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Liouville theorems for weakly F-stationary maps with potential

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Abstract

In this paper, we introduce the notion of weakly F-stationary map with potential which is a critical point of the functional $\Phi_{F,H}$ with respect to variations in the domain. It is a generalization of F-stationary maps with potential. We obtain some Liouville theorems for these maps under some curvature conditions of the domain manifolds and some conditions on H. We obtain similar theorems for maps obeying a class of integral equations involving the stress-energy tensor.

Key Words: Weakly F-stationary map with potential, Liouville theorem, stress-energy tensor.
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1 Introduction

Let $F : [0,\infty) \to [0,\infty)$ be a C^2 function such that $F(0) = 0$ and $F'(t) > 0$ on $[0,\infty)$. For a smooth map $u : (M^m, g) \to (N^n, h)$ between two Riemannian manifolds (M, g) and (N, h), Asserda in [1] introduced the following functional

$$\Phi_F(u) = \int_M F(\frac{||u^*h||^2}{4})dv_g,$$

(see[10, 11, 12, 6]) where u^*h is the symmetric 2-tensor defined by

$$(u^*h)(X,Y) = h(du(X), du(Y))$$

for any vector fields X, Y on M and $||u^*h||$ is given by

$$||u^*h||^2 = \sum_{i,j=1}^{m} [h(du(e_i), du(e_j))]^2$$

with respect to a local orthonormal frame $(e_1, \cdots, e_m)$ on (M, g). They derived that first variation formula of Φ_F, then, by using stress-energy tensor, they obtained some monotonicity formulas and some Liouville theorems for stationary maps for the functional Φ_F. Following [1], Han and Feng in [5] introduced the following functional $\Phi_{F,H}$ by

$$\Phi_{F,H}(u) = \int_M [F(\frac{||u^*h||^2}{4}) - H \circ u] dv_g,$$

where H is a smooth function on N^n. The map u is F-stationary with potential H for $\Phi_{F,H}$ if it is a critical point of $\Phi_{F,H}$ with respect to any compact supported variation of u. They obtained some Liouville theorems for F-stationary maps with potential H and also investigated the stability for F-stationary maps with potential H from or into the standard sphere.

Let $u_t : (M^m, g) \to (N^n, h)$ $(-\epsilon < t < \epsilon)$ be a variation of u, i.e. $u_t = \Psi(t, .)$ with $u_0 = u$, where $\Psi : (-\epsilon, \epsilon) \times M \to N$ is a smooth map. Let $\psi = \frac{d\Psi}{dt}|_{t=0} \in \Gamma(u^{-1}TN)$ be the variational field, where $\Gamma(u^{-1}TN)$ is the set of all smooth cross sections of the bundle. Let $\Gamma_0(u^{-1}TN)$ be a subset of $\Gamma(u^{-1}TN)$ consisting of all elements with compact supports contained in the interior of M. For each $\psi \in \Gamma_0(u^{-1}TN)$, there exists a variation $u_t(x) = \exp_{u(x)}(t\psi)$ (for t small enough) of u, which has the variational field ψ. Such a variation is said to have a compact support. Let $D_\psi \Phi_{F,H}(u) = \frac{d\Phi_{F,H}(u_t)}{dt}|_{t=0}$.

Remark 1. *From the definition of F-stationary map with potential H, we know that a smooth map u from M to N is called F-stationary map with potential H for the functional $\Phi_{F,H}$ if $D_V \Phi_{F,H}(u) = \frac{d\Phi_{F,H}(u_t)}{dt}|_{t=0} = 0$ for $V \in \Gamma_0(u^{-1}TN)$.*

It is known that $du(X) \in \Gamma(u^{-1}TN)$ for any vector field X of M. If X has a compact support which is contained in the interior of M, then $du(X) \in \Gamma_0(u^{-1}TN)$.

Definition 1. *A smooth map $u : (M^m, g) \to (N^n, h)$ is said to be a weakly F-stationary map with potential H for the functional $\Phi_{F,H}(u)$ if $D_{du(X)}\Phi_{F,H}(u) = 0$ for all $X \in \Gamma_0(TM)$.*

Remark 2. *From Remark 1 and Definition 1, we know that F-stationary maps with potential H must be weakly F-stationary maps with potential H, that is, the weakly F-stationary maps with potential H are the generalization of the F-stationary maps with potential H.*

In this paper, we investigate weakly F-stationary maps with potential H and obtain some Liouville theorems for these maps under some curvature conditions of the domain manifolds and some conditions on H. We also investigate some special maps, i.e. maps obeying the integral eqation (3.11) or (3.19) and obtain the Liouville theorems for these maps.

2 Preliminaries

Let ∇ and ${}^N\nabla$ always denote the Levi-Civita connections of M and N respectively. We choose a local orthonormal frame field $\{e_i\}$ on M. We define the F-H-tension field $\tau_{\Phi_{F,H}}(u)$ of u by

$$\tau_{\Phi_{F,H}}(u) \quad = \quad \tau_{\Phi_F}(u) +^N \nabla H \circ u,$$

where $\tau_{\Phi_F}(u) = F'(\frac{||u^*h||^2}{4})div_g(\sigma_u)+\sigma_u(grad(F'(\frac{||u^*h||^2}{4})))$ as defined in [1], and $\sigma_u = \sum_j h(du(.), du(e_j))du(e_j)$ as defined in [10].

Lemma 1. *[5](The first variation formula) Let $u : M \to N$ be a smooth map. Then*

$$D_\psi \Phi_{F,H}(u) = -\int_M h(\tau_{\Phi_{F,H}}(u), \psi)dv_g, \tag{2.1}$$

where $\psi = \Gamma_0(u^{-1}TN)$.

Let $u : M \to N$ be a weakly F-stationary map with potential H and $X \in \Gamma_0(TM)$. Then by (2.1) and the definition of weakly F-stationary maps with potential H, we have

$$D_{du(X)}\Phi_{F,H}(u) = -\int_M h(\tau_{\Phi_{F,H}}(u), du(X))dv_g = 0. \tag{2.2}$$

Recall that for a 2-tensor field $T \in \Gamma(T^*M \otimes T^*M)$, its divergence $divT \in \Gamma(T^*M)$ is defined by

$$(divT)(X) = \sum_i (\nabla_{e_i}T)(e_i, X), \tag{2.3}$$

where X is any smooth vector field on M. For two 2-tensors T_1, $T_2 \in \Gamma(T^*M \otimes T^*M)$, their inner product is defined as follows;

$$< T_1, T_2 >= \sum_{ij} T(e_i, e_j)T_2(e_i, e_j), \tag{2.4}$$

where $\{e_i\}$ is an orthonormal basis of with respect to g. For a vector field $X \in \Gamma(TM)$, we denote by θ_X is dual one form i.e. $\theta_X(Y) = g(X, Y)$. The covariant derivative of θ_X gives a 2-tensor field $\nabla\theta_X$:

$$(\nabla\theta_X)(Y, Z) = (\nabla_Z\theta_X)(Y) = g(\nabla_Z X, Y). \tag{2.5}$$

If $X = \nabla\varphi$ is the gradient of some function φ on M, then $\theta_X = d\varphi$ and $\nabla\theta_X = Hess\varphi$.

Lemma 2. *(cf.[2, 3]). Let T be a symmetric $(0,2)$-type tensor field and let X be a vector field, then*

$$div(i_X T) = (divT)(X) + < T, \nabla\theta_X > = (divT)(X) + \frac{1}{2} < T, L_X g > . \qquad (2.6)$$

where L_X is the Lie derivative of the metric g in the direction of X. Indeed, let $\{e_1, \cdots, e_m\}$ be a local orthonormal frame field on M. Then

$$\frac{1}{2} < T, L_X g > = \sum_{i,j=1}^{m} \frac{1}{2} < T(e_i, e_j), L_X g(e_i, e_j) >$$
$$= \sum_{i,j=1}^{m} T(e_i, e_j) g(\nabla_{e_i} X, e_j) = < T, \nabla\theta_X > .$$

Let D be any bounded domain of M with C^1 boundary. By using the stokes' theorem, we immediately have the following integral formula:

$$\int_{\partial D} T(X, \nu) ds_g = \int_D [< T, \frac{1}{2} L_X g > + div(T)(X)] dv_g, \qquad (2.7)$$

where ν is the unit outward normal vector field along ∂D.

From the equation (2.7), we have

Corollary 1. *If X is a smooth vector field with a compact contained in the interior of M, then*

$$\int_M [< T, \frac{1}{2} L_X g > + div(T)(X)] dv_g = 0. \qquad (2.8)$$

Asserda in [1] introduced a symmetric 2-tensor $S_{\Phi_{F,u}}$ to the functional $\Phi_F(u)$ by

$$S_{\Phi_{F,u}} = F(\frac{||u^* h||^2}{4}) g - F'(\frac{||u^* h||^2}{4}) h(\sigma_u(.), du(.)). \qquad (2.9)$$

which is called the stress-energy tensor.

Lemma 3. *[1] For any smooth vector field X of M, we have*

$$(divS_{\Phi_{F,u}})(X) = -h(\tau_{\Phi_F}(u), du(X)). \qquad (2.10)$$

By using the equations (2.2), (2.8) and (2.10), we know that if $u : M \to N$ is a weakly F-stationary map with potential H, then we have

$$\begin{aligned} 0 &= \int_M < S_{\Phi_{F,u}}, \frac{1}{2} L_X g > dv_g - \int_M h(\tau_{\Phi_F}(u) +^N \nabla H(u) -^N \nabla H(u), du(X)) dv_g \\ &= \int_M < S_{\Phi_{F,u}}, \frac{1}{2} L_X g > dv_g - \int_M h(\tau_{\Phi_{F,H}}(u) -^N \nabla H(u), du(X)) dv_g \\ &= \int_M < S_{\Phi_{F,u}}, \frac{1}{2} L_X g > dv_g + \int_M h(^N\nabla H(u), du(X)) dv_g \end{aligned}$$

i.e.

$$\int_M < S_{\Phi_{F,u}}, \frac{1}{2} L_X g > dv_g + \int_M h({}^N\nabla H(u), du(X)) dv_g = 0 \tag{2.11}$$

for any $X \in \Gamma_0(TM)$.

Han and Feng in [5] introduced a symmetric 2-tensor $S_{\Phi_{F,H,u}}$ to the functional $\Phi_{F,H}(u)$ by

$$S_{\Phi_{F,H,u}} = [F(\frac{||u^*h||^2}{4}) - H \circ u]g - F'(\frac{||u^*h||^2}{4}) h(\sigma_u(.), du(.)). \tag{2.12}$$

which is called the stress-energy tensor.

Lemma 4. *[5] For any smooth vector field X of M, we have*

$$(div S_{\Phi_{F,H,u}})(X) = -h(\tau_{\Phi_{F,H}}(u), du(X)). \tag{2.13}$$

By using the equations (2.2), (2.8) and (2.13), we know that if $u : M \to N$ is a weakly F-stationary map with potential H, then we have

$$\int_M < S_{\Phi_{F,H,u}}, \frac{1}{2} L_X g > dv_g = 0 \tag{2.14}$$

for any $X \in \Gamma_0(TM)$.

3 Liouville theorems

Let (M, g_0) be a complete Riemannian manifold with a pole x_0. Denote by $r(x)$ the g_0-distance function relative to the pole x_0, that is $r(x) = dist_{g_0}(x, x_0)$. Set $B(r) = \{x \in M^m : r(x) \leq r\}$. It is known that $\frac{\partial}{\partial r}$ is always an eigenvector of $Hess_{g_0}(r^2)$ associated to eigenvalue 2. Denote by $\lambda_{\max}$ (resp. $\lambda_{\min}$) the maximum (resp. minimal) eigenvalues of $Hess_{g_0}(r^2) - 2dr \otimes dr$ at each point of $M - \{x_0\}$. Let (N^n, h) be a Riemannian manifold, and H be a smooth function on N.

From now on, we suppose that $u : (M^m, g) \to (N, h)$ is a smooth map, where $g = f^2 g_0$, $0 < f \in C^\infty(M)$. Clearly the vector field $\nu = f^{-1}\frac{\partial}{\partial r}$ is an outer normal vector field along $\partial B(r) \subset (M, g)$. The following conditions that we will assume for f are as follows:

(f_1)

$$\frac{\partial \log f}{\partial r} \geq 0,$$

(f_2) there is a constant $C > 0$ such that

$$(m - 4d_F) r \frac{\partial \log f}{\partial r} + \frac{m-1}{2} \lambda_{\min} + 1 - 2d_F \max\{2, \lambda_{\max}\} \geq C,$$

where d_F is defined as follows: $d_F = \sup_{t \geq 0} \frac{tF'(t)}{F(t)}$ ($l_F = \inf_{t \geq 0} \frac{tF'(t)}{F(t)}$) (cf. [9], [3]). In this paper we assume that d_F is finite.

Theorem 1. *Let $u : (M, f^2 g_0) \to (N, h)$ be a weakly F-stationary map with potential H. If f satisfies $(f_1)(f_2)$, $H \leq 0$ (or $H|_{u(M)} \leq 0$) and $\int_M [F(\frac{||u^*h||^2}{4}) - H \circ u] dv_g < \infty$, then u is a constant.*

Proof: We take $X = \phi(r) r \frac{\partial}{\partial r} = \frac{1}{2}\phi(r)\nabla^0 r^2$, where ∇^0 denotes the covariant derivative determined by g_0 and $\phi(r)$ is a nonnegative function determined later. By direct computation, we have

$$< S_{\Phi_{F,H,u}}, \frac{1}{2} L_X g >= \phi(r) r \frac{\partial \log f}{\partial r} < S_{\Phi_{F,H,u}}, g > + \frac{1}{2} f^2 < S_{\Phi_{F,H,u}}, L_{\phi(r) r \frac{\partial}{\partial r}} g_0 > \quad (3.1)$$

Let $\{e_i\}_{i=1}^m$ be an orthonormal basis with respect to g_0 and $e_m = \frac{\partial}{\partial r}$. We may assume that $Hess_{g_0}(r^2)$ becomes a diagonal matrix with respect to $\{e_i\}_{i=1}^m$. Then $\{\widetilde{e_i} = f^{-1} e_i\}$ ia an orthonormal basis with respect to g.

Now we compute

$$\begin{aligned}
< S_{\Phi_{F,H,u}}, g > &= \sum_{i,j} S_{\Phi_{F,H,u}}(\widetilde{e_i}, \widetilde{e_j}) g(\widetilde{e_i}, \widetilde{e_j}) \\
&= m[F(\frac{||u^*h||^2}{4}) - H \circ u] - F'(\frac{||u^*h||^2}{4}) \sum_i h(\sigma_u(\widetilde{e_i}), du(\widetilde{e_i})) \\
&= m[F(\frac{||u^*h||^2}{4}) - H \circ u] - F'(\frac{||u^*h||^2}{4}) ||u^*h||^2 \\
&\geq m[F(\frac{||u^*h||^2}{4}) - H \circ u] - 4 d_F F(\frac{||u^*h||^2}{4}) \\
&\geq m[F(\frac{||u^*h||^2}{4}) - H \circ u] - 4 d_F [F(\frac{||u^*h||^2}{4}) - H \circ u] \\
&\geq (m - 4 d_F)[F(\frac{||u^*h||^2}{4}) - H \circ u] \qquad (3.2)
\end{aligned}$$

and

$$\begin{aligned}
& f^2 < S_{\Phi_{F,H,u}}, L_{\phi(r) r \frac{\partial}{\partial r}} g_0 > \\
&= f^2 \sum_{i,j} S_{\Phi_{F,H,u}}(\widetilde{e_i}, \widetilde{e_j})(L_{\phi(r) r \frac{\partial}{\partial r}} g_0)(\widetilde{e_i}, \widetilde{e_j}) \\
&= f^2 \{ \sum_{i,j} [F(\frac{||u^*h||^2}{4}) - H \circ u] g(\widetilde{e_i}, \widetilde{e_j})(L_{\phi(r) r \frac{\partial}{\partial r}} g_0)(\widetilde{e_i}, \widetilde{e_j}) \\
&- \sum_{i,j} F'(\frac{||u^*h||^2}{4}) h(\sigma_u(\widetilde{e_i}), du(\widetilde{e_j}))(L_{\phi(r) r \frac{\partial}{\partial r}} g_0)(\widetilde{e_i}, \widetilde{e_j}) \} \\
&= \sum_i [F(\frac{||u^*h||^2}{4}) - H \circ u](L_{\phi(r) r \frac{\partial}{\partial r}} g_0)(e_i, e_i) \\
&- \sum_{i,j} F'(\frac{||u^*h||^2}{4}) h(\sigma_u(\widetilde{e_i}), du(\widetilde{e_j}))(L_{\phi(r) r \frac{\partial}{\partial r}} g_0)(e_i, e_j) \}
\end{aligned}$$

$$
\begin{aligned}
&= \phi(r)\sum_i [F(\frac{||u^*h||^2}{4}) - H\circ u] Hess_{g_0}(r^2)(e_i,e_i) \\
&+2[F(\frac{||u^*h||^2}{4}) - H\circ u] r\phi'(r) \\
&-\phi(r)\sum_{i,j} F'(\frac{||u^*h||^2}{4}) h(\sigma_u(\widetilde{e_i}), du(\widetilde{e_j})) Hess_{g_0}(r^2)(e_i,e_j) \\
&-2F'(\frac{||u^*h||^2}{4}) r\phi'(r) h(\sigma_u(\widetilde{e_m}), du(\widetilde{e_m})) \\
&\geq \phi(r)[F(\frac{||u^*h||^2}{4}) - H\circ u][2+(m-1)\lambda_{\min}] \\
&-\phi(r)F'(\frac{||u^*h||^2}{4}) \max\{2,\lambda_{\max}\} \sum_i h(\sigma_u(\widetilde{e_i}), du(\widetilde{e_i})) \\
&+2[F(\frac{||u^*h||^2}{4}) - H\circ u] r\phi'(r) - 2F'(\frac{||u^*h||^2}{4}) r\phi'(r) h(\sigma_u(\widetilde{e_m}), du(\widetilde{e_m})) \\
&= \phi(r)[F(\frac{||u^*h||^2}{4}) - H\circ u][2+(m-1)\lambda_{\min}] \\
&-\phi(r)F'(\frac{||u^*h||^2}{4}) \max\{2,\lambda_{\max}\} ||u^*h||^2 \\
&+2[F(\frac{||u^*h||^2}{4}) - H\circ u] r\phi'(r) - 2F'(\frac{||u^*h||^2}{4}) r\phi'(r) h(\sigma_u(\widetilde{e_m}), du(\widetilde{e_m})) \\
&\geq \phi(r)[F(\frac{||u^*h||^2}{4}) - H\circ u][2+(m-1)\lambda_{\min}] \\
&-\phi(r) 4\max\{2,\lambda_{\max}\}[F(\frac{||u^*h||^2}{4}) - H\circ u] \\
&+2[F(\frac{||u^*h||^2}{4}) - H\circ u] r\phi'(r) - 2F'(\frac{||u^*h||^2}{4}) r\phi'(r) h(\sigma_u(\widetilde{e_m}), du(\widetilde{e_m})) \\
&\geq \phi(r)[F(\frac{||u^*h||^2}{4}) - H\circ u][2+(m-1)\lambda_{\min} - 4d_F \max\{2,\lambda_{\max}\}] \\
&+2[F(\frac{||u^*h||^2}{4}) - H\circ u] r\phi'(r) \qquad (3.3) \\
&-2F'(\frac{||u^*h||^2}{4}) r\phi'(r) h(\sigma_u(\widetilde{e_m}), du(\widetilde{e_m})).
\end{aligned}
$$

From (3.1), (3.2), (3.3), (f_1) and (f_2), we have

$$
\begin{aligned}
< S_{\Phi_{F,H,u}}, \frac{1}{2}L_X g > \quad &\geq \quad \phi(r)[F(\frac{||u^*h||^2}{4}) - H\circ u][(m-4d_F) r\frac{\partial \log f}{\partial r} + 1 \\
&+ \quad \frac{(m-1)}{2}\lambda_{\min} - 2d_F \max\{2,\lambda_{\max}\}] \\
&+ \quad [F(\frac{||u^*h||^2}{4}) - H\circ u] r\phi'(r)
\end{aligned}
$$

$$\begin{aligned}
&-F'(\frac{||u^*h||^2}{4})r\phi'(r)h(\sigma_u(\widetilde{e_m}), du(\widetilde{e_m}))\\
&\geq C\phi(r)[F(\frac{||u^*h||^2}{4}) - H\circ u] + [F(\frac{||u^*h||^2}{4}) - H\circ u]r\phi'(r)\\
&-F'(\frac{||u^*h||^2}{4})r\phi'(r)h(\sigma_u(\widetilde{e_m}), du(\widetilde{e_m})). \qquad (3.4)
\end{aligned}$$

For any fixed $R > 0$, we take a smooth function $\phi(r)$ which takes value 1 on $B(\frac{R}{2})$, 0 outside $B(R)$ and $0 \leq \phi(r) \leq 1$ on $T(R) = B(R) - B(\frac{R}{2})$. And $\phi(r)$ also satisfies the condition: $|\phi'(r)| \leq \frac{C_0}{r}$ on M, where C_0 is a positive constant. From (2.14) and (3.4), we have

$$\begin{aligned}
0 \;&\geq\; \int_M [C\phi(r)[F(\frac{||u^*h||^2}{4}) - H\circ u] + [F(\frac{||u^*h||^2}{4}) - H\circ u]r\phi'(r)]dv_g\\
&-\; \int_M F'(\frac{||u^*h||^2}{4})r\phi'(r)h(\sigma_u(\widetilde{e_m}), du(\widetilde{e_m}))dv_g\\
&\geq\; \int_{B(\frac{R}{2})} C[F(\frac{||u^*h||^2}{4}) - H\circ u]dv_g + \int_{T(R)} [F(\frac{||u^*h||^2}{4}) - H\circ u]r\phi'(r)dv_g\\
&-\; \int_{T(R)} F'(\frac{||u^*h||^2}{4})r\phi'(r)h(\sigma_u(\widetilde{e_m}), du(\widetilde{e_m}))dv_g\\
&\geq\; \int_{B(\frac{R}{2})} C[F(\frac{||u^*h||^2}{4}) - H\circ u]dv_g - C_0\int_{T(R)} [F(\frac{||u^*h||^2}{4}) - H\circ u]dv_g\\
&-\; C_0\int_{T(R)} F'(\frac{||u^*h||^2}{4})||u^*h||^2 dv_g \qquad (3.5)\\
&\geq\; \int_{B(\frac{R}{2})} C[F(\frac{||u^*h||^2}{4}) - H\circ u]dv_g - C_0(1+4d_F)\int_{T(R)} [F(\frac{||u^*h||^2}{4}) - H\circ u]dv_g.
\end{aligned}$$

From $\int_M [F(\frac{||u^*h||^2}{4}) - H\circ u]dv_g < \infty$, we have

$$\lim_{R\to\infty}\int_{T(R)} [F(\frac{||u^*h||^2}{4}) - H\circ u]dv_g = 0. \qquad (3.6)$$

From (3.5) and (3.6), we have

$$0 \geq C\int_M [F(\frac{||u^*h||^2}{4}) - H\circ u]dv_g \geq C\int_M F(\frac{||u^*h||^2}{4})dv_g,$$

so we know that u is a constant. □

Theorem 2. *Let $u : (M, f^2g_0) \to (N, h)$ be a weakly F-stationary map with potential H. If f satisfies $(f_1)(f_2)$, $\frac{\partial H\circ u}{\partial r} \geq 0$ and $\int_M F(\frac{||u^*h||^2}{4})dv_g < \infty$, then u is a constant.*

Proof: By using the similar method in the proof in Theorem 1, we can obtain the following:

$$\begin{aligned} < S_{\Phi_{F,u}}, \frac{1}{2}L_X g > \quad \geq \quad & C\phi(r)F(\frac{||u^*h||^2}{4}) + F(\frac{||u^*h||^2}{4})r\phi'(r) \\ - \quad & F'(\frac{||u^*h||^2}{4})r\phi'(r)h(\sigma_u(\widetilde{e_m}), du(\widetilde{e_m})). \end{aligned}$$

From $\frac{\partial H\circ u}{\partial r} \geq 0$, we have

$$\begin{aligned} & < S_{\Phi_{F,u}}, \frac{1}{2}L_X g > +\phi(r)r\frac{\partial H\circ u}{\partial r} \\ & \geq C\phi(r)F(\frac{||u^*h||^2}{4}) + F(\frac{||u^*h||^2}{4})r\phi'(r) \\ & -F'(\frac{||u^*h||^2}{4})r\phi'(r)h(\sigma_u(\widetilde{e_m}), du(\widetilde{e_m})). \end{aligned} \tag{3.7}$$

For any fixed $R > 0$, we take a smooth function $\phi(r)$ which takes value 1 on $B(\frac{R}{2})$, 0 outside $B(R)$ and $0 \leq \phi(r) \leq 1$ on $T(R) = B(R) - B(\frac{R}{2})$. And $\phi(r)$ also satisfies the condition: $|\phi'(r)| \leq \frac{C_0}{r}$ on M, where C_0 is a positive constant. From (2.11) and (3.7), we have

$$\begin{aligned} 0 \quad \geq \quad & \int_M [C\phi(r)F(\frac{||u^*h||^2}{4}) + F(\frac{||u^*h||^2}{4})r\phi'(r)]dv_g \\ - \quad & \int_M F'(\frac{||u^*h||^2}{4})r\phi'(r)h(\sigma_u(\widetilde{e_m}), du(\widetilde{e_m}))dv_g \\ \geq \quad & \int_{B(\frac{R}{2})} CF(\frac{||u^*h||^2}{4})dv_g - C_0\int_{T(R)} F(\frac{||u^*h||^2}{4})dv_g \\ - \quad & C_0\int_{T(R)} F'(\frac{||u^*h||^2}{4})||u^*h||^2 dv_g \\ \geq \quad & \int_{B(\frac{R}{2})} CF(\frac{||u^*h||^2}{4})dv_g - C_0(1+4d_F)\int_{T(R)} F(\frac{||u^*h||^2}{4})dv_g. \end{aligned} \tag{3.8}$$

From $\int_M F(\frac{||u^*h||^2}{4})dv_g < \infty$, we have

$$\lim_{R\to\infty}\int_{T(R)} F(\frac{||u^*h||^2}{4})dv_g = 0. \tag{3.9}$$

From (3.8) and (3.9), we have

$$0 \geq C\int_M F(\frac{||u^*h||^2}{4})dv_g,$$

so we know that u is a constant. □

Lemma 5. *[5, 7, 8] Let (M^m, g) be a complete Riemannian manifold with a pole x_0. Denote by K_r the radial curvature of M.*
(i) if $-\alpha^2 \leq K_r \leq -\beta^2$ with $\alpha \geq \beta > 0$ and $(m-1)\beta - 4d_F\alpha > 0$, then

$$[(m-1)\lambda_{\min} + 2 - 4d_F \max\{2, \lambda_{\max}\}] \geq 2(m - \frac{4d_F\alpha}{\beta}),$$

(ii) if $-\frac{A}{(1+r^2)^{1+\varepsilon}} \leq K_r \leq \frac{B}{(1+r^2)^{1+\varepsilon}}$ with $\varepsilon > 0$, $A \geq 0$ and $0 \leq B < 2\varepsilon$, then

$$[(m-1)\lambda_{\min} + 2 - 4d_F \max\{2, \lambda_{\max}\}] \geq 2[1 + (m-1)(1 - \frac{B}{2\varepsilon}) - 4d_F e^{\frac{A}{2\varepsilon}}],$$

(iii) if $-\frac{a^2}{c^2+r^2} \leq K_r \leq \frac{b^2}{c^2+r^2}$ with $a \geq 0$, $b^2 \in [0, \frac{1}{4}]$ and $c^2 \geq 0$, then

$$\begin{aligned}&[(m-1)\lambda_{\min} + 2 - 4d_F \max\{2, \lambda_{\max}\}]\\ &\geq 2[1 + (m-1)\frac{1+\sqrt{1-4b^2}}{2} - 4d_F\frac{1+\sqrt{1+4a^2}}{2}].\end{aligned}$$

Theorem 3. *Let (M, g) be an m-dimensional complete manifold with a pole x_0. Assume that the radial curvature K_r of M satisfies one of the following three conditions:*

(i) if $-\alpha^2 \leq K_r \leq -\beta^2$ with $\alpha \geq \beta > 0$ and $(m-1)\beta - 4d_F\alpha > 0$,

(ii) if $-\frac{A}{(1+r^2)^{1+\varepsilon}} \leq K_r \leq \frac{B}{(1+r^2)^{1+\varepsilon}}$ with $\varepsilon > 0$, $A \geq 0$, $0 \leq B < 2\varepsilon$ and $1 + (m-1)(1 - \frac{B}{2\varepsilon}) - 4d_F e^{\frac{A}{2\varepsilon}} > 0$,

(iii) if $-\frac{a^2}{c^2+r^2} \leq K_r \leq \frac{b^2}{c^2+r^2}$ with $a \geq 0$, $b^2 \in [0, \frac{1}{4}]$, $c^2 \geq 0$ and $1 + (m-1)\frac{1+\sqrt{1-4b^2}}{2} - 4d_F\frac{1+\sqrt{1+4a^2}}{2} > 0$.

If $u : (M, g) \to (N, h)$ is a weakly F-stationary map with potential H, $H \leq 0$ (or $H|_{u(M)} \leq 0$) and $\int_M [F(\frac{||u^*h||^2}{4}) - H \circ u] dv_g < \infty$, then u is a constant.

Proof: From the proof of theorem 1 for $f = 1$ and Lemma 5, we know that theorem 3 is true. □

Theorem 4. *Let M and K_r be as in Theorem 3. If $u : (M, g) \to (N, h)$ is a weakly F-stationary map with potential H, $\frac{\partial H \circ u}{\partial r} \geq 0$ and $\int_M F(\frac{||u^*h||^2}{4}) dv_g < \infty$, then u is a constant.*

Proof: From the proof of theorem 2 for $f = 1$ and Lemma 5, we know that theorem 4 is true. □

We say the functional $\Phi_{F,H}(u)$ $(\Phi_F(u))$ of u is slowly divergent if there exists a positive function $\varphi(r)$ with $\int_{R_0}^{\infty}\frac{dr}{r\varphi(r)} = +\infty$ $(R_0 > 0)$, such that

$$\lim_{R\to\infty}\int_{B(R)}\frac{[F(\frac{||u^*h||^2}{4}) - H\circ u]}{\varphi(r(x))}dv_g < \infty (\lim_{R\to\infty}\int_{B(R)}\frac{F(\frac{||u^*h||^2}{4})}{\varphi(r(x))}dv_g < \infty)\quad(3.10)$$

Theorem 5. *Let $u : (M, f^2g_0) \to (N,h)$ be a smooth map which satisfies the following*

$$\int_M (divS_{\Phi_{F,H,u}})(X)dv_g = 0 \qquad (3.11)$$

for any $X \in \Gamma(TM)$. If f satisfies $(f_1)(f_2)$, $H \leq 0$ (or $H|_{u(M)} \leq 0$) and $\Phi_{F,H}(u)$ of u is slowly divergent, then u is a constant.

Proof: From the inequality (3.4) for $\phi(r) = 1$, we have

$$< S_{\Phi_{F,H,u}}, \frac{1}{2}L_Xg > \geq C[F(\frac{||u^*h||^2}{4}) - H\circ u]. \qquad (3.12)$$

On the other hand, taking $D = B(r)$ and $T = S_{\Phi_{F,H,u}}$ in (2.7), we have

$$\begin{aligned}
&\int_{B(r)} < S_{\Phi_{F,H,u}}, \frac{1}{2}L_Xg > dv_g + \int_{B(r)}(divS_{\Phi_{F,H,u}})(X)dv_g\\
&= \int_{\partial B(r)} S_{\Phi_{F,H,u}}(X,\nu)ds_g\\
&= \int_{\partial B(r)}[F(\frac{||u^*h||^2}{4}) - H\circ u]g(X,\nu)ds_g\\
&- \int_{\partial B(r)} F'(\frac{||u^*h||^2}{4})h(du(X),\sigma_u(\nu))ds_g\\
&= \int_{\partial B(r)}[F(\frac{||u^*h||^2}{4}) - H\circ u]f^2g_0(r\frac{\partial}{\partial r}, f^{-1}\frac{\partial}{\partial r})ds_g\\
&- \int_{\partial B(r)} F'(\frac{||u^*h||^2}{4})f^{-1}rh(du(\frac{\partial}{\partial r}),\sigma_u(\frac{\partial}{\partial r}))ds_g\\
&= r\int_{\partial B(r)}[F(\frac{||u^*h||^2}{4}) - H\circ u]fds_g\\
&- \int_{\partial B(r)} F'(\frac{||u^*h||^2}{4})f^{-1}r\sum_i h(du(\widetilde{e_i}), du(\frac{\partial}{\partial r}))^2ds_g\\
&\leq r\int_{\partial B(r)}[F(\frac{||u^*h||^2}{4}) - H\circ u]fds_g. \qquad (3.13)
\end{aligned}$$

Now suppose that u is a nonconstant map, so there exists $R_1 \geq R_0 > 0$ such that for $R \geq R_1$,

$$\int_{B(R)} [F(\frac{||u^*h||^2}{4}) - H \circ u] dv_g \geq c_0, \tag{3.14}$$

where c_0 is a positive constant. From (3.12), (3.13) and (3.14), we have

$$\int_{\partial B(R)} [F(\frac{||u^*h||^2}{4}) - H \circ u] f ds_g \geq \frac{c_0 C}{R} + \frac{1}{R} \int_{B(R)} (div S_{\Phi_{F,H,u}})(X) dv_g \tag{3.15}$$

for $R \geq R_1$. From (3.11), we have

$$\lim_{R \to \infty} \int_{B(R)} (div S_{\Phi_{F,H,u}})(X) dv_g = 0,$$

so we know that there exists a positive constant $R_2 \geq R_1$ such that for $R \geq R_2$, we have

$$-\frac{c_0 C}{2} \leq \int_{B(R)} (div S_{\Phi_{F,,Hu}})(X) dv_g \leq \frac{c_0 C}{2}. \tag{3.16}$$

From (3.15) and (3.16), we have

$$\int_{\partial B(R)} [F(\frac{||u^*h||^2}{4}) - H \circ u] f ds_g \geq \frac{c_0 C}{2R} \tag{3.17}$$

for $R \geq R_2$. From (3.17) and $|\nabla r| = f^{-1}$, we have

$$\begin{aligned} &\lim_{R \to \infty} \int_{B(R)} \frac{[F(\frac{||u^*h||^2}{4}) - H \circ u]}{\varphi(r(x))} dv_g \\ &= \int_0^\infty \frac{dR}{\varphi(R)} \int_{\partial B(R)} [F(\frac{||u^*h||^2}{4}) - H \circ u] / |\nabla r| ds_g \\ &= \int_0^\infty \frac{dR}{\varphi(R)} \int_{\partial B(R)} [F(\frac{||u^*h||^2}{4}) - H \circ u] f ds_g \\ &\geq \int_{R_2}^\infty \frac{dR}{\varphi(R)} \int_{\partial B(R)} [F(\frac{||u^*h||^2}{4}) - H \circ u] f ds_g \\ &\geq \int_{R_2}^\infty \frac{c_0 C dR}{2R\varphi(R)} = +\infty. \end{aligned} \tag{3.18}$$

This contradicts (3.10), therefore u is a constant.

∎

From the proof of Theorem 5 for $f = 1$, we immediately get the following:

Corollary 2. *Let M and K_r be as in Theorem 3. If $u : (M, g) \to (N, h)$ is a smooth map which satisfies the following*

$$\int_M (divS_{\Phi_{F,H,u}})(X)dv_g = 0$$

for any $X \in \Gamma(TM)$,$H \leq 0$ (or $H|_{u(M)} \leq 0$) and $\Phi_{F,H}(u)$ of u is slowly divergent, then u is a constant.

Theorem 6. *Let $u : (M, f^2 g_0) \to (N, h)$ be a smooth map which satisfies the following*

$$\int_M divS_{\Phi_{F,u}}dv_g = \int_M h(^N\nabla H \circ u, du(X))dv_g \tag{3.19}$$

for any $X \in \Gamma(TM)$. If f satisfies $(f_1)(f_2)$, $\frac{\partial H\circ u}{\partial r} \geq 0$ and $\Phi_F(u)$ of u is slowly divergent, then u is a constant.

Proof: From the inequality (3.7) for $\phi(r) = 1$, we have

$$< S_{\Phi_{F,u}}, \frac{1}{2}L_X g > +h(^N\nabla H \circ u, du(X)) \geq CF(\frac{||u^*h||^2}{4}). \tag{3.20}$$

On the other hand, taking $D = B(r)$ and $T = S_{\Phi_{F,u}}$ in (2.7), we have

$$\begin{aligned}
&\int_{B(r)} < S_{\Phi_{F,u}}, \frac{1}{2}L_X g > dv_g + \int_{B(r)} (divS_{\Phi_{F,u}})(X)dv_g = \int_{\partial B(r)} S_{\Phi_{F,u}}(X,\nu)ds_g \\
&= \int_{\partial B(r)} F(\frac{||u^*h||^2}{4})g(X,\nu)ds_g - \int_{\partial B(r)} F'(\frac{||u^*h||^2}{4})h(du(X),\sigma_u(\nu))ds_g \\
&= r\int_{\partial B(r)} F(\frac{||u^*h||^2}{4})fds_g - \int_{\partial B(r)} F'(\frac{||u^*h||^2}{4})f^{-1}r\sum_i h(du(\widetilde{e_i}),du(\frac{\partial}{\partial r}))^2 ds_g \\
&\leq r\int_{\partial B(r)} F(\frac{||u^*h||^2}{4})fds_g.
\end{aligned} \tag{3.21}$$

Now suppose that u is a nonconstant map, so there exists $R_3 \geq R_0 > 0$ such that for $R \geq R_3$,

$$\int_{B(R)} F(\frac{||u^*h||^2}{4})dv_g \geq c_1, \tag{3.22}$$

where c_1 is a positive constant. From (3.19), we have

$$\lim_{R\to\infty} \int_{B(R)} (divS_{\Phi_{F,H}})dv_g = \lim_{R\to\infty} \int_{B(R)} h(^N\nabla H \circ u, du(X))dv_g, \tag{3.23}$$

so we know that there exists a positive constant $R_4 > R_3$ such that for $R \geq R_4$, we have

$$-\frac{Cc_1}{2} \leq \int_{B(R)} (divS_{\Phi_{F,H}})dv_g - \int_{B(R)} h(^N\nabla H \circ u, du(X))dv_g \leq \frac{Cc_1}{2} \quad (3.24)$$

From (3.21)(3.22) and (3.24), we have for $R > R_4$

$$\begin{aligned} & R\int_{\partial B(R)} F(\frac{||u^*h||^2}{4})fds_g \\ & \geq \int_{B(R)} < S_{\Phi_{F,u}}, \frac{1}{2}L_X g > dv_g + \int_{B(R)} (divS_{\Phi_{F,u}})(X)dv_g \\ & \geq \int_{B(R)} < S_{\Phi_{F,u}}, \frac{1}{2}L_X g > dv_g \\ & + \int_{B(R)} h(^N\nabla H \circ u, du(X))dv_g - \frac{Cc_1}{2} \\ & \geq C\int_{B(R)} F(\frac{||u^*h||^2}{4})dv_g - \frac{Cc_1}{2} \geq \frac{Cc_1}{2} \end{aligned} \quad (3.25)$$

From (3.25) and $|\nabla r| = f^{-1}$, we have

$$\begin{aligned} \lim_{R\to\infty} \int_{B(R)} \frac{F(\frac{||u^*h||^2}{4})}{\varphi(r(x))}dv_g &= \int_0^\infty \frac{dR}{\varphi(R)} \int_{\partial B(R)} F(\frac{||u^*h||^2}{4})/|\nabla r|ds_g \\ &= \int_0^\infty \frac{dR}{\varphi(R)} \int_{\partial B(R)} F(\frac{||u^*h||^2}{4})fds_g \\ &\geq \int_{R_4}^\infty \frac{dR}{\varphi(R)} \int_{\partial B(R)} F(\frac{||u^*h||^2}{4})fds_g \\ &\geq \int_{R_4}^\infty \frac{c_1 C dR}{2R\varphi(R)} = +\infty. \end{aligned} \quad (3.26)$$

This contradicts (3.10), therefore u is a constant. □

From the proof of Theorem 6 for $f = 1$, we immediately get the following:

Corollary 3. *Let M and K_r be as in Theorem 3. If $u : (M, g) \to (N, h)$ is a smooth map which satisfies the following*

$$\int_M divS_{\Phi_{F,u}}dv_g = \int_M h(^N\nabla H \circ u, du(X))dv_g$$

for any $X \in \Gamma(TM)$, $\frac{\partial H\circ u}{\partial r} \geq 0$ and $\Phi_F(u)$ of u is slowly divergent, then u is a constant.

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