf{e}_{\mathrm{m}(u)}. \tag{4.1}$$

We call the set $\mathrm{aShad}(\mathcal{N})$ the *almost shadow* of $\mathcal{N}$. We define the i-th *almost shadow* recursively by $\mathrm{aShad}^i(\mathcal{N}) = \mathrm{aShad}(\mathrm{aShad}^{i-1}(\mathcal{N}))$.

Remark 2. *If $u \in \mathrm{Mon}_d(E)$ and $\mathcal{N} = \{w \in \mathrm{Mon}_d(E) : w \ge_{\mathrm{lex}} u\}$, then $\mathrm{aShad}(\mathcal{N}) = \mathrm{Shad}(\mathcal{N})$.*

Example 4. *Let $\mathcal{N} = \{e_1e_5, e_1e_6, e_2e_3, e_2e_4\} \subsetneq \mathrm{Mon}_2(E)$, $E = K\langle e_1, \dots, e_6\rangle$, then $\mathrm{aShad}(\mathcal{N}) = \{e_1e_5e_6, e_2e_3e_4, e_2e_3e_5, e_2e_3e_6, e_2e_4e_5, e_2e_4e_6\}$.*

For a given graded ideal $I \subsetneq E$, we denote by I^{lex} the unique lexsegment ideal in E with the same Hilbert function as I.

The next definition, introduced in [8], was motivated by [2, Theorem 4.1].

Definition 5. *Let $H : \mathbb{N} \to \mathbb{N}$ be a numerical function. We call H a 0^*-sequence if H satisfies the properties (1.1).*

0^*-sequence with $H(p) = 0$ for $p \geq q$ will be written as:

$$(1, H(1), H(2), \dots, H(q-1), 0).$$

Definition 6. *Let $H \neq (1,0)$ be a 0^*-sequence, we set*

$$\text{indeg}H = \min\left\{d \,:\, H(d) \neq \binom{H(1)}{d}\right\},$$

and call it the initial degree *of H.*

Following [13], we give the following definition.

Definition 7. *Let H be a 0^*-sequence. H is* critical *if the lexsegment ideal I of E with $H_{E/I} = H$ is a ULI.*

Example 5. *The 0^*-sequence $H = (1,4,5,1,0)$ is critical. Indeed, there exists the ULI $I = (e_1e_2, e_1e_3e_4)$ of $E = K\langle e_1, e_2, e_3, e_4\rangle$ such that $H_{E/I} = H$.*

Example 6. *The 0^*-sequence $H = (1,4,5,0)$ is not critical. Indeed, the lexsegment ideal $I = (e_1e_2, e_1e_3e_4, e_2e_3e_4)$ of $E = K\langle e_1, e_2, e_3, e_4\rangle$ such that $H_{E/I} = H$ is not a ULI (Example 2).*

The next lemmas will be crucial in the sequel.

For a subset $\mathcal{N}$ of $\text{Mon}_d(E)$, we denote by $\max(\mathcal{N})$ the greatest monomial in $\mathcal{N}$ with respect to the lex order. Moreover, for a subset $\mathcal{N}$ of monomials in E, we define

$$\text{supp}(\mathcal{N}) = \{i \in [n] : i \in \text{supp}(u), \forall u \in \mathcal{N}\}.$$

Lemma 1. *Let $I \subsetneq E$ be a ULI generated in degree d. Then*

$$\dim_K I_{d+i} = \sum_{q=0}^{|G(I)|-1-c_d} \binom{n-d-q}{i},$$

where $c_d = 0$, for i=0 and for $1 \leq i \leq n-d$ if $|G(I)| < n-d+1$, whereas $c_d = 1$ for $1 \leq i \leq n-d$ if $|G(I)| = n-d+1$.

Proof: Set $k_d = |G(I)|$, and $s ::= \max\{\text{m}(u) \,:\, u \in G(I)_d\}$. From (3.1), it is $d \leq s \leq n$. For $i = 0$, $\dim_K I_d = |G(I)|$. For $i \geq 1$, we have

$$\dim_K I_{d+i} = |\text{Shad}^i(I_d)| = \sum_{q=0}^{k_d-1-c_d} \binom{n-d-q}{i} \tag{4.2}$$

where $c_d = 0$, if $s < n$ and $c_d = 1$, if $s = n$. ∎

Lemma 2. *Let $I \subsetneq E$ be a ULI generated in degrees $d_1 < d_2 < \cdots < d_t$, $t > 1$. Set*

$$r_p = d_p - d_{p-1},\ 2 \le p \le t,\ \text{and } r_{t+1} = 1.$$

Then

(1) for $1 \le p \le t$, $0 \le i \le r_{p+1} - 1$,

$$\dim_K I_{d_p+i} = \sum_{q=0}^{k_{d_1}-1} \binom{n-d_1-q}{\sum_{\ell=2}^{p} r_\ell + i} + \sum_{j=2}^{p} \left[\sum_{q=0}^{k_{d_j}-1} \binom{n - \tilde{s}_{d_{j-1}} - s_{d_j} - q}{\sum_{\ell=j+1}^{p} r_\ell + i} \right];$$

(2) for $p = t$, $1 \le i \le n - d_t$,

$$\begin{aligned} \dim_K I_{d_t+i} &= \sum_{q=0}^{k_{d_1}-1} \binom{n-d_1-q}{\sum_{\ell=2}^{t} r_\ell + i} + \sum_{j=2}^{t-1} \left[\sum_{q=0}^{k_{d_j}-1} \binom{n - \tilde{s}_{d_{j-1}} - s_{d_j} - q}{\sum_{\ell=j+1}^{t} r_\ell + i} \right] \\ &+ \sum_{q=0}^{k_{d_t}-1-c_{d_t}} \binom{n - \tilde{s}_{d_{t-1}} - s_{d_t} - q}{i}, \end{aligned}$$

where $k_{d_p} = |G(I)_{d_p}|$, $1 \le p \le t$; $\tilde{s}_{d_{\ell-1}} = |\{i \in [n] : i \in \operatorname{supp}(\bigcup_{r=1}^{\ell} G(I)_{d_r})\}|$, $s_{d_\ell} = |\{i \in [n] : i \in \operatorname{supp}(\max(G(I)_{d_\ell}), i \notin \operatorname{supp}(G(I)_{d_{\ell-1}})\}|$, for $2 \le \ell \le t$; and c_{d_t} is 0 (1, respectively) if $\max\{\mathrm{m}(u) : u \in G(I)_{d_t}\} < n$ ($\max\{\mathrm{m}(u) : u \in G(I)_{d_t}\} = n$, respectively).

Proof: First of all, observe that since $n \ge d_t + |G(I)| - 1$, then $\max\{\mathrm{m}(u) : u \in G(I)_{d_i}\} < n$, $1 \le i \le t-1$.
(1). For $p = 1$, the assert follows from Lemma 1. By hypothesis, $d_p = d_{p-1} + r_p$ $(2 \le p \le t)$, with $r_p \ge 1$. Hence, $d_p = d_1 + \sum_{\ell=2}^{p} r_\ell$, and consequentaly

$$\begin{aligned} \dim_K I_{d_p+i} &= |\mathrm{Shad}^{\sum_{\ell=2}^{p} r_\ell + i}(I_{d_1})| + \sum_{j=2}^{p} |\mathrm{aShad}^{\sum_{\ell=j+1}^{p} r_\ell + i}(G(I)_{d_j})| = \\ &= \sum_{q=0}^{k_{d_1}-1} \binom{n-d_1-q}{\sum_{\ell=2}^{p} r_\ell + i} + \sum_{j=2}^{p} \left[\sum_{q=0}^{k_{d_j}-1} \binom{n - \tilde{s}_{d_{j-1}} - s_{d_j} - q}{\sum_{\ell=j+1}^{p} r_\ell + i} \right], \end{aligned}$$

for $0 \le i \le r_{p+1} - 1$; where $\tilde{s}_{d_{\ell-1}} = |\{i \in [n] : i \in \operatorname{supp}(\bigcup_{r=1}^{\ell} G(I)_{d_r})\}|$, $s_{d_\ell} = |\{i \in [n] : i \in \operatorname{supp}(\max(G(I)_{d_\ell}), i \notin \operatorname{supp}(G(I)_{d_{\ell-1}})\}|$, $2 \le \ell \le t$.
(2). Let $p = t$, $1 \le i \le n - d_t$. With the same notations as in statement (1), one has

$$\begin{aligned} \dim_K I_{d_t+i} &= \sum_{q=0}^{k_{d_1}-1} \binom{n-d_1-q}{\sum_{\ell=2}^{t} r_\ell + i} + \sum_{j=2}^{t-1} \left[\sum_{q=0}^{k_{d_j}-1} \binom{n - \tilde{s}_{d_{j-1}} - s_{d_j} - q}{\sum_{\ell=j+1}^{t} r_\ell + i} \right] \\ &+ \sum_{q=0}^{k_{d_t}-1-c_{d_t}} \binom{n - \tilde{s}_{d_{t-1}} - s_{d_t} - q}{i}, \end{aligned}$$

where c_{d_t} is equal to 0 if $\max\{\mathrm{m}(u) : u \in G(I)_{d_t}\} < n$, and equals 1 if $\max\{\mathrm{m}(u) : u \in G(I)_{d_t}\} = n$. □

Theorem 1. *Let $I \subsetneq E$ be a ULI generated in degrees $d_1 < d_2 < \cdots < d_t$. Set $k_{d_p} = |G(I)_{d_p}|$, $1 \leq p \leq t$. Then*

(1) for $1 \leq p \leq t$, $0 \leq i \leq d_{p+1} - d_p - 1$,

$$\dim_K I_{d_p+i} = \sum_{j=1}^{p} \left[\sum_{q=0}^{k_{d_j}-1} \binom{n - d_j - q - \sum_{\ell=1}^{d_{j-1}} k_\ell}{d_p - d_j + i} \right];$$

(2) for $p = t$, $1 \leq i \leq n - d_t$,

$$\begin{aligned} \dim_K I_{d_t+i} \quad &= \quad \sum_{j=1}^{t-1} \left[\sum_{q=0}^{k_{d_j}-1} \binom{n - d_j - q - \sum_{\ell=1}^{d_{j-1}} k_\ell}{d_t - d_j + i} \right] \\ &+ \quad \sum_{q=0}^{k_{d_t}-1-c_{d_t}} \binom{n - d_t - q - \sum_{\ell=1}^{d_{t-1}} k_\ell}{i}, \end{aligned}$$

where c_{d_t} is the integer defined in Lemma 2.

Proof: Since I is a ULI generated in degrees $d_1 < d_2 < \cdots < d_t$, then with the same notations as in Characterization 1 and in Lemma 2, one has

$$\tilde{s}_{d_{j-1}} = R_{d_{j-1}} - 1 = d_{j-1} + \sum_{\ell=1}^{d_{j-1}} k_\ell - 1, \quad s_{d_j} = d_j - (d_{j-1} - 1),\ 2 \leq j \leq t.$$

Hence $\tilde{s}_{d_{j-1}} + s_{d_j} = \sum_{\ell=1}^{d_{j-1}} k_\ell + d_j$, for $2 \leq j \leq t$. Moreover, it is easily verified that $r_j + r_{j+1} = d_{j+1} - d_{j-1}$, for $2 \leq j \leq t$, and $r_i + r_{i+1} + \cdots + r_j = d_j - d_{i-1}$, for $2 \leq i < j \leq t$. □

Theorem 1 gives a systematic description of the Hilbert function of a ULI and yields the following result.

Theorem 2. *Let n be a positive integer. A 0^*-sequence $H = (1, H(1), H(2), \ldots, H(n-1), 0)$ is critical of initial degree d if and only if there exists an integer $1 \leq t \leq n-1$ together with a sequence of non negative integers $(k_i)_{i\in\mathbb{N}}$ with $\mathrm{supp}(k_i)_{i\in\mathbb{N}} = \{d = d_1 < d_2 < \cdots < d_t\}$ such that*

(1) $n \geq d_t + \sum_{i=1}^{t} k_{d_i} - 1$;

(2) *for* $1 \le p \le t$, $0 \le i \le d_{p+1} - d_p - 1$,

$$H_{E/I}(d_p + i) = \binom{n}{d_p + i} - \sum_{j=1}^{p} \left[\sum_{q=0}^{k_{d_j}-1} \binom{n - d_j - q - \sum_{\ell=1}^{d_j - 1} k_\ell}{d_p - d_j + i} \right];$$

(3) *for* $1 \le i \le n - d_t$,

$$H_{E/I}(d_t + i) = \binom{n}{d_t + i} - \left[\sum_{j=1}^{t-1} \left[\sum_{q=0}^{k_{d_j}-1} \binom{n - d_j - q - \sum_{\ell=1}^{d_j - 1} k_\ell}{d_t - d_j + i} \right] + \right.$$
$$\left. + \sum_{q=0}^{k_{d_t} - 1 - c_{d_t}} \binom{n - d_t - q - \sum_{\ell=1}^{d_t - 1} k_\ell}{i} \right], c_{d_t} \in \{0, 1\}.$$

Moreover, $\sum_{i=1}^{d_t} k_i$ *is equal to the number of minimal monomial generators of the ULI* I *of* $E = K\langle e_1, \ldots, e_n\rangle$ *with* $H_{E/I} = H$.

Proof: If H is critical of initial degree d, then there exists a ULI $I \subsetneq E = K\langle e_1, \ldots, e_n\rangle$ of initial degree d such that $H = H_{E/I}$. Let I be an ideal generated in degrees $d = d_1 < d_2 < \ldots < d_t$. Set $k_{d_p} = |G(I)_{d_p}|$, $1 \le p \le t$. Therefore, condition (1) follows from Remark 1, whereas, as a consequence of Theorem 1, it follows that H is of the type described in (2) and (3).

Conversely, suppose there exists a sequence of non negative integers $(k_i)_{i\in\mathbb{N}}$ with $\mathrm{supp}(k_i)_{i\in\mathbb{N}} = \{d = d_1 < d_2 < \cdots < d_t\}$ and such that $n \ge d_t + \sum_{i=1}^{t} k_{d_i} - 1$. Let H be a numerical function satisfying conditions (2) and (3). From Remark 1, there exists a ULI $I \subsetneq E = K\langle e_1, \ldots, e_n\rangle$ with $k_{d_i} = |G(I)_{d_i}|$, $1 \le i \le t$. More precisely, for $t = 1$, I is a ULI generated in one degree $d = d_1$ with $|G(I)| = n - d + 1$ if $c_d = 1$, and $|G(I)| < n - d + 1$, if $c_d = 0$. For $t > 1$, I is a ULI generated in several degrees $d_1 < d_2 < \cdots < d_t$, with $|G(I)_{d_t}| = n - d_t - \sum_{i=1}^{t} k_{d_i} + 1$ if $c_{d_t} = 1$, and $|G(I)_{d_t}| < n - d_t - \sum_{i=1}^{t} k_{d_i} + 1$ if $c_{d_t} = 0$. From Theorem 1, $H_{E/I} = H$, and so H, is critical. ∎

5 The depth of a graded ideal with a given Hilbert function

In this Section, we analyze the depth of a graded ideal with a given Hilbert function in $E = K\langle e_1, \ldots, e_n\rangle$.

We give the following definition.

Definition 8. *Let* $I \subsetneq E$ *be a graded ideal.* I *is said* critical *if the Hilbert function of the graded algebra* E/I *is critical.*

In other words, a graded ideal $I \subsetneq E$ is critical if the lexsegment ideal I^{lex} is a ULI.

Example 7. *Let* $I = (e_1e_2, e_2e_3e_4, e_2e_3e_5)$ *be a stable ideal in* $E = K\langle e_1, \ldots, e_5\rangle$. *The Hilbert function of* E/I *is* $H_{E/I} = (1, 5, 9, 5, 1, 0)$. *I is critical. In fact, there exists the ULI* $I^{\text{lex}} = (e_1e_2, e_1e_3e_4, e_1e_3e_5)$ *of* E *such that* $H_{E/I} = H_{E/I^{\text{lex}}}$.

Theorem 3. *Let* $I \subsetneq E$ *be a critical graded ideal with* $|K| = \infty$. *Then* $\text{depth}_E E/I = \text{depth}_E E/I^{\text{lex}}$.

Proof: Since the depth and also the complexity are preserved by the passage to the generic initial ideal [1, 10], we may assume that I is a strongly stable ideal in E. Therefore, from [1, Theorem 3.2], $\text{depth}_E E/I = n - \text{cx}_E E/I$ and $\text{depth}_E E/I^{\text{lex}} = n - \text{cx}_E E/I^{\text{lex}}$. On the other hand, Theorem 2 guarantees that the ideal I is generated in the same degrees $d_1 < d_2 < \cdots < d_t$ as those of I^{lex} and that $|G(I)_{d_i}| = |G(I^{\text{lex}})_{d_i}|$, $1 \leq i \leq t$. In particular, $\nu(I) = \nu(I^{\text{lex}})$. Moreover, for $1 \leq i \leq t$, $\max\{\text{m}(u) : u \in G(I)_{d_i}\} = \max\{\text{m}(u) : u \in G(I^{\text{lex}})_{d_i}\}$. Hence, from [12, Lemma 3.14], $\text{cx}_E E/I = \text{cx}_E E/I^{\text{lex}}$, and the assert follows. □

In general, if $I \subsetneq E$ is a graded ideal and I^{lex} is the unique lexsegment ideal of E such that $H_{E/I} = H_{E/I^{\text{lex}}}$, the equality in Theorem 3 does not hold, as the following example clearly shows.

Example 8. *Let* $I = (e_1e_2, e_1e_3, e_2e_3e_4, e_2e_3e_5)$ *be a stable ideal of the exterior algebra* $E = K\langle e_1, \ldots, e_6\rangle$. *The Hilbert function of* E/I *is* $H = (1, 6, 13, 11, 3, 0)$. *We have* $I^{\text{lex}} = (e_1e_2, e_1e_3, e_1e_4e_5, e_1e_4e_6, e_2e_3e_4e_5, e_2e_3e_4e_6)$. *It follows that* $\text{cx}_E E/I = 5 < \text{cx}_E E/I^{\text{lex}} = 6$. *Hence,* $\text{depth}_E E/I = 1 > \text{depth}_E E/I^{\text{lex}} = 0$. *Note that I is not a stable critical ideal since* I^{lex} *is not a ULI.*

As a consequence of Theorem 3 and Proposition 1, we obtain the following corollary.

Corollary 1. *Let* $|K| = \infty$ *and* $I \subsetneq E$ *be a critical stable ideal. Then* I *and* I^{lex} *have the same graded Betti numbers.*

We close this Section with some formulas that show the relation between the depth, the Castelnuovo-Mumford regularity and the minimal system of monomial generators of a ULI.

Our first result is the following.

Proposition 2. *Let* $|K| = \infty$ *and* $0 \neq I \subsetneq E$ *be a ULI generated in degrees* $d_1 < d_2 < \cdots < d_t$. *Then* $\text{depth}_E E/I + |G(I)| = n + 1 - d_t$.

Proof: From [1, Theorem 3.2], $\text{depth}_E E/I = n - \text{cx}_E E/I$. Hence, under the same notations of Characterization 1, set $R_t = d_t + \sum_{i=1}^{t} |G(I)_{d_i}|$, from [12, Lemma 3.14], one has $\text{depth}_E E/I = n - \text{cx}_E E/I = n - R_t + 1 = n - d_t - |G(I)| + 1$. □

Therefore, we finally get the following corollary.

Corollary 2. *Let* $|K| = \infty$ *and* $0 \neq I \subsetneq E = K\langle e_1, \ldots, e_n\rangle$ *be a ULI. Then* $\operatorname{depth}_E E/I + \operatorname{reg}_E(E/I) + |G(I)| = n$.

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