GAZETA MATEMATICĂ

SERIA A

ANUL XXVIII(CVII) **Nr. 3 – 4/ 2010**

ARTICOLE

Picard and Krasnoselski sequences: applications to fixed point problems[1)]

Teodora–Liliana T. Rădulescu[2)] and Vicenţiu D. Rădulescu[3)]

Abstract. We are concerned with the role of *Picard* and *Krasnoselski* sequences in the approximation of fixed points in various classes of nonlinear equations. We also give a connection with the cobweb method that describes equilibrium phenomena in mathematical economics.

Keywords: fixed point; Brouwer fixed point theorem; Knaster fixed point theorem; successive approximation; Picard sequence; Krasnoselski sequence; asymptotic regularity; cobweb method.

MSC : 26A15; 26A18; 47H10.

1. Introduction

The development of fixed point theory is closely related with the study of various problems arising in the theory of ordinary differential equations. One of the first contributions to this field is due to the French mathematician *Henri Poincaré*[4)] (1854-1912) in his famous paper [14] of 1890 on the three-body problem crowned by *King Oscar Prize.* This problem concerns the free

---

[1)]This paper contains the main topics of a plenary conference delivered at the XIVth Conference of the Romanian Mathematical Society, Alba Iulia, 15–17 October 2010.

[2)]Department of Mathematics, „Fraţii Buzeşti“ National College, 200352 Craiova, Romania, teodoraradulescu@yahoo.com

[3)]Institute of Mathematics „Simion Stoilow“ of the Romanian Academy, Bucharest, Romania, Department of Mathematics, University of Craiova, 200585 Craiova, Romania, vicentiu.radulescu@imar.ro

[4)]One of the most influential mathematician of modern times, with crucial contributions to the development of applied mathematics, mathematical physics, and celestial mechanics. According to *H. Brezis* and *F. Browder* [3], "Poincaré emphasized that a wide variety of physically significant problems arising in very different areas (such as electricity, hydrodynamics, heat, magnetism, optics, elasticity, etc...) have a family resemblance – un “air de famille” in Poincaré's words – and should be treated by common methods”.

motion of multiple orbiting bodies and *Poincaré* reduced the study to the qualitative analysis of the T–periodic solutions of a differential system in $\mathbb{R}^n$

$$x' = f(t,x) \tag{1}$$

to the study of the fixed points of the operator $P_T : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$P_T(y) = p(T;0,y),$$

where $p(t;s,y)$ denotes the solution of equation (1) verifying the initial condition $x(s) = y$. We refer to the survey paper by *Mawhin* [11] for more details and related results.

As early as 1883, *Poincaré* stated in [13] a theorem shown much later to be equivalent with a fixed point theorem for continuous functions on a closed ball into itself, published by *L.E.J. Brouwer*[1)] in 1912 (see [4]). In its simplest one-dimensional case (see Figure), the *Brouwer* fixed point theorem asserts the following property: *any continuous function* $f : [a,b] \to [a,b]$ *has at least a fixed point.* The proof combines very simple arguments, which strongly rely

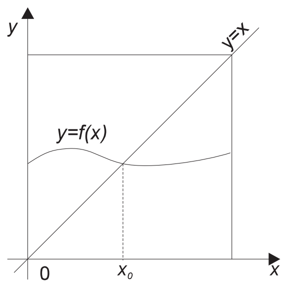

Fig. 1. *Brouwer*'s fixed point theorem (one dimensional case)

on the main continuity assumption combined with the order structure of the set of real numbers. In the general form, the *Brouwer* fixed point theorem asserts that any continuous function with domain the closed unit ball B in $\mathbb{R}^N$ and range contained in B must have at least one fixed point. This result was first applied in 1943 to some forced *Liénard* equations by *Lefschetz* [9] and *Levinson* [10]. If $N \geq 2$, the proof of the the *Brouwer* theorem is much more complicated. However, simpler proofs have been found by means of powerful topological tools, such as the topological degree.

In this paper we are concerned with the following natural related questions:

– what happens if the continuity hypothesis in the *Brouwer* fixed point theorem is replaced with a monotonicity assumption;

– what about the approximation of the fixed point by means of two classical successive approximations:

$$x_{n+1} = f(x_n) \qquad (\textit{Picard sequence})$$

1) Dutch mathematician (1881–1966).

$$x_{n+1} = \frac{x_n + f(x_n)}{2} \qquad (\textit{Krasnoselski} \text{ sequence}).$$

If f is continuous, these sequences provide fixed points, provided that they *are convergent*. Indeed, taking the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 1 - x$, let us consider $x_0 \in [0,1]$ and $x_{n+1} = f(x_n)$, for all $n \geq 0$. Then the *Picard* sequence (x_n) converges if and only if $x_0 = 1/2$. However, if we construct the *Krasnoselski* sequence $x_{n+1} = [x_n + f(x_n)]/2$, then (x_n) converges for any initial value $x_0 \in [0,1]$.

We also establish in the present paper related fixed point properties. A particular interest is given to the cobweb method arising in mathematical economics, in strong relationship with successive approximations. We refer to the recent problem books [15], [16] for further results and relevant applications.

2. Knaster fixed point theorem

In this section we argue that the fixed point property stated in the *Brouwer* theorem remains true if the continuity assumption is replaced with the hypothesis that the function $f : [a,b] \to [a,b]$ is non-decreasing. The same property does not hold provided that if f is *decreasing*. In the non-decreasing case, the proof strongly relies on the order structure of the real axis.

Theorem 1. *Let $f : [a,b] \to [a,b]$ be a non-decreasing function.*
(i) *Then f has at least one fixed point.*
(ii) *There are decreasing function $f : [a,b] \to [a,b]$ with no fixed points.*

Proof. (i) Set

$$A = \{a \leq x \leq b;\ f(x) \geq x\}$$

and $x_0 = \sup A$. The following situations may occur.

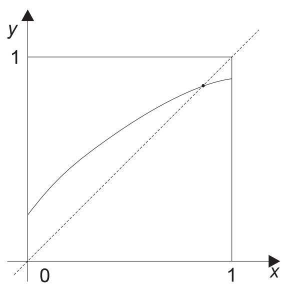

Fig. 2. Knaster's fixed point theorem

Case 1: $x_0 \in A$. By the definition of x_0 it follows that $f(x_0) \geq x_0$. If $f(x_0) = x_0$, then the proof is concluded. If not, we argue by contradiction and assume that $f(x_0) > x_0$. By the definition of x_0 we obtain $f(x) < x$, $\forall x > x_0$.

On the other hand, for any $x_0 < x < f(x_0)$ we have $x > f(x) \geq f(x_0)$, contradiction since $x \in (x_0, f(x_0))$, that is, $x < f(x_0)$. It follows that the assumption $f(x_0) > x_0$ is false, so f has a fixed point.

Case 2: $x_0 \notin A$. We prove that, in fact, it is impossible to have $x_0 \notin A$, so $x_0 \in A$, which reduces the problem to Case 1. If $x_0 \notin A$ then there exists a sequence $x_n \to x_0$, $x_n < x_0$, such that $x_n \in A$. Since f is increasing, it follows that $\lim\limits_{n\to\infty} f(x_n) = x_0$. On the other hand, from $f(x_0) < x_0$ we deduce that there exists $x_n < x_0$ such that $f(x_n) > f(x_0)$, contradiction with the fact that f is increasing.

(ii) Consider the function

$$f(x) = \begin{cases} 1 - x & \text{if } 0 \leq x < \dfrac{1}{2} \\ \dfrac{1}{2} - \dfrac{x}{2} & \text{if } \dfrac{1}{2} \leq x \leq 1. \end{cases}$$

Then $f : [0,1] \to [0,1]$ is decreasing but does not have any fixed point. □

A related counterexample is depicted in Figure 3.

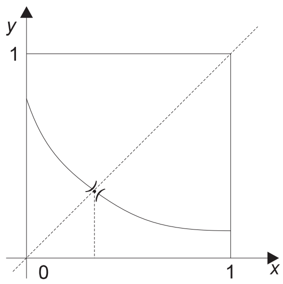

Fig. 3. *Knaster*'s fixed point theorem fails for decreasing functions

3. Further fixed point properties

We start with some simple facts regarding the hypotheses of the *Brouwer* fixed point theorem on the real axis.

(i) While a fixed point in $[a, b]$ exists for a continuous function $f : [a, b] \to [a, b]$, it need **not** be unique. Indeed, any point $x \in [a, b]$ is a fixed point of the function $f : [a, b] \to [a, b]$ defined by $f(x) = x$.

(ii) The condition that f is defined on a closed subset of $\mathbb{R}$ is essential for the existence of a fixed point. For example, if $f : [0, 1) \to \mathbb{R}$ is defined by $f(x) = (1+x)/2$, then f maps $[0, 1)$ into itself, and f is continuous. However, f has **no** fixed point in $[0, 1)$.

(iii) The condition that f be defined on a bounded subset of $\mathbb{R}$ is essential for the existence of a fixed point. For example, if $f : [1, \infty) \to \mathbb{R}$ is defined by $f(x) = x + x^{-1}$, then f maps $[1, \infty)$ into itself, f is continuous, but f has **no** fixed point in $[1, \infty)$.

(iv) The condition that f be defined on an interval in $\mathbb{R}$ is essential for the existence of a fixed point. For example, if $D = [-2, -1] \cup [1, 2]$ and $f : D \to \mathbb{R}$ is defined by $f(x) = -x$, then f maps D into itself, f is continuous, but f has **no** fixed point in D.

We prove in what follows some elementary fixed point properties of real-valued functions.

Proposition 1. *Let $f : [0, 1] \to [0, 1]$ be a continuous function such that $f(0) = 0$, $f(1) = 1$. Denote $f^n := f \circ f \circ \cdots \circ f$ (n times) and assume that there exists a positive integer m such that $f^m(x) = x$ for all $x \in [0, 1]$. Then $f(x) = x$ for any $x \in [0, 1]$.*

Proof. Our hypothesis implies that f is one–to–one, so increasing (since f is continuous). Assume, by contradiction, that there exists $x \in (0, 1)$ such that $f(x) > x$. Then, for any $n \in \mathbb{N}$, we have $f^n(x) > f^{n-1}(x) > \cdots > f(x) > x$. For $n = m$ we find a contradiction. A similar argument shows that the case $f(x) < x$ (for some x) is not possible. □

Proposition 2. *Let a, b be real numbers, $a < b$ and consider a continuous function $f : [a, b] \to \mathbb{R}$.*

(i) If $[a, b] \subset f([a, b])$ then f has a fixed point.

(ii) Assume that there exists a closed interval $I' \subset f([a, b])$. Then $I' = f(J)$, where J is a closed interval contained in $[a, b]$.

(iii) Assume that there exists n closed intervals $I_0, \ldots, I_{n-1}$ contained in $[a, b]$ such that for all $0 \le k \le n-2$, $I_{k+1} \subset f(I_k)$ and $I_0 \subset f(I_{n-1})$. Then f^n has a fixed point ($f^n = f \circ \ldots \circ f$).

Proof. (i) Denote $f([a, b]) = [m, M]$ and let x_m, $x_M \in [a, b]$ be such that $f(x_m) = m$ and $f(x_M) = M$. Since $f(x_m) - x_m \le 0$ and $f(x_M) - x_m \ge 0$, it follows by the intermediate value property that f has at least a fixed point.

(ii) Set $I' = [c, d]$ and consider u, $v \in I$ such that $f(u) = c$ and $f(v) = d$. Assume, without loss of generality, that $u \le v$.

The set $A = \{x \in [u, v];\ f(x) = c\}$ is compact and non-empty, so there exists $\alpha = \max\{x;\ x \in A\}$ and, moreover, $\alpha \in A$. Similarly, the set $B = \{x \in [\alpha, v];\ f(x) = d\}$ has a minimum point β. Then $f(\alpha) = c$, $f(\beta) = d$ and for all $x \in (\alpha, \beta)$ we have $f(x) \ne c$ and $f(x) \ne d$. So, by the intermediate value property, $[c, d] \subset f((\alpha, \beta))$ and $f((\alpha, \beta))$ is an interval which contains neither c nor d. It follows that $I' = f(J)$, where $J = [\alpha, \beta]$.

(iii) Since $I_0 \subset f(I_{n-1})$, it follows by b) that there exists a closed interval $J_{n-1} \subset I_{n-1}$ such that $I_0 = f(J_{n-1})$. But $J_{n-1} \subset I_{n-1} \subset f(I_{n-2})$. So, by (ii), there exists a closed interval $J_{n-2} \subset I_{n-2}$ such that $J_{n-1} = f(J_{n-2})$. Thus, we obtain n closed intervals $J_0, \ldots, J_{n-1}$ such that

$$J_k \subset I_k, \qquad \text{for all } 0 \le k \le n-1$$

and

$$J_{k+1} = f(J_k), \qquad \text{for all } 0 \le k \le n-2 \text{ and } I_0 = f(J_{n-1}).$$

Consequently, J_0 is included in the domain of the nth iterate f^n and $J_0 \subset I_0 = f^n(J_0)$. By a) we deduce that f^n has a unique fixed point in J_0. □

Proposition 3. *Suppose that* $f : \mathbb{R} \to \mathbb{R}$ *satisfies* $|f(x) - f(y)| < |x - y|$ *whenever* $x \neq y$. *Then there is some* ξ *in* $[-\infty, +\infty]$ *such that, for any real* x, $f^n(x) \to \xi$ *as* $n \to \infty$.

Proof. We suppose first that f has a fixed point, say ξ, in $\mathbb{R}$. Then, from the contracting property of f, ξ is the only fixed point of f. We may assume that $\xi = 0$, and this implies that $|f(x)| < |x|$ for all nonzero x. Thus, for any x, the sequence $|f^n(x)|$ is decreasing, so converges to some nonnegative number $\mu(x)$. We want to show that $\mu(x) = 0$ for every x, so suppose now that x is such that $\mu(x) > 0$. Then f maps $\mu(x)$ and $-\mu(x)$ to points y_1 and y_2, say, where $y_j < |\mu(x)|$ for each j. Thus, as f is continuous, there are open neighborhoods of $\pm\mu(x)$ that are mapped by f into the open interval $I = (-\mu(x), \mu(x))$ that contains y_1 and y_2. This implies that, for sufficiently large n, $f^n(x)$ lies in I, which contradicts the fact that $|f^n(x)| \ge \mu(x)$ for all n. Thus, for all x, $\mu(x) = 0$ and $f^n(x) \to 0$.

Now suppose that f has no fixed point in $\mathbb{R}$. Then the function $f(x) - x$ is continuous and nonzero in $\mathbb{R}$. By the intermediate value theorem, $f(x) > x$ for al x, or $f(x) < x$ for all x. We may assume that $f(x) > x$ for all x, as similar argument holds in the other case. Now the sequence $f^n(x)$ is strictly increasing, hence converges to some ξ in $\mathbb{R} \cup \{+\infty\}$. Moreover, $\xi \notin \mathbb{R}$, else ξ would be a fixed point of f. Thus $f^n(x) \to +\infty$ for all x. □

We conclude this paper with the following elementary property, which is due to *M.W. Botsko* [2].

Proposition 4. *Let* $f : [0,1] \to [0,1]$ *be a function such that* $|f(x) - f(y)| \le |x - y|$ *for all* $x, y \in [0,1]$. *Then the set of all fixed points of* f *is either a single point or an interval.*

Proof. Let $F = \{x \in [0,1];\ f(x) = x\}$. Since F is continuous, it follows that F is compact. Let a be the smallest number in F and b the largest number in F. It follows that $F \subset [a,b]$. Fix arbitrarily $x_0 \in [a,b]$. Since a is a fixed point of f, we have

$$f(x_0) - a \le |f(x_0) - a| = |f(x_0) - f(a)| \le x_0 - a\,.$$

Therefore, $f(x_0) \le x_0$. Similarly,

$$b - f(x_0) \le |b - f(x_0)| = |f(b) - f(x_0)| \le b - x_0\,,$$

which shows that $f(x_0) \ge x_0$. It follows that $f(x_0) = x_0$, hence x_0 is a fixed point of f. Thus, $F = [a,b]$. □

4. Approximation of fixed points

We have observed that if $f : [a, b] \to [a, b]$ is a continuous function then f must have at least one fixed point, that is, a point $x \in [a, b]$ such that $f(x) = x$. A natural question in applications is to provide an algorithm for finding (or approximating) this point. One method of finding such a fixed point is by successive approximation. This technique is due to the French mathematician *Émile Picard* (1856–1941) and was introduced in his classical textbook on analysis [12]. More precisely, if $x_1 \in [a, b]$ is chosen arbitrarily, define $x_{n+1} = f(x_n)$ and the resulting sequence $(x_n)_{n\geq 1}$ is called the *sequence of successive approximations* of f (or a *Picard sequence* for the function f). If the sequence $(x_n)_{n\geq 1}$ converges to some x, then a direct argument based on the continuity of f shows that x is a fixed point of f. Indeed,

$$f(x) = f\left(\lim_{n\to\infty} x_n\right) = f\left(\lim_{n\to\infty} x_{n-1}\right) = \lim_{n\to\infty} f(x_{n-1}) = \lim_{n\to\infty} x_n = x\,.$$

The usual method of showing that the sequence $(x_n)_{n\geq 1}$ of successive approximations converges is to show that it satisfies the *Cauchy* convergence criterion: for every $\varepsilon > 0$ there is an integer N, such that for all integers $j, k \geq N$, we have $|x_j - x_k| < \varepsilon$. The next exercise asserts that it is enough to set $j = k + 1$ in the *Cauchy* criterion.

Proposition 5. *Let $f : [a, b] \to [a, b]$ be a continuous function. Let x_1 be a point in $[a, b]$ and let $(x_n)_{n\geq 1}$ denote the resulting sequence of successive approximations. Then the sequence $(x_n)_{n\geq 1}$ converges to a fixed point of f if and only if* $\lim_{n\to\infty}(x_{n+1} - x_n) = 0$.

Proof. Clearly $\lim\limits_{n\to\infty}(x_{n+1} - x_n) = 0$ if $(x_n)_{n\geq 1}$ converges to a fixed point. Suppose $\lim\limits_{n\to\infty}(x_{n+1} - x_n) = 0$ and the sequence $(x_n)_{n\geq 1}$ does not converge. Since $[a, b]$ is compact, there exist two subsequences of $(x_n)_{n\geq 1}$ that converge to ξ_1 and ξ_2 respectively. We may assume $\xi_1 < \xi_2$. It suffices to show that $f(x) = x$ for all $x \in (\xi_1, \xi_2)$. Suppose this is not the case, hence there is some $x^* \in (\xi_1, \xi_2)$ such that $f(x^*) \neq x^*$. Then a $\delta > 0$ could be found such that $[x^* - \delta, x^* + \delta] \subset (\xi_1, \xi_2)$ and $f(\tilde{x}) \neq \tilde{x}$ whenever $\tilde{x} \in (x^* - \delta, x^* + \delta)$. Assume $\tilde{x} - f(\tilde{x}) > 0$ (the proof in the other case being analogous) and choose N so that $|f^n(x) - f^{n+1}(x)| < \delta$ for $n > N$. Since ξ_2 is a cluster point, there exists a positive integer $n > N$ such that $f^n(x) > x^*$. Let n_0 be the smallest such integer. Then, clearly,

$$f^{n_0-1}(x) < x^* < f^{n_0}(x)$$

and since $f^{n_0}(x) - f^{n_0-1}(x) < \delta$ we must have

$$f^{n_0-1}(x) - f^{n_0}(x) > 0 \text{ so that } f^{n_0}(x) < f^{n_0-1}(x) < x^*,$$

a contradiction. $\square$

The usual method of showing that the sequence $(x_n)_{n\geq 0}$ of successive approximations converges is to show that it satisfies the *Cauchy* convergence criterion. The next result establishes that this happens if and only if the difference of two consecutive terms in this iteration converges to zero. The American mathematician *Felix Browder* has called this condition *asymptotic regularity.*

The next result is due to *H.G. Barone* [1] and was established in 1939.

Theorem 2. *Let $(x_n)_{n\geq 0}$ be a sequence of real numbers such that the sequence $(x_{n+1}-x_n)$ converges to zero. Then the set of cluster points of $(x_n)_{n\geq 0}$ is a closed interval in $\overline{\mathbb{R}}$, eventually degenerated.*

Proof. Set $\ell_- := \liminf_{n\to\infty} x_n$, $\ell_+ := \limsup_{n\to\infty} x_n$ and choose $a \in (\ell_-, \ell_+)$. By the definition of ℓ_-, there exists $x_{n_1} < a$. Let n_2 be the least integer greater than n_1 such that $x_{n_2} > a$ (the existence of n_2 follows by the definition of ℓ_+). Thus, $x_{n_2-1} \leq a < x_{n_2}$. Since $\ell_- < a$, there exists a positive integer $n_3 > n_2$ such that $x_{n_3} < a$. Next, by the definition of ℓ_+, there exists an integer $N_4 > n_3$ such that $x_{N_4} > a$. If n_4 denotes the least integer with these properties, then $x_{n_4-1} \leq a < x_{n_4}$. In this manner we construct an increasing sequence of positive numbers $(n_{2k})_{k\geq 1}$ such that, for all $k \geq 1$, $x_{n_{2k}-1} \leq a < x_{n_{2k}}$. Using the hypothesis we deduce that the sequences $(x_{n_{2k}-1})_{k\geq 1}$ and $(x_{n_{2k}})_{k\geq 1}$ converge to a, so a is a cluster point. □

The following convergence result was established by *B.P. Hillam* [7] in 1976.

Theorem 3. *Let $f : [a,b] \to [a,b]$ be a continuous function. Consider the sequence $(x_n)_{n\geq 0}$ defined by $x_0 \in [a,b]$ and, for any positive integer n, $x_n = f(x_{n-1})$. Then the sequence $(x_n)_{n\geq 0}$ converges if and only if $(x_{n+1}-x_n)$ converges to zero.*

Proof. Assume that the sequence of successive approximations $(x_n)_{n\geq 0}$ satisfies $x_{n+1} - x_n \to 0$, as $n \to \infty$. With the same notations as above, assume that $\ell_- < \ell_+$. The proof of (i) combined with the continuity of f imply $a = f(a)$, for all $a \in (\ell_-, \ell_+)$. But this contradicts our assumption $\ell_- < \ell_+$. Indeed, choose $\ell_- < c < d < \ell_+$ and $0 < \varepsilon < (d-c)/3$. Since $x_{n+1} - x_n \to 0$, there exists N_ε such that for all $n \geq N_\varepsilon$, $-\varepsilon < x_{n+1} - x_n < \varepsilon$. Let $N_2 > N_1 > N_\varepsilon$ be such that $x_{N_1} < c < d < x_{N_2}$. Our choice of ε implies that there exists an integer $n \in (N_1, N_2)$ such that $a := x_n \in (c,d)$. Hence $x_{n+1} = f(a) = a$, $x_{n+2} = a$, and so on. Therefore $x_{N_2} = a$, contradiction.

The reversed assertion is obvious. □

The following result is a particular case of a fixed point theorem due to *Krasnoselski* (see [8]). We refer to [6] for the general framework corresponding to functions defined on a closed convex subset of strictly convex *Banach* spaces.

Theorem 4. *Let* $f : [a,b] \to [a,b]$ *a function satisfying* $|f(x) - f(y)| \leq |x-y|$, *for all* $x,\ y \in [a,b]$. *Define the sequence* $(x_n)_{n\geq 1}$ *by* $x_1 \in [a,b]$ *and, for all* $n \geq 1$, $x_{n+1} = [x_n + f(x_n)]/2$. *Then the sequence* $(x_n)_{n\geq 1}$ *converges to some fixed point of* f.

Proof. We observe that it is enough to show that $(x_n)_{n\geq 1}$ converges. In this case, by the recurrence relation and the continuity of f, it follows that the limit of $(x_n)_{n\geq 1}$ is a fixed point of f. We argue by contradiction and denote by A the set of all limit points of $(x_n)_{n\geq 1}$, that is,

$$A := \{\ell \in [a,b];\ \text{there exists a subsequence } (x_{n_k})_{k\geq 1} \text{ of } (x_n)_{n\geq 1} \text{ such that } x_{n_k} \to \ell\}.$$

By our hypothesis and the compactness of $[a,b]$, we deduce that A contains at least two elements and is a closed set.

We split the proof into several steps.

(i) For any $\ell \in A$ we have $f(\ell) \neq \ell$. Indeed, assume that $\ell \in A$ and fix $\varepsilon > 0$ and $n_k \in \mathbb{N}$ such that $|x_{n_k} - \ell| \leq \varepsilon$. Then

$$|\ell - x_{n_k+1}| = \left| \frac{\ell + f(\ell)}{2} - \frac{x_{n_k} + f(x_{n_k})}{2} \right| \leq$$

$$\leq \frac{|\ell - f(x_{n_k})|}{2} + \frac{|f(\ell) - f(x_{n_k})|}{2} \leq |\ell - x_{n_k}| \leq \varepsilon$$

and so on. This shows that $|x_n - \ell| \leq \varepsilon$, for all $n \geq n_k$. Hence $(x_n)_{n\geq 1}$ converges to ℓ, contradiction.

(ii) There exists $\ell_0 \in A$ such that $f(\ell_0) > \ell_0$. Indeed, arguing by contradiction, set $\ell_- = \min_{\ell \in A} \ell$. Then $\ell_- \in A$ and $f(\ell_-) \leq \ell_-$. The variant $f(\ell_-) = \ell_-$ is excluded, by (i). But $f(\ell_-) < \ell_-$ implies that $\frac{[\ell_- + f(\ell_-)]}{2} \in A$ and $\frac{[\ell_- + f(\ell_-)]}{2} < \ell_-$, which contradicts the definition of ℓ_-.

(iii) There exists $\varepsilon > 0$ such that $|f(\ell) - \ell| \geq \varepsilon$, for all $\ell \in A$. For if not, let $\ell_n \in A$ such that $|f(\ell_n) - \ell_n| < \frac{1}{n}$, for all $n \geq 1$. This implies that any limit point of $(\ell_n)_{n\geq 1}$ (which lies in A, too) is a fixed point of f. This contradicts (i).

(iv) Conclusion. By (ii) and (iii), there exists a largest $\ell_+ \in A$ such that $f(\ell_+) > \ell_+$. Let $\ell' = \frac{[\ell_+ + f(\ell_+)]}{2}$ and observe that $f(\ell_+) > \ell' > \ell_+$ and $f(\ell') < \ell'$. By (iii), there exists a smallest $\ell'' \in A$ such that $\ell'' > \ell_+$ and $f(\ell'') < \ell''$. It follows that $\ell_+ < \ell'' < f(\ell_+)$. Next note that $f(\ell'') < \ell_+$; for, if not, $\ell''' := \frac{[\ell'' + f(\ell'')]}{2}$ satisfies $\ell_+ < \ell''' < \ell''$ and, by definitions of ℓ_+ and ℓ'', it follows that $f(\ell''') = \ell'''$, contrary to (i). Thus $f(\ell'') < \ell_+ < \ell'' < f(\ell_+)$. It then follows that $|f(\ell'') - f(\ell_+)| > |\ell'' - \ell_+|$. This contradicts the hypothesis and concludes the proof. □

Remark. The iteration scheme described in the above *Krasnoselski*'s property does not apply to arbitrary continuous mappings of a closed interval into itself. Indeed, consider the function $f : [0,1] \to [0,1]$ defined by

$$f(x) = \begin{cases} \dfrac{3}{4}, & \text{if} \quad 0 \le x \le \dfrac{1}{4} \\ -3x + \dfrac{3}{2}, & \text{if} \quad \dfrac{1}{4} < x < \dfrac{1}{2} \\ 0, & \text{if} \quad \dfrac{1}{2} \le x \le 1. \end{cases}$$

Then the sequence defined in the above statement is defined by $x_{2n} = \frac{1}{2}$ and $x_{2n+1} = \frac{1}{4}$, for any $n \ge 1$. So, $(x_n)_{n\ge 1}$ is a divergent sequence.

The contraction mapping theorem states that if $f : \mathbb{R} \to \mathbb{R}$ (or $f : [a,\infty) \to [a,\infty)$) is a map such that for some k in $(0,1)$ and all x and y in $\mathbb{R}$, $|f(x) - f(y)| \le k\,|x-y|$, then the iterates $f^n = f \circ \ldots \circ f$ (n terms) of f converge to a (unique) fixed point ξ of f. This theorem can be accompanied by an example to show that the inequality cannot be replaced by the weaker condition $|f(x) - f(y)| < |x-y|$. The most common example of this type is $f(x) = x + 1/x$ acting on $[1,\infty)$. Then $f(x) > x$, so that f has no fixed points. Also, for every x, the sequence x, $f(x)$, $f^2(x)$, $\ldots$ is strictly increasing and so must converge in the space $[-\infty,+\infty]$. In fact, $f^n(x) \to +\infty$, for otherwise $f^n(x) \to a$ for some real a, and then $f(f^n(x)) \to f(a)$ (because f is continuous) so that $f(a) = a$, which is not so. Thus we define $f(+\infty)$ to be $+\infty$ and deduce that this example is no longer a counterexample. The following property clarifies these ideas and provides an elementary, but interesting, adjunct to the contraction mapping theorem. We just point out that a mapping $f : \mathbb{R} \to \mathbb{R}$ satisfying $|f(x) - f(y)| < |x-y|$ for all $x \ne y$ is called a *contractive* function.

We say that a function $f : [a,b] \to \mathbb{R}$ satisfies the Lipschitz condition with constant $L > 0$ if for all x and y in $[a,b]$, $|f(x)-f(y)| \le L\,|x-y|$. A function that satisfies a *Lipschitz* condition is clearly continuous. Geometrically, if $f : [a,b] \to \mathbb{R}$ satisfies the Lipschitz condition

$$|f(x) - f(y)| \le L\,|x-y| \qquad \text{for all } x,\, y \in [a,b],$$

then for any x, $y \in [a,b]$, $x \ne y$, the inequality

$$\left|\frac{f(x) - f(y)}{x-y}\right| \le L$$

indicates that the *slope* of the chord joining the points $(x, f(x))$ and $(y, f(y))$ on the graph of f is bounded by L.

Using the fact that the real line is totally ordered, the following more general theorem with much more elementary proof is possible.

Proposition 6. *Let $f : [a,b] \to [a,b]$ be a function that satisfies a Lipschitz condition with constant L. Let x_1 in $[a,b]$ be arbitrary and define $x_{n+1} =$ $= (1-\lambda)x_n + \lambda f(x_n)$ where $\lambda = 1/(L+1)$. If $(x_n)_{n\ge1}$ denotes the resulting* *sequence then* $(x_n)_{n\geq 1}$ *converges monotonically to a point* z *in* $[a,b]$ *where* $f(z)=z$.

Proof. Without loss of generality we can assume $f(x_n)\neq x_n$ for all n. Suppose $f(x_1)>x_1$ and let p be the first point greater than x_1 such that $f(p)=p$. Since $f(x_1)>x_1$ and $f(b)\leq b$, the continuity of f implies there is such a point.

Next, we prove the following claim. If $x_1<x_2<\ldots<x_n<p$ and $f(x_i)>x_i$ for $i=1,2,\ldots,n$, then $f(x_{n+1})>x_{n+1}$ and $x_{n+1}<p$. Indeed, suppose $p<x_{n+1}$, then $x_n<p<x_{n+1}$, hence $0<p-x_n<x_{n+1}-x_n=\lambda(f(x_n)-x_n)$. Therefore

$$\begin{aligned}0 &< \frac{1}{\lambda}\left|x_n-p\right|=(L+1)\left|x_n-p\right|<\left|f(x_n)-x_n\right|\leq\\ &\leq \left|f(x_n)-f(p)\right|+\left|p-x_n\right|.\end{aligned}$$

It follows that

$$L|x_n-p|<\left|f(x_n)-f(p)\right|,$$

which contradicts the fact that f is a Lipschitz function. Thus $x_{n+1}<p$ and $f(x_{n+1})>x_{n+1}$ by the choice of p, and the claim is proved.

Using the induction hypothesis it follows that $x_n<x_{n+1}<p$ for all integers n. Since a bounded monotonic sequence converges, $(x_n)_{n\geq 1}$ converges to some point z. By the triangle inequality it follows that

$$\begin{aligned}|z-f(z)| &\leq |z-x_n|+|x_n-f(x_n)|+|f(x_n)-f(z)|=\\ &= |z-x_n|+\frac{1}{\lambda}\left|x_{n+1}-x_n\right|+\left|f(x_n)-f(z)\right|.\end{aligned}$$

Since the right-hand side tends to 0 as $n\to\infty$, we conclude that $f(z)=z$. If $f(x_1)<x_1$ a similar argument holds. □

Applying a somewhat more sophisticated argument, one can allow λ to be any number less than $2/(L+1)$ but the resulting sequence $(x_n)_{n\geq 1}$ need not converge monotonically. The following example shows this last result is best possible.

Let $f:[0,1]\to[0,1]$ be defined by

$$f(x)=\begin{cases}1, & 0\leq x<\dfrac{L-1}{2L}\\ -Lx+\dfrac{1}{2}(L+1), & \dfrac{L-1}{2L}\leq x\leq\dfrac{L+1}{2L}\\ 0, & \dfrac{L+1}{2L}<x\leq 1,\end{cases}$$

where $L>1$ is arbitrary. Note that f satisfies a *Lipschitz* condition with constant L. Let $\lambda=2/(L+1)$ and let $x_1=(L-1)/2L$. Then $x_2==(1-\lambda)x_1+\lambda f(x_1)=(L+1)/2L$, $x_3=(1-\lambda)x_2+\lambda f(x_2)=(L-1)/2L$, etc.

5. Picard sequences versus the cobweb model and qualitative analysis of markets

We start with the following elementary geometric interpretation of the *Picard* method. First, take a point $A(x_1, f(x_1))$ on the curve $y = f(x)$. Next, consider the point $B_1(f(x_1), f(x_1))$ on the diagonal line $y = x$ and then, project the point B_1 vertically onto the curve $y = f(x)$ to obtain a point $A_2(x_2, f(x_2))$. Again, project A_2 horizontally to B_2 on $y = x$ and then, project B_2 vertically onto $y = f(x)$ to obtain $A_3(x_3, f(x_3))$. This process can be repeated a number of times. Often, it will weave a cobweb in which the fixed point of f, that is, the point of intersection of the curve $y = f(x)$ and the diagonal line $y = x$, gets trapped. In fact, such trapping occurs if the slopes of tangents to the curve $y = f(x)$ are smaller (in absolute value) than the slope of the diagonal line $y = x$. The situation described above is illustrated in Figure 4.

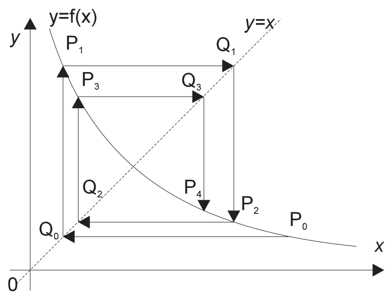

Fig. 4. *Picard* sequence converging to a fixed point

When the slope condition is not met, then the points A_1, $A_2, \ldots$ may move away from a fixed point. This case is depicted in Figure 5.

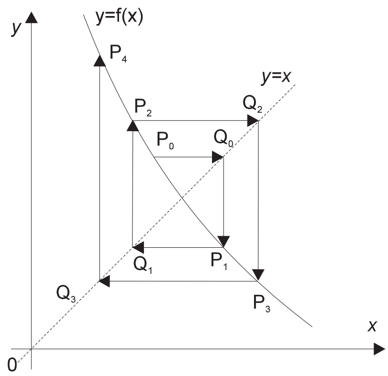

Fig. 5. *Picard* sequence diverging away from a fixed point

In mathematical economics, the behaviour described in Figure 4 corresponds to the convergence to an equilibrium point, while the framework described in Figure 5 describes the divergence from equilibrium.

A sufficient condition for the convergence of a *Picard* sequence, which is a formal analogue of the geometric condition of slopes mentioned above, is stated in the following result, which is also referred as the *Picard convergence theorem.*

Theorem 5. *Let $f : [a, b] \to [a, b]$ be a continuous function which is differentiable on (a, b), with $|f'(x)| < 1$ for all $x \in (a, b)$.*

Then f has a unique fixed point. Moreover, any Picard sequence for f is convergent and converges to the unique fixed point of f.

Proof. We first observe that the *Brouwer* fixed point theorem implies that f has at least one fixed point. Next, assuming that f has two fixed points x_* and x^*, the *Lagrange* mean value theorem implies that there exists $\xi \in (a, b)$ such that

$$|x_* - x^*| = |f(x_*) - f(x^*)| = |f'(\xi)| \cdot |x_* - x^*| < |x_* - x^*| \, ,$$

a contradiction. Thus, f has a unique fixed point.

We point out that the condition $|f'(x)| < 1$ for all $x \in (a, b)$ is essential for the uniqueness of a fixed point. For example, if $f : [a, b] \to [a, b]$ is defined by $f(x) = x$, then $f'(x) = 1$ for all $x \in [a, b]$ and every point of $[a, b]$ is a fixed point of f.

We prove in what follows that the corresponding *Picard* sequence converges. Let x^* denote the unique fixed point of f. Consider arbitrarily $x_1 \in [a, b]$ and let $(x_n)_{n\geq 1} \subset [a, b]$ be the *Picard* sequence for f with its initial point x_1. This means that $x_n = f(x_{n-1})$ for all $n \geq 2$. Fix an integer $n \geq 1$. Thus, by the *Lagrange* mean value theorem, there exists ξ_n between x_n and x^* such that

$$x_{n+1} - x^* = f(x_n) - f(x^*) = f'(\xi_n)(x_n - x^*) \, .$$

This implies that $|x_{n+1} - x^*| < |x_n - x^*|$. Next, we prove that $x_n \to x^*$ as $n \to \infty$. Since $(x_n)_{n\geq 1}$ is bounded, it suffices to show that every convergent subsequence of $(x_n)_{n\geq 1}$ converges to x^*. Let $x \in \mathbb{R}$ and $(x_{n_k})_{k\geq 1}$ be a subsequence of $(x_n)_{n\geq 1}$ converging to x. Then

$$|x_{n_{k+1}} - x^*| \leq |x_{n_k+1} - x^*| \leq |x_{n_k} - x^*| \, .$$

But $|x_{n_{k+1}} - x^*| \to |x - x^*|$ and

$$|x_{n_{k+1}} - x^*| = |f(x_{n_k}) - f(x^*)| \to |f(x) - f(x^*)|, \qquad \text{as } k \to \infty \, .$$

It follows that $|f(x) - f(x^*)| = |x - x^*|$. Now, if $x \neq x^*$, then by the *Lagrange* mean value theorem, there exists $\xi \in (a, b)$ such that

$$|x - x^*| = |f(x) - f(x^*)| = |f'(\xi)| \cdot |x - x^*| < |x - x^*| \, ,$$

which is a contradiction. This proves that $x \neq x^*$. □

Remark. If the condition $|f'(x)| < 1$ for all $x \in (a,b)$ is not satisfied, then f can still have a unique fixed point x^* but the *Picard* sequence $(x_n)_{n\geq 1}$ with initial point $x_1 \neq x^*$ may not converge to x^*.

Example. Let $f : [-1,1] \to [-1,1]$, $f(x) = -x$. Then f is differentiable, $|f'(x)| = 1$ for all $x \in [-1,1]$, and $x^* = 0$ is the unique fixed point of f. If $x_1 \neq 0$ then the corresponding *Picard* sequence is $x_1, -x_1, x_1, -x_1, \ldots$, which oscillates between x_1 and $-x_1$ and never reaches the fixed point. In geometric terms, the cobweb that we hope to weave just traces out a square over and over again.

When the hypotheses of the *Picard* convergence theorem are satisfied, a *Picard* sequence for $f : [a,b] \to [a,b]$ with arbitrary $x_1 \in [a,b]$ as its initial point, converges to a fixed point of f. It is natural to expect that if x_1 is closer to the fixed point, then the convergence rate will be better. A fixed point of f lies not only in the range of f but also in the ranges of the iterates $f \circ f$, $f \circ f \circ f$, and so on. Thus, if $\mathcal{R}_n$ is the range of the n–fold composite $f \circ \cdots \circ f$ (n times), then a fixed point is in each $\mathcal{R}_n$. If only a single point belongs to $\bigcap_{n=1}^{\infty} \mathcal{R}_n$, then we have found our fixed point. In fact, the *Picard* method amounts to starting with any $x_1 \in [a,b]$ and considering the image of x_1 under the n–fold composite $f \circ \cdots \circ f$.

Example. If $f : [0,1] \to [0,1]$ is defined by $f(x) = \frac{x+1}{4}$, then the nth iterate of f is given by

$$f \circ \cdots \circ f(n \text{ times})(x) = \frac{3x + 4^n - 1}{3 \cdot 4^n}$$

and

$$\mathcal{R}_n = \left[\frac{1}{3}\left(1 - \frac{1}{4^n}\right), \frac{1}{3}\left(1 + \frac{2}{4^n}\right)\right].$$

Thus, $\bigcap_{n=1}^{\infty} \mathcal{R}_n = \{\frac{1}{3}\}$, hence $\frac{1}{3}$ is the unique fixed point of f. In general, it is not convenient to determine the ranges $\mathcal{R}_n$ for all n. So, it is simpler to use the *Picard* method, but this tool will be more effective if the above observations are used to some extent in choosing the initial point.

The *Picard* convergence theorem was extended in [5] to a framework arising frequently in mathematical economics. This corresponds to the *cobweb model* that concerns a qualitative analysis of markets in which supply adjustments have a time lag and demand adjustments occur with no delay. We briefly describe in what follows the cobweb model and we conclude with the *cobweb theorem*, which is a generalization of Theorem 5. Let $s(p)$ denote the total quantity of the product that sellers are willing to supply at a given price level $p > 0$. Assume the demand function $d(p)$ represent the total quantity of the product that buyers are willing to purchase at a given price level p. The situation described in the next result corresponds to price converging to an equilibrium price and it is described by economists as a "stable equilibrium". This means that if a small extraneous disturbance occurs in the market, eventually price will again converges to some equilibrium price. The same result shows that disturbance should be be large enough to remove the equilibrium. Such a disturbance might be a depression, drought, or large recession.

Theorem 6. *(Cobweb Theorem) Let s and d be real-valued functions of the real variable $p > 0$, and suppose that the graphs of s and d intersect at the point (p^*, q^*) where $q^* > 0$. Let I be a closed interval centered at p^* on which functions s and d have nonvanishing continuous derivatives. Define sequences (p_n) and (q_n) by letting p_0 be any element of I, $q_n = s(p_{n-1})$ and $p_n = d^{-1}(q_n)$ for all $n \geq 1$. Assume that $|s'(p)| < |d'(p)|$ for all p in I. Then $\lim_{n\to\infty} p_n = p^*$ and $\lim_{n\to\infty} q_n = q^*$.*

The proof of Theorem 6 relies on the *Cauchy* mean value theorem; we refer to [5] for details and related properties.

Acknowledgments. The authors are greatly indebted to Professor *Jean Mawhin* for his interest in this work and for his numerous remarks, which have considerably improved the initial version of this paper.

References

[1] H.G. Barone, Limit points of sequences and their transforms by methods of summability, *Duke Math. J.* **5** (1939). 740–752.

[2] M.W. Botsko, Problem Q982, *Mathematics Magazine* **81** (2008), p. 304.

[3] H. Brezis, F. Browder, Partial differential equations in the 20th century, *Adv. Math.* **135** (1998), 76–144.

[4] L.E.J. Brouwer, Ueber Abbildungen von Mannigfaltigkeiten, *Math. Ann.* **71** (1912), 97–115.

[5] E.H. Davis, Cauchy and economic cobwebs, *Mathematics Magazine* **53** (1980), 171–173.

[6] M. Edelstein, A remark on a theorem of M. A. Krasnoselski, *Amer. Math. Monthly* **73** (1966), 509–510.

[7] B.P. Hillam, A characterization of the convergence of successive approximations, *Amer. Math. Monthly* **83** (1976), p. 273.

[8] M.A. Krasnoselski, Two remarks on the method of successive approximations, *Math. Nauk (N. S.)* **10** (1955), 123–127.

[9] S. Lefschetz, Existence of periodic solutions for certain differential equations, *Proc. Nat. Acad. Sci.* **29** (1943), 1–4.

[10] N. Levinson, On the existence of periodic solutions for second order differential equations with a forcing term, *J. Math. Phys.* **22** (1943), 41–48.

[11] J. Mawhin, Topological fixed point theory and nonlinear differential equations, in *Handbook of Topological Fixed Point Theory* (R.F. Brown, M. Furi, L. Górniewicz, B. Jiang, Eds.), pp. 867–904, Springer, Dordrecht, 2005.

[12] É. Picard, *Traité d'Analyse*, Gauthier-Villars, Paris, 1891.

[13] H. Poincaré, Sur certaines solutions particulières du problème de trois corps, *C.R. Acad. Sci. Paris Sér. I* **97** (1883), 251–252.

[14] H. Poincaré, Sur le problème de trois corps et les équations de la dynamique, *Acta Math.* **13** (1890), 1–270.

[15] M. Rădulescu, S. Rădulescu, V. Rădulescu: *Selected Problems in Mathematical Analysis*, Problem Books in Mathematics, Springer-Verlag, Heidelberg, in preparation.

[16] T.-L. Rădulescu, V. Rădulescu, T. Andreescu, *Problems in Real Analysis: Advanced Calculus on the Real Axis*, Springer, New York, 2009.

Proprietăţi de derivabilitate de ordin superior a funcţiei-punct intermediar din teorema de medie a lui Lagrange

Dorel I. Duca[1)]

Abstract. If the function $f : I \to \mathbb{R}$ is differentiable on the interval $I \subseteq \mathbb{R}$, and $a \in I$, then for each $x \in I$, according to the mean value theorem, there exists a point $c(x)$ belonging to the open interval determined by x and a, and there exists a real number $\theta(x) \in]0, 1[$ such that

$$f(x) - f(a) = (x - a) f^{(1)}(c(x))$$

and

$$f(x) - f(a) = (x - a) f^{(1)}(a + (x - a)\theta(x)).$$

In this paper we shall study the differentiability of high order of the functions c and θ in a neighbourhood of a.

Keywords: intermediate point, mean-value theorem.
MSC : 26A24.

În lucrările anterioare [8] şi [9], apărute în această revistă, dedicate studiului punctului intermediar din teorema de medie a lui *Lagrange* am demonstrat, printre altele, următoarea afirmaţie:

Teorema 1. *Fie I un interval din $\mathbb{R}$, $a \in I$ şi $f : I \longrightarrow \mathbb{R}$. Dacă funcţia f este derivabilă pe intervalul I, atunci*

1^0 *Există cel puţin o funcţie $c : I\backslash\{a\} \to I\backslash\{a\}$ care satisface relaţia*

$$f(x) - f(a) = (x - a) f^{(1)}(c(x)), \text{ oricare ar fi } x \in I\backslash\{a\}. \quad (1)$$

Dacă, în plus, funcţia $f^{(1)}$ este injectivă, atunci funcţia c este unică.

2^0 *Există cel puţin o funcţie $\theta : I\backslash\{a\} \longrightarrow (0, 1)$ care satisface relaţia*

$$f(x) - f(a) = (x - a) f^{(1)}(a + (x - a)\theta(x)), \text{ oricare ar fi } x \in I\backslash\{a\}. \quad (2)$$

Dacă, în plus, funcţia $f^{(1)}$ este injectivă, atunci funcţia θ este unică.

În cele ce urmează avem nevoie de următoarele rezultate:

Teorema 2. *Fie n un număr natural, I şi J două intervale din $\mathbb{R}$ şi $u : I \to \mathbb{R}$ şi $f : J \to \mathbb{R}$ astfel încât $u(I) \subseteq J$.*

Dacă:

(a) *funcţia u este derivabilă de n ori pe intervalul I;*

(b) *funcţia f este derivabilă de n ori pe mulţimea J,*

atunci funcţia compusă $f \circ u : I \to \mathbb{R}$ este derivabilă de n ori pe intervalul I şi

[1)] Prof univ. dr., Facultatea de Matematică şi Informatică, Universitatea Babeş-Bolyai, Cluj-Napoca, România, dorelduca@yahoo.com

$$\left(f\circ u\right)^{(n)}(x)=\sum_{k=1}^{n}\left(f^{(k)}\circ u\right)(x)\,A_k^{\langle n\rangle}(x)\,,\ \text{oricare ar fi } x\in I, \tag{3}$$

unde, pentru fiecare $k\in\{1,...,n\}$ *şi* $x\in I$, *avem*

$$A_k^{\langle n\rangle}(x)=\sum_{\substack{i_1+2i_2+...+ni_n=n\\ i_1+i_2+...+i_n=k}} a_{i_1,i_2,...,i_n}^{\langle n\rangle}G_{i_1,i_2,...,i_n}^{\langle n\rangle}(x)\,, \tag{4}$$

cu

$$a_{i_1,i_2,...,i_n}^{\langle n\rangle}=\frac{n!}{(i_1)!\cdots(i_n)!\,(1!)^{i_1}\cdots(n!)^{i_n}} \tag{5}$$

$$G_{i_1,i_2,...,i_n}^{\langle n\rangle}(x)=\left(u^{(1)}(x)\right)^{i_1}\left(u^{(2)}(x)\right)^{i_2}...\left(u^{(n)}(x)\right)^{i_n} \tag{6}$$

oricare ar fi $i_1,...,i_n\in\{1,...,n\}$ *şi* $x\in I$.

Demonstraţie. Demonstraţia se găseşte, de exemplu, în [12]. □

Teorema 3. *Fie* I *şi* J *două intervale din* $\mathbb{R}$ *şi* $f:I\to J$ *o funcţie bijectivă. Dacă*

(i) *funcţia* f *este derivabilă de* $n\geq 2$ *ori pe* I,

(ii) $f^{(1)}(x)\neq 0$, *oricare ar fi* $x\in I$,

atunci funcţia $f^{-1}:J\to I$ *este derivabilă de* n *ori pe* J *şi*

$$\left(f^{-1}\right)^{(n)}=$$

$$=\frac{-1}{\left(f^{(1)}\circ f^{-1}\right)^{n}}\sum_{k=1}^{n-1}\left(f^{-1}\right)^{(k)}\times\left(\sum_{\substack{i_1+2i_2+...+ni_n=n\\ i_1+i_2+...+i_n=k}}F_{i_1,i_2,...,i_n}\circ f^{-1}\right), \tag{7}$$

unde $F_{i_1,i_2,...,i_n}:I\to\mathbb{R}$ *este definită prin*

$$F_{i_1,i_2,...,i_n}(x)=\frac{n!}{(i_1)!\,(i_2)!...\,(i_n)!}\left(\frac{f^{(1)}(x)}{1!}\right)^{i_1}\left(\frac{f^{(2)}(x)}{2!}\right)^{i_2}...\left(\frac{f^{(n)}(x)}{n!}\right)^{i_n},$$

oricare ar fi $x\in I$.

Demonstraţie. Demonstraţia se găseşte, de exemplu, în [7]. □

Teorema următoare, a cărei demonstraţie se găseşte în [9], [7], ne dă, în anumite condiţii, comportarea punctului intermediar din teorema de medie a lui *Lagrange*.

Teorema 4. *Fie* I *un interval din* $\mathbb{R}$, *a un punct interior intervalului* I *şi* $f:I\to\mathbb{R}$ *o funcţie care satisface următoarele condiţii:*

(i) *funcţia* f *este derivabilă de două ori pe intervalul* I*;*

(ii) *funcţia* $f^{(2)}$ *este continuă în punctul* a*;*

(iii) $f^{(2)}(a)\neq 0$.

Atunci au loc următoarele afirmaţii:

1^0 *Există un număr real* $\delta>0$ *astfel încât*

a) $(a-\delta, a+\delta) \subseteq I$;
b) $f^{(2)}(x) \neq 0$, *oricare ar fi* $x \in (a-\delta, a+\delta)$;
c) $f^{(1)}$ *este injectivă pe* $(a-\delta, a+\delta)$.

2^0 *Există o funcţie* $c : (a-\delta, a+\delta)\backslash\{a\} \to (a-\delta, a+\delta)\backslash\{a\}$, *şi una singură, cu proprietatea că:*

$$f(x) - f(a) = (x-a)f^{(1)}(c(x)), \tag{8}$$

oricare ar fi $x \in (a-\delta, a+\delta)\backslash\{a\}$.

3^0 *Există o funcţie* $\theta : (a-\delta, a+\delta)\backslash\{a\} \to (0,1)$, *şi una singură, cu proprietatea că:*

$$f(x) - f(a) = (x-a)f^{(1)}(a + (x-a)\,\theta(x)), \tag{9}$$

oricare ar fi $x \in (a-\delta, a+\delta)\backslash\{a\}$.

4^0 *Funcţia* θ *are limită în punctul* $x = a$ *şi*

$$\lim_{x\to a} \theta(x) = \frac{1}{2}.$$

5^0 *Funcţia* $\overline{c} : (a-\delta, a+\delta) \to (a-\delta, a+\delta)$ *definită prin*

$$\overline{c}\,(x) = \begin{cases} c\,(x), & \text{dacă } x \in (a-\delta, a+\delta)\backslash\{a\} \\ a, & \text{dacă } x = a, \end{cases}$$

este derivabilă în punctul a şi

$$\overline{c}^{(1)}\,(a) = \frac{1}{2}.$$

Scopul urmărit în continuare este acela de a stabili condiţii în care funcţia $\overline{c}$ este derivabilă de ordin superior şi de a calcula derivatele ei. Are loc următoarea teoremă:

Teorema 5. *Fie* I *un interval din* $\mathbb{R}$, a *un punct din interiorul intervalului* I *şi* $f : I \longrightarrow \mathbb{R}$ *o funcţie care îndeplineşte condiţiile:*
(i) *funcţia* f *este derivabilă de trei ori pe intervalul* I;
(ii) *funcţia* $f^{(3)}$ *este continuă în punctul* a;
(iii) $f^{\,(2)}(a) \neq 0$.
Atunci:
1^0 *Există un număr real* $\delta > 0$ *astfel încât:*
a) $(a-\delta, a+\delta) \subseteq I$;
b) $f^{(2)}(x) \neq 0$, *oricare ar fi* $x \in (a-\delta, a+\delta)$;
c) funcţia $f^{(1)}$ *este injectivă pe* $(a-\delta, a+\delta)$.

2^0 *Există o funcţie* $c : (a-\delta, a+\delta)\backslash\{a\} \to (a-\delta, a+\delta)\backslash\{a\}$, *şi una singură, cu proprietatea că:*

$$f(x) - f(a) = (x-a)f^{(1)}(c(x)),$$

oricare ar fi $x \in (a-\delta, a+\delta)\backslash\{a\}$.

3^0 *Există o funcţie* $\theta : (a-\delta, a+\delta) \setminus \{a\} \to (0,1)$, *şi una singură, astfel încât*

$$f(x) - f(a) = (x-a)f^{(1)}(a + (x-a)\theta(x)), \tag{10}$$

oricare ar fi $x \in (a-\delta, a+\delta) \setminus \{a\}$.

4^0 *Funcţia* $\overline{\theta} : (a-\delta, a+\delta) \to (0,1)$ *definită prin*

$$\overline{\theta}(x) = \begin{cases} \theta(x), & \textit{dacă } x \in (a-\delta, a+\delta) \setminus \{a\} \\ 1/2, & \textit{dacă } x = a, \end{cases}$$

este derivabilă în punctul $x = a$ *şi*

$$\overline{\theta}^{(1)}(a) = \frac{f^{(3)}(a)}{24 f^{(2)}(a)}.$$

5^0 *Funcţia* $\overline{c} : (a-\delta, a+\delta) \to (a-\delta, a+\delta)$ *definită prin*

$$\overline{c}\,(x) = \begin{cases} c\,(x)\,, & \textit{dacă } x \in (a-\delta, a+\delta)\setminus\{a\} \\ a, & \textit{dacă } x = a, \end{cases}$$

este derivabilă de două ori în punctul a *şi*

$$\overline{c}^{(2)}\,(a) = \frac{f^{(3)}(a)}{12 f^{(2)}(a)}.$$

Demonstraţie. $1^0 - 3^0$. Afirmaţiile 1^0, 2^0 şi 3^0 rezultă din teorema 4.

4^0 Fie $J = f^{(1)}((a-\delta, a+\delta))$. Deoarece $f^{(1)}$ este continuă pe $(a-\delta, a+\delta)$, mulţimea J este interval. Considerăm funcţia $\varphi : (a-\delta, a+\delta) \to J$, definită prin

$$\varphi(x) = f^{(1)}(x), \text{ oricare ar fi } x \in (a-\delta, a+\delta).$$

Din afirmaţia 1^0 rezultă că funcţia φ este bijectivă. Atunci funcţia inversă $\varphi^{-1} : J \to (a-\delta, a+\delta)$ este continuă pe J. Pe de altă parte din (10) rezultă că

$$\frac{f(x) - f(a)}{x - a} = f^{(1)}(a + (x-a)\theta(x)), \tag{11}$$

oricare ar fi $x \in (a-\delta, a+\delta) \setminus \{a\}$ şi deci

$$\frac{f(x) - f(a)}{x - a} \in J, \text{ oricare ar fi } x \in (a-\delta, a+\delta) \setminus \{a\}.$$

Din (11) obţinem

$$\theta(x) = \frac{1}{x-a}[\varphi^{-1}(r(x)) - a], \text{ oricare ar fi } x \in (a-\delta, a+\delta) \setminus \{a\},$$

unde $r : I \setminus \{a\} \to \mathbb{R}$ este definită prin

$$r(x) = \frac{f(x) - f(a)}{x - a}, \text{ oricare ar fi } x \in (a-\delta, a+\delta) \setminus \{a\}.$$

Deoarece funcţiile φ^{-1} şi r sunt continue, rezultă că funcţia $\overline{\theta}$ este continuă pe $(a-\delta, a+\delta) \setminus \{a\}$. Din teorema 4 avem

$$\lim_{x \to a} \overline{\theta}(x) = \lim_{x \to a} \theta(x) = \frac{1}{2} = \overline{\theta}(a).$$

Prin urmare funcţia $\overline{\theta}$ este continuă şi în punctul $x = a$. Urmează că funcţia $\overline{\theta}$ este continuă pe mulţimea $(a-\delta, a+\delta)$. Intrucât funcţia $f^{(1)}$este derivabilă pe I şi derivata

$$\varphi^{(1)}(x) = f^{(2)}(x) \neq 0, \text{ oricare ar fi } x \in (a-\delta, a+\delta),$$

obţinem că funcţia φ^{-1} este derivabilă în punctul $\varphi(x) = f^{(1)}(x)$, oricare ar fi $x \in (a-\delta, a+\delta)$, şi

$$(\varphi^{-1})^{(1)}(\varphi(x)) = \frac{1}{\varphi^{(1)}(x)} = \frac{1}{f^{(2)}(x)}, \text{ oricare ar fi } x \in (a-\delta, a+\delta).$$

Atunci funcţia $c = \varphi^{-1} \circ r$ este derivabilă pe mulţimea $(a-\delta, a+\delta) \setminus \{a\}$.

Calculăm derivata lui $\overline{\theta}$ în punctul $x = a$ folosind definiţia derivatei şi regula lui l'*Hôpital.* Avem

$$\overline{\theta}^{(1)}(a) = \lim_{x\to a} \frac{\overline{\theta}(x) - \overline{\theta}(a)}{x-a} = \lim_{x\to a} \frac{\theta(x) - \frac{1}{2}}{x-a} = \lim_{x\to a} \frac{\frac{1}{x-a}(\varphi^{-1}(r(x)) - a) - \frac{1}{2}}{(x-a)} =$$

$$= \lim_{x\to a} \frac{\left(\varphi^{-1}\left(r(x)\right) - a - \frac{x-a}{2}\right)^{(1)}}{((x-a)^2)^{(1)}} = \lim_{x\to a} \frac{(\varphi^{-1})^{(1)}\left((r(x))r^{(1)}(x) - \frac{1}{2}\right)}{2(x-a)}.$$

Întrucât

$$\lim_{x\to a} r^{(1)}(x) = \lim_{x\to a} \left(\frac{f(x) - f(a)}{x-a}\right)^{(1)} = \frac{1}{2} f^{(2)}(a),$$

şi

$$\lim_{x\to a} (\varphi^{-1})^{(1)}(r(x)) = \frac{1}{f^{(2)}(a)},$$

putem aplica, din nou, teorema lui l'*Hôpital.* Atunci

$$\overline{\theta}^{(1)}(a) = \lim_{x\to a} \frac{1}{2} \left[\left(\varphi^{-1}\right)^{(2)} (r(x)) \left(r^{(1)}(x)\right)^2 + \left(\varphi^{-1}\right)^{(1)} (r(x)) r^{(2)}(x) \right].$$

Deoarece

$$\lim x \to a r^{(2)}(x) = \lim_{x\to a} \left(\frac{f(x) - f(a)}{x-a}\right)^{(2)} = \frac{1}{3} f^{(3)}(a)$$

şi

$$(\varphi^{-1})^{(2)}(r(x)) = -\tfrac{f^{(3)}(a)}{(f^{(2)}(a))^3},$$

obţinem

$$\theta^{(1)}(a) = \frac{1}{2} \left(-\frac{f^{(3)}(a)}{4(f^{(2)}(a))^3} (f^{(2)}(a))^2 + \frac{1}{3f^{(2)}(a)} f^{(3)}(a) \right) = \frac{f^{(3)}(a)}{24 f^{(2)}(a)}.$$

5^0 Pentru fiecare $x \in (a-\delta, a+\delta)$ avem

$$\overline{c}\,(x) = \overline{c}\,(a) + (x-a)\,\overline{\theta}\,(x)\,.$$

Urmează că funcţia $\overline{c}$ este derivabilă de două ori în punctul a şi

$$\overline{c}^{(2)}(a) = 2 \cdot \overline{\theta}^{(1)}(a).$$

Teorema este demonstrată. □

Derivabilitatea de ordin superior a funcţiei punct intermediar este analizată în teorema următoare:

Teorema 6. *Fie I un interval din $\mathbb{R}$, a un punct din interiorul intervalului I şi $f : I \to \mathbb{R}$ o funcţie care satisface condiţiile:*

(i) funcţia f este derivabilă de $(n+1)$ ori pe intervalul I;

(ii) funcţia $f^{(n+1)}$ este continuă în punctul $x = a$;

(iii) $f^{(2)}(a) \neq 0$.

Atunci :

1^0 Există un număr real $\delta > 0$ astfel încât

a) $(a-\delta, a+\delta) \subseteq I$;

b) $f^{(2)}(x) \neq 0$, oricare ar fi $x \in (a-\delta, a+\delta)$;

c) funcţia $f^{(1)}$ este injectivă pe $(a-\delta, a+\delta)$.

2^0 Există o funcţie $c : (a-\delta, a+\delta)\backslash\{a\} \to (a-\delta, a+\delta)\backslash\{a\}$, şi una singură, cu proprietatea că

$$f(x) - f(a) = (x-a) f^{(1)}(c(x)), \tag{12}$$

oricare ar fi $x \in (a-\delta, a+\delta)\backslash\{a\}$.

3^0 Există o funcţie $\theta : (a-\delta, a+\delta)\backslash\{a\} \to (0,1)$, şi una singură, cu proprietatea că

$$f(x) - f(a) = (x-a) f^{(1)}(a + (x-a)\,\theta(x)), \tag{13}$$

oricare ar fi $x \in (a-\delta, a+\delta)\backslash\{a\}$.

4^0 Funcţia $\varphi : (a-\delta, a+\delta) \to f^{(1)}((a-\delta, a+\delta))$ definită prin

$$\varphi(x) = f^{(1)}(x), \text{ oricare ar fi } x \in (a-\delta, a+\delta),$$

este bijectivă.

5^0 Funcţia $\overline{c} : (a-\delta, a+\delta) \to (a-\delta, a+\delta)$ definită prin

$$\overline{c}(x) = \begin{cases} c(x), & \text{dacă } x \in (a-\delta, a+\delta) \setminus \{a\} \\ a\,, & \text{dacă } x = a, \end{cases} \tag{14}$$

este derivabilă de n ori pe $(a-\delta, a+\delta)$ şi, pentru fiecare $x \in (a-\delta, a+\delta)\setminus\{a\}$ şi $m \in \{1, \ldots, n\}$, avem

$$\overline{c}^{(m)}(x) = \sum_{k=1}^{m} (\varphi^{-1})^{(k)} \left(\frac{f(x) - f(a)}{x-a} \right) \times \sum_{\substack{i_1 + 2i_2 + \ldots + m i_m = m \\ i_1 + i_2 + \ldots + i_m = k}} \frac{m!}{i_1! i_2! \ldots i_m!} F_{i_1, \ldots, i_m}(x)$$

unde

$$F_{i_1,\dots,i_m}(x)=\left(\frac{1}{1!}\left(\frac{f(x)-f(a)}{x-a}\right)^{(1)}\right)^{i_1}\left(\frac{1}{2!}\left(\frac{f(x)-f(a)}{x-a}\right)^{(2)}\right)^{i_2}\times\cdots$$

$$\cdots\times\left(\frac{1}{m!}\left(\frac{f(x)-f(a)}{x-a}\right)^{(m)}\right)^{i_m}$$

oricare ar fi $x\in(a-\delta,a+\delta)\backslash\{a\}$, *şi* $i_1,...,i_m\in\{1,...,m\}$ *şi*

$$\overline{c}^{(m)}(a)=\sum_{k=1}^{m}\left(\varphi^{-1}\right)^{(k)}\left(f^{(1)}(a)\right)\times \tag{15}$$

$$\times\sum_{\substack{i_1+2i_2+...+mi_m=m\\ i_1+i_2+...+i_m=k}}\frac{m!}{i_1!i_2!...i_m!}\left(\frac{f^{(2)}(a)}{2!}\right)^{i_1}\left(\frac{f^{(3)}(a)}{3!}\right)^{i_2}...\left(\frac{f^{(m+1)}(a)}{(m+1)!}\right)^{i_m}=$$

$$=\sum_{k=1}^{m}\left(\sum_{\substack{i_2+2i_3+...+(k-1)i_k=k-1\\ i_1+i_2+...+i_k=k-1}}\frac{(-1)^{k-1+i_1}(2k-2-i_1)!}{i_2!i_3!...i_k!\left(f^{(2)}(a)\right)^{2k-1}}\times\right.$$

$$\times\left(\frac{f^{(2)}(a)}{1!}\right)^{i_1}\left(\frac{f^{(3)}(a)}{2!}\right)^{i_2}\cdots\left(\frac{f^{(k+1)}(a)}{k!}\right)^{i_k}\times$$

$$\left.\times\sum_{\substack{i_1+2i_2+...+mi_m=m\\ i_1+i_2+...+i_m=k}}\frac{m!}{i_1!i_2!...i_m!}\left(\frac{f^{(2)}(a)}{2!}\right)^{i_1}\left(\frac{f^{(3)}(a)}{3!}\right)^{i_2}\cdots\left(\frac{f^{(m+1)}(a)}{(m+1)!}\right)^{i_m}\right),$$

oricare ar fi $m\in\{1,...,n\}$.

6^0 *Funcţia* $\overline{\theta}:(a-\delta,a+\delta)\to(0,1)$ *definită prin*

$$\overline{\theta}(x)=\begin{cases}\theta(x), & \text{dacă } x\in(a-\delta,a+\delta)\setminus\{a\}\\ \frac{1}{2}, & \text{dacă } x=a\end{cases}$$

este derivabilă de $n-1$ *ori pe* $(a-\delta,a+\delta)$ *şi pentru fiecare* $x\in(a-\delta,a+\delta)\backslash\{a\}$

$$\overline{\theta}^{(m)}(x)=\left(\frac{c(x)-a}{x-a}\right)^{(m)},\ \text{oricare ar fi } m\in\{1,...,n-1\} \tag{16}$$

şi

$$\overline{\theta}^{(m)}(a)=\frac{1}{m+1}\overline{c}^{(m+1)}(a)= \tag{17}$$

$$= \frac{1}{m+1}\sum_{k=1}^{m+1}\left(\varphi^{-1}\right)^{(k)}\left(f^{(1)}(a)\right)\times \sum_{\substack{i_1+2i_2+...+(m+1)i_{m+1}=m+1\\ i_1+i_2+...+i_{m+1}=k}} \frac{(m+1)!}{(i_1)!...(i_{m+1})!}\times$$

$$\times\left(\frac{f^{(2)}(a)}{2!}\right)^{i_1}\left(\frac{f^{(3)}(a)}{3!}\right)^{i_2}...\left(\frac{f^{(m+2)}(a)}{(m+2)!}\right)^{i_{m+1}}.$$

oricare ar fi $m \in \{1, ..., n-1\}$.

Demonstraţie. $1^0 - 3^0$. Afirmaţiile 1^0, 2^0 şi 3^0 rezultă din teorema 4.

4^0 Din afirmaţia 1^0, c) avem că φ este injectivă, iar din modul cum a fost definită, funcţia φ este surjectivă.

5^0 Din (12), rezultă că funcţia c are următoarea expresie

$$c(x) = \varphi^{-1}\left(\frac{f(x)-f(a)}{x-a}\right), \text{ oricare ar fi } x \in (a-\delta, a+\delta)\setminus\{a\}. \quad (18)$$

Întrucât

$$\lim_{x\to a} c(x) = a = \overline{c}(a),$$

funcţia $\overline{c}$ este continuă în punctul $x = a$.

Fie $r : (a-\delta, a+\delta) \to \mathbb{R}$ funcţia definită prin

$$r(x) = \begin{cases} \dfrac{f(x)-f(a)}{x-a}, & \text{dacă } x \in (a-\delta, a+\delta)\setminus\{a\} \\ f^{(1)}(a), & \text{dacă } x = a. \end{cases}$$

Atunci, din (18) rezultă că

$$\overline{c}(x) = \begin{cases} \left(\varphi^{-1}\circ r\right)(x), & \text{dacă } x \in (a-\delta, a+\delta)\setminus\{a\} \\ a, & \text{dacă } x = a. \end{cases}$$

Din (i), definiţia lui r şi proprietăţile funcţiei f, deducem că funcţia $\overline{c}$ este derivabilă de n ori pe $(a-\delta, a+\delta)$ şi, în baza formulei de derivare a compusei a două funcţii, pentru fiecare $x \in (a-\delta, a+\delta)\setminus\{a\}$ şi $m \in \{1, ..., n\}$, avem

$$\overline{c}^{(m)}(x) = \sum_{k=1}^{m}\left((\varphi^{-1})^{(k)}\circ r\right)(x)\times\left(\sum_{\substack{i_1+2i_2+...+mi_m=m\\ i_1+i_2+...+i_m=k}} F_{i_1,...,i_m}(x)\right),$$

unde, pentru fiecare $i_1, i_2, ..., i_m \in \{0, 1, ..., m\}$,

$$F_{i_1,...,i_m}(x) = \frac{m!}{i_1!i_2!...i_m!}\left(\frac{r^{(1)}(x)}{1!}\right)^{i_1}\left(\frac{r^{(2)}(x)}{2!}\right)^{i_2}\cdots\left(\frac{r^{(m)}(x)}{m!}\right)^{i_m}.$$

Deoarece, pentru fiecare $i \in \{1, ..., n\}$,

$$r^{(i)}(a) = \lim_{x \to a} r^{(i)}(x) = \lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} \right)^{(i)} = \frac{1}{i+1} f^{\ (i+1)}(a). \qquad (19)$$

deducem că

$$\overline{c}^{(m)}(a) = \lim_{x \to a} \overline{c}^{(m)}(x) =$$

$$= \sum_{k=1}^{m} (\varphi^{-1})^{(k)} \left(f^{(1)}(a) \right) \times \left(\sum_{\substack{i_1 + 2i_2 + ... + m i_m = m \\ i_1 + i_2 + ... + i_m = k}} F_{i_1, ..., i_m}(a) \right),$$

unde

$$F_{i_1, ..., i_m}(a) = \frac{m!}{i_1! i_2! ... i_m!} \left(\frac{f^{(2)}(a)}{2!} \right)^{i_1} \left(\frac{f^{(3)}(a)}{3!} \right)^{i_2} \cdots \left(\frac{f^{(m+1)}(a)}{(m+1)!} \right)^{i_m},$$

oricare ar fi $i_1, \ i_2, ..., \ i_m \in \{0, 1, ..., m\}$.

6^0 Din (13) şi (14) deducem că funcţia $\overline{\theta}$ este derivabilă de $n-1$ ori pe $(a - \delta, a + \delta)$ şi

$$\overline{\theta}^{(m)}(x) = \left(\frac{\overline{c}(x) - \overline{c}(a)}{x - a} \right)^{(m)} = \left(\frac{c(x) - a}{x - a} \right)^{(m)}, \text{ oricare ar fi } m \in \{1, ..., n-1\}$$

oricare ar fi $x \in (a - \delta, a + \delta) \setminus \{a\}$ şi deci

$$\overline{\theta}^{(m)}(a) = \lim_{x \to a} \left(\frac{\overline{c}(x) - \overline{c}(a)}{x - a} \right)^{(m)}, \text{ oricare ar fi } m \in \{1, ..., n-1\}.$$

Dacă acum ţinem seama de (18), teorema este demonstrată. □

Exemplul 1. *Pentru* $n = 1$ *din* (15) *obţinem că*

$$\overline{c}^{(1)}(a) = \frac{1}{2},$$

şi atunci, din (17), *rezultă că*

$$\overline{\theta}(a) = c^{(1)}(a) = \frac{1}{2}.$$

Exemplul 2. *Pentru* $n = 2$ *din* (15) *obţinem că*

$$\overline{c}^{(2)}(a) = \frac{f^{(3)}(a)}{12\ f^{(2)}(a)},$$

şi atunci

$$\overline{\theta}^{(1)}(a) = \frac{f^{(3)}(a)}{24\ f^{(2)}(a)}.$$

Exemplul 3. *Pentru $n = 3$ din (15) obţinem că*

$$\overline{c}^{(3)}(a) = \frac{f^{(2)}(a)f^{(4)}(a) - \left(f^{(3)}(a)\right)^2}{8\left(f^{(2)}(a)\right)^2}$$

şi atunci

$$\overline{\theta}^{(2)}(a) = \frac{f^{(2)}(a)f^{(4)}(a) - \left(f^{(3)}(a)\right)^2}{24\left(f^{(2)}(a)\right)^2}.$$

Observaţie. Dacă punctul a este extremitatea stângă a intervalului I, atunci are loc următoarea afirmaţie:

Teorema 7. *Fie I un interval din $\mathbb{R}$, $a \in I$, extremitatea stângă a intervalului I, şi $f : I \to \mathbb{R}$ o funcţie care satisface condiţiile:*

(i) *funcţia f este derivabilă de $(n+1)$ ori pe intervalul I;*

(ii) *funcţia $f^{(n+1)}$ este continuă în punctul $x = a$;*

(iii) $f^{(2)}(a) \neq 0$.

Atunci:

1^0 *Există un număr real $\delta > 0$ astfel încât*

a) $[a, a+\delta) \subseteq I$;

b) $f^{(2)}(x) \neq 0$, *oricare ar fi $x \in [a, a+\delta)$;*

c) funcţia $f^{(1)}$ este injectivă pe $[a, a+\delta)$.

2^0 *Există o funcţie $c : (a, a+\delta) \to (a, a+\delta)$, şi una singură, cu proprietatea că*

$$f(x) - f(a) = (x-a)f^{(1)}(c(x)),$$

oricare ar fi $x \in (a, a+\delta)$.

3^0 *Există o funcţie $\theta : (a, a+\delta) \to (0,1)$, şi una singură, cu proprietatea că*

$$f(x) - f(a) = (x-a)f^{(1)}(a + (x-a)\,\theta(x)),$$

oricare ar fi $x \in (a, a+\delta)$.

4^0 *Funcţia $\varphi : [a, a+\delta) \to f^{(1)}([a, a+\delta))$ definită prin*

$$\varphi(x) = f^{(1)}(x), \text{ oricare ar fi } x \in [a, a+\delta),$$

este bijectivă.

5^0 *Funcţia $\overline{c} : [a, a+\delta) \to [a, a+\delta)$ definită prin*

$$\overline{c}(x) = \begin{cases} c(x), & \text{dacă } x \in (a, a+\delta) \\ a, & \text{dacă } x = a, \end{cases}$$

este derivabilă de n ori pe $[a, a+\delta)$ şi, pentru fiecare $x \in (a, a+\delta)$ şi $m \in \{1, ..., n\}$, avem

$$\overline{c}^{(m)}(x) = \sum_{k=1}^{m} (\varphi^{-1})^{(k)}\left(\frac{f(x)-f(a)}{x-a}\right) \times \sum_{\substack{i_1+2i_2+...+mi_m=m \\ i_1+i_2+...+i_m=k}} \frac{m!}{i_1!i_2!...i_m!} F_{i_1,...,i_m}(x)$$

unde

$$F_{i_1,\ldots,i_m}(x) = \left(\frac{1}{1!}\left(\frac{f(x)-f(a)}{x-a}\right)^{(1)}\right)^{i_1}\left(\frac{1}{2!}\left(\frac{f(x)-f(a)}{x-a}\right)^{(2)}\right)^{i_2}\times\cdots$$

$$\cdots\times\left(\frac{1}{m!}\left(\frac{f(x)-f(a)}{x-a}\right)^{(m)}\right)^{i_m}$$

oricare ar fi $x \in (a, a+\delta)$, *şi* $i_1, ..., i_m \in \{0, 1, ..., m\}$ *şi*

$$\overline{c}^{(m)}(a) = \sum_{k=1}^{m} \left(\varphi^{-1}\right)^{(k)}\left(f^{(1)}(a)\right)\times$$

$$\times \sum_{\substack{i_1+2i_2+...+mi_m=m\\ i_1+i_2+...+i_m=k}} \frac{m!}{i_1!i_2!...i_m!}\left(\frac{f^{(2)}(a)}{2!}\right)^{i_1}\left(\frac{f^{(3)}(a)}{3!}\right)^{i_2}...\left(\frac{f^{(m+1)}(a)}{(m+1)!}\right)^{i_m} =$$

$$= \sum_{k=1}^{m}\left(\sum_{\substack{i_2+2i_3+...+(k-1)i_k=k-1\\ i_1+i_2+...+i_k=k-1}} \frac{(-1)^{k-1+i_1}(2k-2-i_1)!}{i_2!i_3!...i_k!\left(f^{(2)}(a)\right)^{2k-1}}\times\right.$$

$$\times\left(\frac{f^{(2)}(a)}{1!}\right)^{i_1}\left(\frac{f^{(3)}(a)}{2!}\right)^{i_2}\cdots\left(\frac{f^{(k+1)}(a)}{k!}\right)^{i_k}\times$$

$$\left.\times \sum_{\substack{i_1+2i_2+...+mi_m=m\\ i_1+i_2+...+i_m=k}} \frac{m!}{i_1!i_2!...i_m!}\left(\frac{f^{(2)}(a)}{2!}\right)^{i_1}\left(\frac{f^{(3)}(a)}{3!}\right)^{i_2}\cdots\left(\frac{f^{(m+1)}(a)}{(m+1)!}\right)^{i_m}\right),$$

oricare ar fi $m \in \{0, 1, ..., n\}$.

6^0 *Funcţia* $\overline{\theta} : [a, a+\delta) \to (0, 1)$ *definită prin*

$$\overline{\theta}(x) = \begin{cases} \theta(x), & \text{dacă } x \in (a, a+\delta) \\ \dfrac{1}{2}, & \text{dacă } x = a \end{cases}$$

este derivabilă de $n-1$ *ori pe* $[a, a+\delta)$ *şi pentru fiecare* $x \in (a, a+\delta)$

$$\overline{\theta}^{(m)}(x) = \left(\frac{c(x)-a}{x-a}\right)^{(m)}, \text{ oricare ar fi } m \in \{1, ..., n-1\}$$

şi

$$\overline{\theta}^{(m)}(a) = \frac{1}{m+1}\overline{c}^{(m+1)}(a), \text{ oricare ar fi } m \in \{1, ..., n-1\} =$$

$$= \frac{1}{m+1}\sum_{k=1}^{m+1}\left(\varphi^{-1}\right)^{(k)}(f^{(1)}(a))\times$$

$$\times \sum_{\substack{i_1+2i_2+...+(m+1)i_{m+1}=m+1\\ i_1+i_2+...+i_{m+1}=k}} \frac{(m+1)!}{(i_1)!...(i_{m+1})!}\left(\frac{f^{(2)}(a)}{2!}\right)^{i_1}\left(\frac{f^{(3)}(a)}{3!}\right)^{i_2}...\left(\frac{f^{(m+2)}(a)}{(m+2)!}\right)^{i_{m+1}}.$$

Demonstraţie. Demonstraţia este similară cu a teoremei anterioare. □

O teoremă asemănătoare se poate demonstra şi în cazul în care a este capătul drept al intervalului I.

Bibliografie

[1] D. Andrica, D.I. Duca, I. Purdea şi I. Pop, *Matematica de bază*, Casa Cărţii de Ştiinţă, Cluj-Napoca, 2004.

[2] D.I. Duca, *A Note on the Mean Value Theorem*, Didactica Matematicii, 19(2003), 91-102.

[3] D.I. Duca şi E. Duca, *Exerciţii şi probleme de analiză matematică,* vol. 1, Casa Cărţii de Ştiinţă, Cluj-Napoca, 2007.

[4] D.I. Duca şi E. Duca, *Exerciţii şi probleme de analiză matematică*, vol. 2, Casa Cărţii de Ştiinţă, Cluj-Napoca, 2009.

[5] D.I. Duca şi O. Pop, *Asupra punctului intermediar din teorema creşterilor finite*, Lucrările Seminarului de Didactica Matematicii, 19 (2003), 91-102.

[6] D.I. Duca and O. Pop, *On the Intermediate Point in Cauchy's Mean-Value Theorem*, Mathematical Inequalities & Applications, 9 (2006) , no. 3, 375-389.

[7] D.I. Duca and O.T. Pop, *Concerning the Intermediate Point in the Mean Value Theorem*, Mathematical Inequalities & Applications, 12 (2009) no. 3, 499-512.

[8] D.I. Duca, *Proprietăţi ale punctului intermediar din teorema de medie a lui Lagrange*, Gazeta matematică, seria A, 28 (57) 2010.

[9] D.I. Duca, *Proprietăţi de derivabilitate ale funcţiei punct intermediar din teorema de medie a lui Lagrange*, Gazeta matematică, seria A, 28(57) (2010) .

[10] J.J. Koliha, *Mean, Meaner, and the Meanest Mean Value Theorem*, The American Mathematical Monthly, 116(2009), no. 4, 356-361.

[11] J.P. Penot, *On the Mean-Value Theorem*, Optimization, 19(1988), no. 1, 147-156.

[12] J. Riordan: *Derivatives of Composite Function*s, The American Mathematical Monthly, 52(1946), 664-667.

[13] T. Trif, *Asymptotic Behaivior of Intermediate Points in certain Main Value Theorems*, Journal of Mathematical Inequalities, 2 (2008), no. 2, 151-161.

Asupra unor criterii de convergenţă

EUGEN PĂLTĂNEA[1)]

Abstract. In this paper, we propose a short trip into the world of classical convergence criteria for real sequences. We point out some connections, extensions and integral versions. This presentation represents the first part of a recent exposure at the conference *A XIV-a Conferinţă Anuală a S.S.M.R., Alba Iulia, 15-17 octombrie 2010.*

Keywords: convergence and divergence

MSC : 40A05

Lucrarea este dedicată comentării unor criterii clasice de convergenţă pentru şiruri şi serii numerice. Vom evidenţia o serie de conexiuni şi extinderi. Vom propune şi unele rezultate originale, printre care o versiune integrală a Lemei lui *Kronecker*.

Prezentarea de faţă urmează firul expunerii susţinute recent la *A XIV-a Conferinţă Anuală a S.S.M.R., Alba Iulia, 15-17 octombrie 2010.*

Un rezultat clasic, cu largă aplicabilitate, al analizei matematice reale asigură determinarea, în anumite condiţii, a limitei raportului a două şiruri cu ajutorul limitei raportului diferenţelor de termeni consecutivi. Rezultatul respectiv este datorat matematicianului austriac *Otto Stolz* (1842-1905) şi matematicianului italian *Ernesto Cesàro* (1859-1906).

Fie două şiruri reale, $(a_n)_{n\geq 1}$ şi $(b_n)_{n\geq 1}$. Presupunem că şirul $(b_n)_{n\geq 1}$ este strict pozitiv, strict crescător şi divergent. În cele ce urmează, vom considera următoarele două şiruri asociate lui $(a_n)_{n\geq 1}$ şi $(b_n)_{n\geq 1}$:

$$w_n = \frac{a_{n+1} - a_n}{b_{n+1} - b_n} \text{ şi } v_n = \frac{a_n}{b_n},\ n \geq 1. \tag{1}$$

Teorema 1. (Stolz-Cesàro). *Dacă există limita* $\lim\limits_{n\to\infty} w_n = L$*, unde* $L \in \overline{\mathbb{R}}$*, atunci există limita* $\lim\limits_{n\to\infty} v_n = L$. În fapt, teorema decurge din următoarea lemă binecunoscută.

Lema 1.

$$\liminf_{n\to\infty} w_n \leq \liminf_{n\to\infty} v_n \leq \limsup_{n\to\infty} v_n \leq \limsup_{n\to\infty} w_n. \tag{2}$$

Limita inferioară şi *limita superioară* ale unui şir reprezintă punctele limită extreme ale şirului considerat. Pentru detalii, pot fi consultate, de exemplu, lucrările [4] şi [7]. Ilustrativ, vom prezenta demonstraţia inegalităţii stângi a relaţiei (2).

Demonstraţia inegalităţii $\liminf\limits_{n\to\infty} w_n \leq \liminf\limits_{n\to\infty} v_n$.

Fie $\alpha_n = \inf\limits_{k\geq n} w_k$, $n \in \mathbb{N}^*$. Avem $w_k \geq \alpha_n$, $\forall\ k \geq n$, de unde $a_{k+1} - a_k \geq$ $\geq \alpha_n(b_{k+1} - b_k)$, $\forall\ k \geq n$. Din aceste relaţii, obţinem $a_k - a_n \geq \alpha_n(b_k - b_n)$,

[1)]Prof. univ., Universitatea „Transilvania“ din Braşov, epaltanea@unitbv.ro

$\forall\ k \geq n$, sau

$$v_k \geq \alpha_n + \frac{a_n - \alpha_n b_n}{b_k},\ \forall\ k \geq n.$$

Dar $\lim\limits_{k\to\infty} \frac{a_n-\alpha_n b_n}{b_k} = 0$. Rezultă $\liminf\limits_{m\to\infty} v_m = \lim\limits_{m\to\infty}\left(\inf\limits_{k\geq m} v_k\right) \geq \alpha_n$, $\forall\ n \in \mathbb{N}^*$, de unde $\liminf\limits_{n\to\infty} w_n = \sup\limits_{n\geq 1} \alpha_n \leq \liminf\limits_{m\to\infty} v_m$. □

Teorema *Stolz-Cesàro* reprezintă o extensie a următorului rezultat.

Teorema 2 (Cesàro). *Dacă şirul* $(x_n)_{n\geq 1}$ *are limita* L *(*$L \in \overline{\mathbb{R}}$*), atunci şirul* $\left(\sum\limits_{k=1}^{n} x_k/n\right)_{n\geq 1}$ *(şirul mediilor Cesàro) are limita* L*.*

Menţionăm de asemenea conceptul de *şir Cesàro sumabil.*

Definiţie. *Fie un şir real* $(x_n)_{n\geq 1}$*. Considerăm şirul* $s_n = \sum\limits_{k=1}^{n} x_k$*,* $n \geq 1$ *(şirul sumelor parţiale ale seriei* $\sum\limits_{n=1}^{\infty} x_n$*). Şirul* $(x_n)_{n\geq 1}$ *se numeşte Cesàro sumabil, cu suma Cesàro* $S \in \mathbb{R}$*, dacă* $\lim\limits_{n\to\infty} \frac{1}{n}\sum\limits_{k=1}^{n} s_k = S$*.*

Vom observa cu uşurinţă că teorema *Stolz-Cesàro* rămâne valabilă şi în cazul şirurilor $(b_n)_{n\geq 1}$ strict descrescătoare, strict negative şi divergente. Printre aplicaţiile notabile ale teoremei *Stolz-Cesàro* amintim varianta ∞/∞ a regulii lui l'Hôpital[1)].

Teorema 3 (l'Hôpital). *Fie* $f, g : (a,\infty) \to \mathbb{R}$*, două funcţii derivabile. Presupunem* $\lim\limits_{x\to\infty} g(x) = \infty$ *şi* $g'(x) > 0,\ \forall\ x \in (a,\infty)$*. Dacă* $\lim\limits_{x\to\infty} \frac{f'(x)}{g'(x)} = L$ *(unde* $L \in \overline{\mathbb{R}}$*), atunci* $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = L$*.*

Demonstraţie. Este suficient să dovedim $\lim\limits_{n\to\infty} \frac{f(t_n)}{g(t_n)} = L$ pentru un şir $(t_n)_{n\geq 1}$ strict crescător şi divergent, cu termenii în intervalul (a,∞). Conform teoremei de medie a lui *Cauchy*, există $s_n \in (t_n, t_{n+1})$ astfel încât

$$\frac{f(t_{n+1}) - f(t_n)}{g(t_{n+1}) - g(t_n)} = \frac{f'(s_n)}{g'(s_n)},\ n \geq 1.$$

Cum $\lim\limits_{n\to\infty} \frac{f'(s_n)}{g'(s_n)} = L$, concluzia se obţine aplicând teorema *Stolz-Cesàro.* □

Vom aminti faptul că teorema *Stolz-Cesàro* admite o binecunoscută versiune de tip 0/0 (a se vedea, spre exemplu, [2]). De asemenea există următoarea reciprocă a teoremei.

---

1) Guillaume François Antoine, Marquis de l'Hôpital (1661-1704), matematician francez.

Teorema 4. *Fie şirurile reale* $(a_n)_{n\geq 1}$ *şi* $(b_n)_{n\geq 1}$. *Presupunem că şirul* $(b_n)_{n\geq 1}$ *este strict pozitiv şi strict crescător. Notăm*:

$$u_n = \frac{b_n}{b_{n+1}},\ v_n = \frac{a_n}{b_n},\ w_n = \frac{a_{n+1} - a_n}{b_{n+1} - b_n},\ n \geq 1.$$

Dacă $\lim_{n\to\infty} v_n = L$ *(unde* $L \in \mathbb{R}$*) şi* $\lim_{n\to\infty} u_n = \rho$, *unde* $\rho \in [0, 1)$, *atunci* $\lim_{n\to\infty} w_n = L$.

Demonstraţie. Şirul $(w_n)_{n\geq 1}$ poate fi reprezentat prin

$$w_n = v_n + \frac{v_{n+1} - v_n}{1 - u_n},\ n \in \mathbb{N}^*,$$

de unde concluzia. □

În Gazeta Matematică, seriile A şi B, au fost publicate recent unele rezultate care extind versiunea clasică a teoremei *Stolz-Cesàro*. Astfel, în [1] este formulată următoarea teoremă interesantă.

Teorema 5. (Bârsan) *Fie* $(a_n)_{n\geq 0}$ *şi* $(b_n)_{n\geq 0}$ *două şiruri de numere reale care satisfac ipotezele*:

(1) *Şirul* $(b_n)_{n\geq 1}$ *este strict monoton şi nemărginit.*

(2) *Există un număr natural nenul k astfel încât* $\lim_{n\to\infty} \frac{a_{n+k}-a_n}{b_{n+k}-b_n} = L$ $(L \in \overline{\mathbb{R}})$.

Atunci şirul $\left(\frac{a_n}{b_n}\right)_{n\geq 0}$ *are limita L.*

Demonstraţie. Se aplică teorema *Stolz-Cesàro* subşirurilor $(a_{nk+r})_{n\geq 0}$ şi $(b_{nk+r})_{n\geq 0}$, unde $r \in \{0, 1, \cdots, k-1\}$, ale şirurilor $(a_n)_{n\geq 0}$ şi respectiv $(b_n)_{n\geq 0}$. □

În lucrarea [6] se propun o serie de generalizări ale teoremei *Stolz-Cesàro*. În fapt, autorul speculează în diverse moduri următorul tip interesant de majorare:

$$|v_n - L| \leq \frac{|a_m - Lb_m| + \sup\limits_{k\geq m} |w_k - L| \sum\limits_{i=m}^{n-1} |b_{i+1} - b_i|}{|b_n|},\ \forall\, n, m \in \mathbb{N}^*,\ n > m,$$

unde L este un număr real arbitrar.

Teorema *Stolz-Cesàro* poate fi regândită din considerente elementare. Astfel, cu notaţiile anterioare, avem

$$a_n = a_1 + \sum_{k=1}^{n-1}(a_{k+1} - a_k) = a_1 + \sum_{k=1}^{n-1} w_k(b_{k+1} - b_k),$$

deci, relaţia dintre şirurile $(v_n)_{n\geq 1}$ şi $(w_n)_{n\geq 1}$ poate fi exprimată prin:

$$v_n = \frac{a_1 + \sum_{k=1}^{n-1} w_k(b_{k+1} - b_k)}{b_n}, \ n \in \mathbb{N}^*.$$

Mai mult, dacă vom considera $a_0 = b_0 = 0$, şi ca urmare $w_0 = a_1/b_1$, $a_1 = w_0(b_1 - b_0)$, relaţia de mai sus se va transcrie:

$$v_n = \frac{\sum_{k=0}^{n-1} w_k(b_{k+1} - b_k)}{b_n} = \sum_{k=0}^{n-1} \left(\frac{b_{k+1} - b_k}{b_n} w_k \right), \ n \in \mathbb{N}^*. \quad (3)$$

Să observăm că $\sum\limits_{k=0}^{n-1} \frac{b_{k+1}-b_k}{b_n} = 1$, iar dacă şirul $(b_n)_{n\geq 1}$ este monoton şi divergent, atunci $\lim\limits_{n\to\infty} \frac{b_{k+1}-b_k}{b_n} = 0$, $\forall\ k \in \mathbb{N}$. Astfel, matematicianul german *Otto Toeplitz* (1881-1940) a obţinut următoarea extindere a teoremei *Stolz-Cesàro*, evidenţiind matricele triunghiulare infinite care conservă limitele de şiruri.

Teorema 6 (Toeplitz). *Fie* $B = (b_{n,k})_{n,k\in\mathbb{N}^*}$ *o matrice infinită (şir dublu) cu proprietăţile:*

(1) *B este o matrice triunghiulară* $(b_{n,k} = 0, \forall\ k > n)$;

(2) $b_{n,k} \geq 0, \forall\ n, k \in \mathbb{N}^*$;

(3) $\sum\limits_{k=1}^{n} b_{n,k} = 1, \ \forall\ n \in \mathbb{N}^*$;

(4) $\lim\limits_{n\to\infty} b_{n,k} = 0, \ \forall\ k \in \mathbb{N}^*$.

Dacă $(w_n)_{n\geq 1}$ *este un şir real cu limita* $L \in \overline{\mathbb{R}}$, *atunci şirul* $v_n = \sum\limits_{k=1}^{n} b_{n,k} w_k$, $n \geq 1$, *are limita L.*

Demonstraţia teoremei poate fi urmărită, de exemplu, în [3]. Menţionăm că un rezultat mai general este oferit de teorema *Silverman-Toeplitz* (a se vedea PlanetMath [5]). Teorema *Silverman-Toeplitz* se referă la conservarea limitei pentru şiruri de numere complexe.

Să revenin la cadrul specific teoremei *Stolz-Cesàro*. Având în vedere reprezentarea (3), propunem următorul enunţ, în care se analizează şi cazul unui şir $(b_n)_{n\geq 1}$ mărginit.

Teorema 7. *Fie şirurile reale* $(w_n)_{n\geq 1}$ *şi* $(b_n)_{n\geq 1}$, *cu* $(b_n)_{n\geq 1}$ *strict pozitiv şi monoton crescător. Fie şirul* $(v_n)_{n\geq 1}$, $v_n = \frac{\sum_{k=1}^{n-1} w_k(b_{k+1}-b_k)}{b_n}$, $n \in \mathbb{N}^*$.

(1) (*Stolz-Cesàro-Toeplitz.*) *Dacă* $\lim_{n\to\infty} w_n = L$ *(unde* $L \in \overline{\mathbb{R}}$*) şi* $\lim_{n\to\infty} b_n = \infty$*, atunci* $\lim_{n\to\infty} v_n = L$.

(2) *Dacă şirurile* $(w_n)_{n\geq 1}$ *şi* $(b_n)_{n\geq 1}$ *sunt mărginite, atunci şirul* $(v_n)_{n\geq 1}$ *este convergent.*

Demonstraţia cazului (2). Şirul $(b_n)_{n\geq 1}$, monoton şi mărginit, este convergent. Fie $u_n = \sum\limits_{k=1}^{n-1} w_k(b_{k+1} - b_k),\ n \geq 1$. Arătăm că şirul $(u_n)_{n\geq 1}$ este fundamental. Conform ipotezei, există $M > 0$ astfel ca $|w_n| \leq M,\ \forall\ n \geq 1$. Atunci, pentru oricare $n, p \in \mathbb{N}^*$, avem

$$|u_{n+p} - u_n| \leq \sum_{k=n}^{n+p-1} |w_k| \cdot |b_{k+1} - b_k| \leq M(b_{n+p} - b_n).$$

Din criteriul lui *Cauchy*[1)] rezultă că, pentru $\varepsilon > 0$, arbitrar, există $n_\varepsilon \in \mathbb{N}^*$ astfel încât $0 \leq b_{n+p} - b_n < \frac{\varepsilon}{M},\ \forall\ n, p \in \mathbb{N}^*,\ n \geq n_\varepsilon$. Atunci $|u_{n+p} - u_n| < \varepsilon,\ \forall\ n, p \in \mathbb{N}^*,\ n \geq n_\varepsilon$.

Astfel, şirul $(u_n)_{n\geq 1}$ este fundamental, deci convergent (criteriul lui *Cauchy*). Rezultă că şirul $(v_n)_{n\geq 1}$ este convergent. □

Următorul rezultat, datorat matematicianului german *Leopold Kronecker* (1823-1891), reprezintă o continuare teoretică a precedentelor teoreme. Importanţa *Lemei lui Kronecker* decurge în principal din implicarea sa în demonstraţia clasică a *legii tari a numerelor mari* în teoria probabilităţilor. Enunţul de mai jos conţine şi o completare a Lemei lui *Kronecker.*

Teorema 8. *Fie* $\sum\limits_{n=1}^{\infty} x_n$ *o serie reală convergentă, iar* $(b_n)_{n\geq 1}$ *un şir real strict pozitiv şi monoton crescător. Considerăm şirul* $(y_n)_{n\geq 1}$,

$$y_n = \frac{\sum\limits_{k=1}^{n} b_k x_k}{b_n},\ n \in \mathbb{N}^*.$$

(1) *Lema lui Kronecker. Dacă* $\lim_{n\to\infty} b_n = \infty$ *atunci* $\lim_{n\to\infty} y_n = 0$.

(2) *Dacă* $(b_n)_{n\geq 1}$ *este mărginit atunci şirul* $(y_n)_{n\geq 1}$ *este convergent.*

[1)] Augustin Louis Cauchy (1789-1857), matematician francez

Demonstraţie. Notăm $w_n = \sum_{k=1}^{n} x_k$, $n \geq 1$, şirul sumelor parţiale ale seriei convergente $\sum_{n=1}^{\infty} x_n$. Aplicând metoda de sumaţie *Abel*[1)], obţinem

$$y_n = \frac{b_1 w_1 + \sum\limits_{k=2}^{n} b_k(w_k - w_{k-1})}{b_n} = \frac{b_n w_n + \sum\limits_{k=1}^{n-1} w_k(b_k - b_{k+1})}{b_n} =$$

$$= w_n - \frac{\sum\limits_{k=1}^{n-1} w_k(b_{k+1} - b_k)}{b_n}.$$

Dar şirul $(w_n)_{n\geq 1}$ este convergent (cu limită egală cu suma seriei $\sum_{n=1}^{\infty} x_n$). Atunci, concluziile decurg direct din Teorema 7. □

În mod natural, teoremele precedente admit versiuni integrale. Astfel, Teorema 7 se poate reformula în „limbaj integral" în modul următor.

Teorema 9. *Fie funcţiile reale $f, g : [0, \infty) \to \mathbb{R}$, integrabile Riemann pe $[0, a]$, pentru oricare $a > 0$. Presupunem că, pentru orice $x > 0$, avem $g(x) \geq 0$ şi $\int_0^x g(t)\,\mathrm{d}t > 0$. Definim $h : [0, \infty) \to \mathbb{R}$,*

$$h(x) = \frac{\int\limits_0^x f(t)g(t)\,\mathrm{d}t}{\int\limits_0^x g(t)\,\mathrm{d}t},\ x > 0.$$

(1) *Dacă $\lim_{x\to\infty} f(x) = L$ (unde $L \in \overline{\mathbb{R}}$ şi $\int_0^\infty g(t)\,\mathrm{d}t = \infty$, atunci $\lim_{x\to\infty} h(x) = L$.*

(2) *Dacă funcţia f este mărginită şi integrala $\int_0^\infty g(t)\,\mathrm{d}t$ este convergentă, atunci funcţia h are limită finită spre ∞.*

Demonstraţie. Fie funcţia $G : (0, \infty) \to (0, \infty)$, $G(x) = \int_0^x g(t)\,\mathrm{d}t$. G este monoton crescătoare, deci are limită la ∞.

Avem $\int_0^\infty g(t)\,\mathrm{d}t = \lim_{x\to\infty} G(x) = \sup_{x>0} G(x)$.

[1)]Niels Henrik Abel (1802-1829), matematician norvegian

(1) Analizăm cazul $L \in \mathbb{R}$. Funcţia f este mărginită deoarece are limită finită spre ∞ şi este integrabilă *Riemann* pe compactele $[0,a]$, $a>0$. Notăm $M = \sup\limits_{x\in[0,\infty)} |f(x)-L|$. Fie $\varepsilon > 0$. Există $a>0$ astfel încât $|f(t)-L| < \frac{\varepsilon}{2}$, $\forall\ t \geq a$. Deoarece $\lim\limits_{x\to\infty} G(x) = \infty$, există $b > a$ astfel ca $G(x) > \frac{2G(a)M}{\varepsilon}$, $\forall\ x > b$. Atunci

$$|h(x)-L| \leq \frac{\int\limits_0^x |f(t)-L| g(t)\,\mathrm{d}t}{\int\limits_0^x g(t)\,\mathrm{d}t} \leq \frac{G(a)M}{G(x)} + \frac{\varepsilon(G(x)-G(a))}{2G(x)} < \varepsilon,\ \forall\ x > b.$$

Rezultă $\lim\limits_{x\to\infty} h(x) = L$.

Presupunem $L = \infty$. Fie $c > 0$. Există $a > 0$ astfel ca $f(t) > 2c$, $\forall\ t \geq a$. Notăm $\alpha = \sup\limits_{t\in[0,a]} |f(t)| \in \mathbb{R}$.

Există $b > a$ astfel ca $G(x) > \frac{G(a)(\alpha+2c)}{c}$, $\forall\ x > b$. Atunci

$$h(x) \geq \frac{-\int\limits_0^a |f(t)| g(t)\,\mathrm{d}t + \int\limits_a^x f(t)g(t)\,\mathrm{d}t}{\int\limits_0^x g(t)\,\mathrm{d}t} \geq 2c - \frac{G(a)(\alpha+2c)}{G(x)} > 2c - c = c,$$

$\forall\ x > b$. Cum $c > 0$ este arbitrar, deducem $\lim\limits_{x\to\infty} h(x) = \infty = L$.

Cazul $L = -\infty$ se reduce la precedentul caz, prin considerarea funcţiei $-f$.

(2) Este suficient să dovedim convergenţa integralei improprii $\int\limits_0^\infty f(t)g(t)\mathrm{d}t$. Notăm $s = \sup\limits_{t\geq 0} |f(t)|$ şi $I = \int\limits_0^\infty g(t)\,\mathrm{d}t$. Avem

$$\int\limits_0^x |f(t)g(t)|\mathrm{d}t = \int\limits_0^x |f(t)| g(t)\mathrm{d}t \leq sG(x) \leq sI, \quad \forall\ x > 0.$$

Rezultă că $\int\limits_0^\infty |f(t)g(t)|\mathrm{d}t$ este convergentă. Atunci $\int\limits_0^\infty f(t)g(t)\mathrm{d}t$ este convergentă. $\square$

În final, prezentăm o versiune integrală a Teoremei 8.

Teorema 10. *Fie funcţiile reale $f, g : [0, \infty) \to \mathbb{R}$, integrabile Riemann pe $[0, a]$, pentru oricare $a > 0$. Presupunem că g este strict pozitivă şi monoton crescătoare, iar integrala improprie $\int\limits_0^\infty f(t)\,\mathrm{d}t$ este convergentă. Definim funcţia $h : (0, \infty) \to \mathbb{R}$,*

$$h(x) = \frac{\int\limits_0^x f(t)g(t)\,\mathrm{d}t}{g(x)}, \quad x \geq 0.$$

(1) *Dacă $\lim\limits_{x\to\infty} g(x) = \infty$, atunci $\lim\limits_{x\to\infty} h(x) = 0$.*

(2) *Dacă g este mărginită, atunci h are limită finită spre ∞.*

Demonstraţie. (1) Este suficient să dovedim $\lim\limits_{n\to\infty} h(t_n) = 0$ pentru un şir $(t_n)_{n\geq 0}$, strict crescător şi divergent, cu $t_0 = 0$.

Avem $g(t_n) \leq g(t_{n+1})$, $\forall\, n \in \mathbb{N}$. Conform teoremei de medie *Bonnet*[1]-*Weierstrass*[2], pentru $k \in \mathbb{N}^*$ există $\xi_k \in [t_{k-1}, t_k]$ astfel încât:

$$\int\limits_{t_{k-1}}^{t_k} f(t)g(t)\,\mathrm{d}t = g(t_{k-1}) \int\limits_{t_{k-1}}^{\xi_k} f(t)\,\mathrm{d}t + g(t_k) \int\limits_{\xi_k}^{t_k} f(t)\,\mathrm{d}t.$$

Aplicând transformarea *Abel*, obţinem:

$$h(t_n) = \frac{\sum\limits_{k=1}^{n} g(t_k) \int\limits_{t_{k-1}}^{t_k} f(t)\,\mathrm{d}t}{g(t_n)} - \frac{\sum\limits_{k=1}^{n} [g(t_k) - g(t_{k-1})] \int\limits_{t_{k-1}}^{\xi_k} f(t)\,\mathrm{d}t}{g(t_n)}.$$

Conform Lemei lui *Kronecker* (Teorema 8),

$$\lim_{n\to\infty} \frac{\sum\limits_{k=1}^{n} g(t_k) \int\limits_{t_{k-1}}^{t_k} f(t)\,\mathrm{d}t}{g(t_n)} = 0,$$

iar, conform Teoremei 7,

$$\lim_{n\to\infty} \frac{\sum\limits_{k=1}^{n} [b(t_k) - b(t_{k-1})] \int\limits_{t_{k-1}}^{\xi_k} f(t)\,\mathrm{d}t}{b(t_n)} = 0.$$

Rezultă $\lim\limits_{n\to\infty} h(t_n) = 0$.

---

[1] Pierre Ossian Bonnet (1819-1892), matematician francez

[2] Karl Theodor Wilhelm Weierstrass (1815-1897), matematician german

(2) Fie $M = \lim\limits_{x\to\infty} g(x) = \sup\limits_{x>0} g(x) \in (0,\infty)$. Pentru a dovedi existenţa limitei finite $\lim\limits_{x\to\infty} \int\limits_0^x f(t)g(t)\,\mathrm{d}t$, deci convergenţa integralei $\int\limits_0^\infty f(t)g(t)\,\mathrm{d}t$, vom utiliza criteriul de convergenţă a lui *Cauchy* pentru integrale improprii. Fie $\varepsilon > 0$. Convergenţa integralei $\int\limits_0^\infty f(t)\,\mathrm{d}t$ asigură existenţa unui $c_\varepsilon > 0$, astfel încât $\left|\int\limits_x^y f(t)\,\mathrm{d}t\right| < \frac{\varepsilon}{2M}$, $\forall\ x, y > c_\varepsilon$. Atunci, pentru $c_\varepsilon < x < y$, obţinem:

$$\left|\int_x^y f(t)g(t)\,\mathrm{d}t\right| = \left|g(x)\int_x^z f(t)\,\mathrm{d}t + g(y)\int_z^y f(t)\,\mathrm{d}t\right| \le$$

$$\le g(x)\left|\int_x^z f(t)\,\mathrm{d}t\right| + g(y)\left|\int_z^y f(t)\,\mathrm{d}t\right| < 2M\frac{\varepsilon}{2M} = \varepsilon,$$

unde $z \in [x, y]$ este punctul intermediar asigurat de teorema de medie *Bonnet-Weierstrass*. Rezultă că $\int\limits_0^\infty f(t)g(t)\,\mathrm{d}t$ este convergentă. Ca urmare, h are o limită finită spre ∞. □

Bibliografie

[1] A. Bârsan, *O extindere a criteriului Stolz-Cesàro*, *Gazeta Matematică Seria B*, nr. 2/2008, 63-65.

[2] D.M. Bătineţu, *Şiruri*, Editura Albatros, Bucureşti, 1979.

[3] D. Duca, E. Copaciu, Gh. Lobonţ, *Analiză matematică, clasele XI-XII, pentru grupele de excelenţă*, Editura Studia, Cluj-Napoca, 2010.

[4] R. Gologan, *Despre limitele superioară şi inferioară ale unui şir*, Gazeta Matematică Seria B nr. 4/2008, 169-177.

[5] `http://planetmath.org/encyclopedia/SilvermanToeplitzTheorem.html`

[6] S. Puşpană, *O generalizare a teoremelor Stolz-Cesàro*, Gazeta Matematică Seria A, nr. 4/2007, 268-279.

[7] Gh. Sireţchi, *Calcul diferenţial şi integral*, Vol. I, Ed. Şt. şi Enc., Bucureşti, 1985.

Multipliers on discrete and locally compact groups

Tania-Luminiţa Costache[1)]

Abstract. This paper contains some results concerning normalized equivalent multipliers on discrete groups. We prove that for a given multiplier ω on a locally compact group G we find a projective representation of G on the *Hilbert* space $L^2(G)$ with ω as the associated multiplier. We also give a necessary condition for a multiplier to be exact.

Keywords: multiplier, projective representation.

MSC : 22Do5, 22D12, 43A70, 20C25

1. Introduction

In Section 2 we prove that any multiplier ω is equivalent with a normalized multiplier ω_1 on a discrete group G and we emphasize the fact that an ω-regular element a is also ω_1-regular for any multiplier ω_1 equivalent with the multiplier ω.

Section 3 is dedicated to the projective representations of locally compact groups and their associated multipliers. First we find a projective representation of a locally compact group G on the *Hilbert* space $H = L^2(G)$. After defining the equivalence between two projective representations of a locally compact group G on a *Hilbert* space H_1, respectively H_2, we prove that a multiplier ω on a locally compact group G is exact if any projective representation of G on a *Hilbert* space H_1 with the associated multiplier ω is equivalent with an ordinary representation of G on a *Hilbert* space H_2.

2. Multipliers on discrete groups

Throughout this section we denote by G a discrete group with the identity e and by $\mathbb{T}$ the group of complex numbers of modulus one.

Definition 2.1. ([5]) *A **multiplier** ω on G is a function $\omega : G \times G \to \mathbb{T}$ with the properties:*

i) $\omega(x, e) = \omega(e, x) = 1$ *for all* $x \in G$*;*
ii) $\omega(x, y)\omega(xy, z) = \omega(x, yz)\omega(y, z)$ *for all* $x, y, z \in G$*.*

Definition 2.2. ([5]) *Two multipliers ω_1 and ω_2 on G are **equivalent** if there is a map $\mu : G \to \mathbb{T}$ such that $\mu(e) = 1$ and*

$$\omega_2(x, y) = \mu(x)\mu(y)\mu(xy)^{-1}\omega_1(x, y),$$

for all $x, y \in G$.

Lemma 2.1. *Let ω_1 and ω_2 be two equivalent multipliers on G. Then*

$$\omega_1(x, y)\omega_1(y, x)^{-1} = \omega_2(x, y)\omega_2(y, x)^{-1},$$

for all $x, y \in G$ with $xy = yx$.

[1)] University of Bucharest, Romania

Proof. Since ω_1 and ω_2 are equivalent, by Definition 2.2, there is a map $\mu : G \to \mathbb{T}$ with $\mu(e) = 1$ such that

$$\omega_2(x, y) = \mu(x)\mu(y)\mu(xy)^{-1}\omega_1(x, y)$$

$$\omega_2(y, x) = \mu(y)\mu(x)\mu(yx)^{-1}\omega_1(y, x)$$

for all $x, y \in G$.

Therefore,

$$\omega_2(x, y)\omega_2(y, x)^{-1} = \mu(x)\mu(y)\mu(xy)^{-1}\omega_1(x, y)(\mu(y)\mu(x)\mu(yx)^{-1}\omega_1(y, x))^{-1} =$$

$$= \mu(x)\mu(y)\mu(xy)^{-1}\omega_1(x, y)\mu(y)^{-1}\mu(x)^{-1}\mu(yx)\omega_1(y, x)^{-1} =$$

$$= \mu(x)\mu(y)\mu(xy)^{-1}\omega_1(x, y)\mu(y)^{-1}\mu(x)^{-1}\mu(xy)\omega_1(y, x)^{-1} = \omega_1(x, y)\omega_1(y, x)^{-1}.$$

□

Definition 2.3. *([7]) Let ω be a multiplier on G. An element $a \in G$ is called ω-**regular** if $\omega(x, a) = \omega(a, x)$ for all $x \in C_G(a) = \{x \in G |\ xa = ax\}$ the centralizer of a in G.*

Lemma 2.2. *If a is ω-regular, then a is ω_1-regular for all multipliers ω_1 equivalent with ω.*

Proof. Since a is ω-regular, by Definition 2.3, $\omega(x, a) = \omega(a, x)$ for all $x \in C_G(a)$.

Since ω and ω_1 are equivalent, by Definition 2.2, there is a map $\mu : G \to$ $\to \mathbb{T}$, $\mu(e) = 1$ such that

$$\omega(x, a) = \mu(x)\mu(a)\mu(xa)^{-1}\omega_1(x, a),$$

$$\omega(a, x) = \mu(a)\mu(x)\mu(ax)^{-1}\omega_1(a, x),$$

for $a \in G$ and $x \in C_G(a)$.

Hence, $\mu(x)\mu(a)\mu(xa)^{-1}\omega_1(x, a) = \mu(a)\mu(x)\mu(ax)^{-1}\omega_1(a, x)$ for all $a \in G$ and $x \in C_G(a)$. Because $x \in C_G(a)$, we have $xa = ax$, so $\mu(xa) =$ $= \mu(ax)$. Therefore $\omega_1(x, a) = \omega_1(a, x)$ for all $a \in G$ and $x \in C_G(a)$. So, a is ω_1-regular. □

Definition 2.4. ([5]) *A multiplier ω is called **normalized** if*

$$\omega(x, x^{-1}) = 1 \quad \textit{for all } x \in G.$$

Remark 2.1. ([5]) If ω is a normalized multiplier, then it satisfies the equality $\omega(x, y)^{-1} = \omega(y^{-1}, x^{-1})$ for all $x, y \in G$.

Remark 2.2. The identity of G is ω-regular and if x is ω-regular, then x^{-1} is ω-regular, for a normalized multiplier ω.

Proof. By Definition 2.1 i), $\omega(x,e)=\omega(e,x)=1$ for all $x\in G$ with $xe=ex=x$, so the identity is ω-regular.

Since x is ω-regular, by Definition 2.3, $\omega(a,x)=\omega(x,a)$ for all $a\in G$ with $ax=xa \Leftrightarrow a=xax^{-1} \Leftrightarrow x^{-1}a=ax^{-1} \Leftrightarrow x^{-1}=ax^{-1}a^{-1} \Leftrightarrow$ $\Leftrightarrow a^{-1}x^{-1}=x^{-1}a^{-1}$ for all $a\in G$.

Since $\omega(a,x)=\omega(x,a)$, we obtain $\omega(a,x)^{-1}=\omega(x,a)^{-1}$ and, by Remark
2.1, it results that $\omega(x^{-1},a^{-1})=\omega(a^{-1},x^{-1})$ for all $a\in G$ with $a^{-1}x^{-1}=x^{-1}a^{-1}$, so x^{-1} is ω-regular. □

Lemma 2.3. *Every multiplier ω on G is equivalent with a normalized multiplier ω_1 on G.*

Proof. Let ω be a multiplier on G. We take

$$\omega_1(x,y)=[\omega(x,x^{-1})\omega(y,y^{-1})]^{-\frac{1}{2}}\omega(xy,y^{-1}x^{-1})^{\frac{1}{2}}\omega(x,y),$$

for all $x,y\in G$.

We show that ω_1 is a normalized multiplier on G equivalent with ω.

We verify the conditions in Definition 2.1:

i) $\omega_1(x,e)=[\omega(x,x^{-1})\omega(e,e^{-1})]^{-\frac{1}{2}}\omega(xe,e^{-1}x^{-1})^{\frac{1}{2}}\omega(x,e)=$ $=\omega(x,x^{-1})^{-\frac{1}{2}}\omega(x,x^{-1})^{\frac{1}{2}}=1$, using Definition 2.1 i) for the multiplier ω and the fact that e is the identity of G.

ii) $\omega_1(x,y)\omega_1(xy,z)=[\omega(x,x^{-1})\omega(y,y^{-1})]^{-\frac{1}{2}}\omega(xy,y^{-1}x^{-1})^{\frac{1}{2}}\omega(x,y)\cdot$

$$\cdot[\omega(xy,(xy)^{-1})\omega(z,z^{-1})]^{-\frac{1}{2}}\omega(xyz,z^{-1}(xy)^{-1})^{\frac{1}{2}}\omega(xy,z)=$$
$$=\omega(x,x^{-1})^{-\frac{1}{2}}\omega(y,y^{-1})^{-\frac{1}{2}}\omega(xy,y^{-1}x^{-1})^{\frac{1}{2}}\omega(x,y)\omega(xy,y^{-1}x^{-1})^{-\frac{1}{2}}\cdot$$
$$\cdot\omega(z,z^{-1})^{-\frac{1}{2}}\omega(xyz,z^{-1}y^{-1}x^{-1})^{\frac{1}{2}}\omega(xy,z)=\omega(x,x^{-1})^{-\frac{1}{2}}\omega(y,y^{-1})^{-\frac{1}{2}}\cdot$$
$$\cdot\omega(z,z^{-1})^{-\frac{1}{2}}\omega(x,y)\omega(xy,z)\omega(xyz,z^{-1}y^{-1}x^{-1})^{\frac{1}{2}}, \tag{2.1}$$

for all $x,y,z\in G$.

On the other hand,

$$\omega_1(x,yz)\omega_1(y,z)=[\omega(x,x^{-1})\omega(yz,(yz)^{-1})]^{-\frac{1}{2}}\omega(xyz,(yz)^{-1}x^{-1})^{\frac{1}{2}}\cdot$$
$$\cdot\omega(x,yz)[\omega(y,y^{-1})\omega(z,z^{-1})]^{-\frac{1}{2}}\omega(yz,z^{-1}y^{-1})^{\frac{1}{2}}\omega(y,z)=$$
$$=\omega(x,x^{-1})^{-\frac{1}{2}}\omega(yz,z^{-1}y^{-1})^{-\frac{1}{2}}\omega(xyz,z^{-1}y^{-1}x^{-1})^{\frac{1}{2}}\omega(x,yz)\omega(y,y^{-1})^{-\frac{1}{2}}\cdot$$
$$\cdot\omega(z,z^{-1})^{-\frac{1}{2}}\omega(yz,z^{-1}y^{-1})^{\frac{1}{2}}\omega(y,z)=\omega(x,x^{-1})^{-\frac{1}{2}}\omega(y,y^{-1})^{-\frac{1}{2}}\omega(z,z^{-1})^{-\frac{1}{2}}\cdot$$
$$\cdot\omega(x,yz)\omega(y,z)\omega(xyz,z^{-1}y^{-1}x^{-1})^{\frac{1}{2}}. \tag{2.2}$$

From relations (2.1) and (2.2) and by Definition 2.1 ii) applied to ω, it results that condition ii) in Definition 2.1 is verified for ω_1.

We have

$$\omega_1(x,x^{-1})=\omega(x,x^{-1})^{-\frac{1}{2}}\omega(x^{-1},(x^{-1})^{-1})^{-\frac{1}{2}}\omega(xx^{-1},(x^{-1})^{-1}x^{-1})^{\frac{1}{2}}\cdot$$
$$\cdot\omega(x,x^{-1})=\omega(x,x^{-1})^{\frac{1}{2}}\omega(x^{-1},x)^{-\frac{1}{2}}\omega(e,e)^{\frac{1}{2}}=\omega(x,x^{-1})^{\frac{1}{2}}\omega(x^{-1},x)^{-\frac{1}{2}}=1,$$

because, applying Definition 2.1 ii) for ω, we have

$$\omega(x,x^{-1})\omega(xx^{-1},x)=\omega(x,x^{-1}x)\omega(x^{-1},x)\Rightarrow$$
$$\Rightarrow\omega(x,x^{-1})\omega(e,x)=\omega(x,e)\omega(x^{-1},x)\Rightarrow\omega(x,x^{-1})=\omega(x^{-1},x),$$

for all $x\in G$, by Definition 2.1 i) applied to ω. Hence, by Definition 2.4, ω_1 is normalized.

In definition of the multiplier ω_1 we take $\mu(x) = \omega(x,x^{-1})^{-\frac{1}{2}}$. Therefore, $\mu : G\to\mathbb{T}$ and $\mu(e)=\omega(e,e^{-1})^{-\frac{1}{2}}=1$. Hence ω and ω_1 are equivalent. □

Definition 2.5. ([7]) *Given a multiplier ω on G, we define*

$$f_\omega(x,a)=\omega(x,a)\omega(xax^{-1},x)^{-1},$$

for all $a\in G$ ω-regular and for all $x\in G$.

Lemma 2.4. *Let ω be a multiplier on G. Then*

$$f_\omega(xy,z)=f_\omega(y,z)f_\omega(x,yzy^{-1}),$$

for all $x,y,z\in G$.

Proof. Using Definition 2.5, the given equality is equivalent with

$$\omega(xy,z)\omega(xyzy^{-1}x^{-1},xy)^{-1}=$$
$$=\omega(y,z)\omega(yzy^{-1},y)^{-1}\omega(x,yzy^{-1})\omega(xyzy^{-1}x^{-1},x)^{-1}.$$

So we have to verify the following equality:

$$\omega(xy,z)\omega(yzy^{-1},y)\omega(xyzy^{-1}x^{-1},x)=$$
$$=\omega(y,z)\omega(x,yzy^{-1})\omega(xyzy^{-1}x^{-1},xy). \tag{2.3}$$

By Definition 2.1 ii), we have:

$$\omega(y,z)\omega(x,yzy^{-1})\omega(xyzy^{-1}x^{-1},xy)=$$
$$=\omega(y,z)\omega(x,yz)\omega(xyz,y^{-1})\omega(yz,y^{-1})^{-1}\omega(xyzy^{-1}x^{-1},xy)$$

The relation (2.3) becomes:

$$\omega(xy,z)\omega(yzy^{-1},y)\omega(xyzy^{-1}x^{-1},x)=$$
$$=\omega(x,yz)\omega(y,z)\omega(xyz,y^{-1})\omega(yz,y^{-1})^{-1}\omega(xyzy^{-1}x^{-1},xy)$$

Hence,

$$\omega(xy,z)\omega(yzy^{-1},y)\omega(xyzy^{-1}x^{-1},x)\omega(yz,y^{-1})=$$
$$=\omega(x,yz)\omega(y,z)\omega(xyz,y^{-1})\omega(xyzy^{-1}x^{-1},xy). \tag{2.4}$$

Applying Definition 2.1 ii) in the right side of the relation (2.4), we obtain:

$$\omega(xy,z)\omega(yzy^{-1},y)\omega(xyzy^{-1}x^{-1},x)\omega(yz,y^{-1})=$$
$$=\omega(x,y)\omega(xy,z)\omega(xyz,y^{-1})\omega(xyzy^{-1}x^{-1},xy).$$

Therefore,

$$\omega(yzy^{-1},y)\omega(xyzy^{-1}x^{-1},x)\omega(yz,y^{-1})=$$

$$= \omega(x, y)\omega(xyz, y^{-1})\omega(xyzy^{-1}x^{-1}, xy) \tag{2.5}$$

Applying again Definition 2.1 ii) in the right side of the relation (2.5), we obtain:

$$\omega(yzy^{-1}, y)\omega(xyzy^{-1}x^{-1}, x)\omega(yz, y^{-1}) =$$

$$= \omega(x, y)\omega(xyzy^{-1}x^{-1}, xy)\omega(xyzy^{-1}x^{-1}xy, y^{-1}) =$$

$$= \omega(x, y)\omega(xyzy^{-1}x^{-1}, xyy^{-1})\omega(xy, y^{-1}) =$$

$$= \omega(x, y)\omega(xyzy^{-1}x^{-1}, x)\omega(xy, y^{-1}).$$

Hence,

$$\omega(yzy^{-1}, y)\omega(yz, y^{-1}) = \omega(x, y)\omega(xy, y^{-1}). \tag{2.6}$$

By Definition 2.1 ii), the relation (2.6) becomes:

$$\omega(yzy^{-1}, yy^{-1})\omega(y, y^{-1}) = \omega(x, yy^{-1})\omega(y, y^{-1}) \Leftrightarrow \omega(yzy^{-1}, e) = \omega(x, e),$$

which is true by Definition 2.1 i). □

Definition 2.6. ([7]) *Let ω be a multiplier on G. We define*

$$\omega'(x, y) = \omega(x, y)\omega(y, x)^{-1},$$

for all $x, y \in G$.

Lemma 2.5. *Let $x, y, z \in G$ such that $y, z \in C_G(x)$. Then*

$$\omega'(x, yz) = \omega'(x, y)\omega'(x, z).$$

Proof. By Definition 2.6, the given relation becomes:

$$\omega(x, yz)\omega(yz, x)^{-1} = \omega(x, y)\omega(y, x)^{-1}\omega(x, z)\omega(z, x)^{-1} \Leftrightarrow$$

$$\Leftrightarrow \omega(y, x)\omega(z, x)\omega(x, yz) = \omega(yz, x)\omega(x, y)\omega(x, z). \tag{2.7}$$

Applying Definition 2.1 ii) in (2.7), we obtain:

$$\omega(y, x)\omega(z, x)\omega(x, y)\omega(xy, z)\omega(y, z)^{-1} = \omega(yz, x)\omega(x, y)\omega(x, z) \Leftrightarrow$$

$$\Leftrightarrow \omega(y, x)\omega(z, x)\omega(x, y)\omega(xy, z) = \omega(y, z)\omega(yz, x)\omega(x, y)\omega(x, z) \Leftrightarrow$$

$$\Leftrightarrow \omega(y, x)\omega(z, x)\omega(xy, z) = \omega(y, z)\omega(yz, x)\omega(x, z) \Leftrightarrow$$

$$\Leftrightarrow \omega(y, x)\omega(z, x)\omega(xy, z) = \omega(y, zx)\omega(z, x)\omega(x, z) \Leftrightarrow$$

$$\Leftrightarrow \omega(y, x)\omega(xy, z) = \omega(y, zx)\omega(x, z). \tag{2.8}$$

Since $y, z \in C_G(x)$, we have $xy = yx$ and $zx = xz$.

So relation (2.8) becomes $\omega(y, x)\omega(yx, z) = \omega(y, xz)\omega(x, z)$, which is true by Definition 2.1 ii). □

3. Multipliers on locally compact groups. Projective representations of locally compact groups on *Hilbert* spaces

Throughout this section G will be a locally compact group with the identity e.

Definition 3.1. ([5]) *Let G be a locally compact group. A **multiplier** ω on G is a Borel function $\omega : G \times G \to \mathbb{T}$ with the properties :*

i) $\omega(x, e) = \omega(e, x) = 1$ *for all* $x \in G$;

ii) $\omega(x, y)\omega(xy, z) = \omega(x, yz)\omega(y, z)$ *for all* $x, y, z \in G$.

Definition 3.2. ([1]) *A **(unitary) projective representation** ρ of a locally compact group G on a Hilbert space H with the multiplier ω is a map $\pi : G \to \to \mathcal{U}(H)$, where $\mathcal{U}(H)$ is the group of all unitary operators on the Hilbert space H onto itself, such that*

i) $\pi(xy) = \omega(x, y)\pi(x)\pi(y)$ *for all* $x, y \in G$;

ii) $\pi(e) = I_H$, *where* I_H *is the identity operator on* H.

A representation is called **ordinary** if $\omega \equiv 1$.

Proposition 3.1. *If ω is a multiplier on G, then, taking $H = L^2(G)$ (relative to the Haar measure on G), we define $\pi : G \to \mathcal{U}(H)$ by*

$$(\pi(x)f)(y) = \overline{\omega(y^{-1}, x)}f(x^{-1}y),$$

for all $x, y \in G, f \in L^2(G)$. Then π is a projective representation of G with the associated multiplier ω.

Proof. Let $y \in G$ and $f \in L^2(G)$. By Definition 3.1 i), we have :

$$(\pi(e)f)(y) = \overline{\omega(y^{-1}, e)}f(y) = f(y) \Rightarrow \pi(e) = I_H.$$

Let $x, y, z \in G$ and $f \in L^2(G)$. Then:

$$(\pi(xy)f)(z) = \overline{\omega(z^{-1}, xy)}f((xy)^{-1}z) = \overline{\omega(z^{-1}, xy)}f(y^{-1}x^{-1}z).$$

On the other hand,

$$\begin{aligned}(\omega(x, y)\pi(x)\pi(y)f)(z) &= \omega(x, y)\overline{\omega(z^{-1}, x)}(\pi(y)f)(x^{-1}z) = \\ &= \omega(x, y)\overline{\omega(z^{-1}, x)\omega((x^{-1}z)^{-1}, y)}f(y^{-1}x^{-1}z) = \\ &= \omega(x, y)\overline{\omega(z^{-1}, x)\omega(z^{-1}x, y)}f(y^{-1}x^{-1}z).\end{aligned}$$

To show i) in Definition 3.2, we verify the equality

$$\overline{\omega(z^{-1}, xy)} = \omega(x, y)\overline{\omega(z^{-1}, x)\omega(z^{-1}x, y)}. \tag{3.1}$$

By Definition 3.1 ii), we have

$$\overline{\omega(z^{-1}, x)\omega(z^{-1}x, y)} = \overline{\omega(z^{-1}, xy)\omega(x, y)}.$$

So the relation (3.1) becomes:

$$\overline{\omega(z^{-1}, xy)} = \omega(x, y)\overline{\omega(z^{-1}, xy)\omega(x, y)}$$

equivalent with $1 = \omega(x,y)\overline{\omega(x,y)}$ (true because of the fact that ω takes values in $\mathbb{T}$). □

Definition 3.3. ([1]) *A multiplier ω on G is called **exact** if there is a Borel function $\nu : G \to \mathbb{T}$ such that*

$$\omega(x,y) = \frac{\nu(x)\nu(y)}{\nu(xy)}$$

for all $x, y \in G$.

Definition 3.4. ([1]) *Two projective representations π_1 and π_2 of G on the Hilbert spaces H_1, respectively H_2 are **equivalent** if there is a unitary operator $U : H_1 \to H_2$ and a Borel function $\mu : G \to \mathbb{T}$ such that*

$$\pi_2(x)U = \mu(x)U\pi_1(x),$$

for all $x \in G$.

Proposition 3.2. *A multiplier ω on G is exact if any projective representation of G on a Hilbert space H_1 with the multiplier ω is equivalent with an ordinary representation of G on a Hilbert space H_2.*

Proof. Let π_1 be a projective representation of G on a *Hilbert* space H_1 with the multiplier ω.

Since π_1 is equivalent with an ordinary representation π_2, by Definition 3.4, there is a unitary operator $U : H_1 \to H_2$ and a *Borel* function $\mu : G \to \mathbb{T}$ such that

$$\pi_2(x)U = \mu(x)U\pi_1(x), \qquad (3.2),$$

for all $x \in G$.

Relation (3.2) is also true for $y \in G$ and $xy \in G$:

$$\pi_2(y)U = \mu(y)U\pi_1(y), \qquad (3.3)$$

$$\pi_2(xy)U = \mu(xy)U\pi_1(xy). \qquad (3.4)$$

Since π_1 is a projective representation with multiplier ω and π_2 is an ordinary representation, we have $\pi_1(xy) = \omega(x,y)\pi_1(x)\pi_1(y)$ and $\pi_2(xy) = \pi_2(x)\pi_2(y)$.

Therefore, relation (3.4) becomes:

$$\pi_2(x)\pi_2(y)U = \mu(xy)U\omega(x,y)\pi_1(x)\pi_1(y)$$

and by relations (3.2) and (3.3), we have:

$$\mu(x)U\pi_1(x)U^*\mu(y)U\pi_1(y)U^*U = \mu(xy)\omega(x,y)U\pi_1(x)\pi_1(y).$$

which is equivalent with $\mu(x)\mu(y)U\pi_1(x)\pi_1(y) = \mu(xy)\omega(x,y)U\pi_1(x)\pi_1(y)$, so $\mu(x)\mu(y) = \mu(xy)\omega(x,y)$, which means that ω is exact. □

References

[1] B. Bagchi, G. Misra, *A note on the multipliers and projective representations of semi-simple Lie groups*, The Indian J. of Statistics, Special Issue on 2000, Vol. **62**, Series A, Pt. 3.

[2] B. Bagchi, G. Misra, *Homogeneous operators and projective representations of the Möbius group: A survey*, Proc. Indian Acad. Sci. (Math. Sci.), Vol. **111**, No.4 (2001), 415-437.

[3] A. Kleppner, *The structure of some induced representations*, Duke Math. J. **29** (1962), 555-572.

[4] A. Kleppner, *Multipliers on Abelian Groups*, Math. Annalen **158** (1965), 11-34.

[5] A. Kleppner, *Continuity and measurability of multiplier and projective representations*, J. Funct. Analysis **17** (1974), 214-226.

[6] E. W. Read, *Projective characters of the Weyl group of type F_4*, J. London Math. Soc. (2), **8** (1974), 83-93.

[7] E. W. Read, *On projective representations of finite reflection groups of type B_l and D_l*, J. London Math. Soc. (2), **10** (1975), 129-142.

[8] E. W. Read, *The Projective Representations of the Generalized Symmetric Group*, J. of Algebra **46** (1977), 102-133.

O proprietate a funcţiilor derivabile

DAN ŞTEFAN MARINESCU[1)], MIHAI MONEA[2)], MIHAI OPINCARIU[3)]
MARIAN STROE[4)]

Abstract. În această notă, vom pune în evidenţă o proprietate a funcţiilor derivabile pe care o vom utiliza apoi pentru a deduce o nouă demonstraţie a teoremei lui *Jarnik* sau a prezenta reciproce ale teoremei de medie a lui *Cauchy*.

Keywords: Funcţii derivabile, Teorema lui Lagrange, Teorema lui Cauchy.

MSC : 26A24,26A06.

1. Introducere

În numărul 2/2010 al revistei American Mathematical Monthly, *Sam B. Nadler Jr.* prezintă o nouă demonstraţie a teoremei lui *Darboux* (vezi[3]). Autorul utilizează rezultatul descris detaliat în propoziţia următoare:

Propoziţia 1.1. *Fie I un interval nedegenerat şi $f : I \to \mathbb{R}$ o funcţie derivabilă. Se consideră mulţimea*

$$A = \left\{ \frac{f(b) - f(a)}{b - a} \mid a, b \in I, a \neq b \right\}.$$

Atunci:

a) Mulţimea A este interval ;

b) Avem $A \subseteq f'(I) \subseteq \overline{A}$, unde cu $\overline{A}$ am notat aderenţa mulţimii A .

Acest rezultat este problema nr. 1, dată la faza finală a Olimpiadei de Matematică din România, ediţia 1996.(vezi [1], pag.120) al cărei autor este *Ioan Raşa*. Demonstraţia se găsesşte în [1], pag. 434. Ulterior, la ediţia din 2008 a fazei finale a Olimpiadei de Matematică din România, problema nr. 3 de la clasa a XI-a, autor *Bogdan Enescu* (vezi [5], pag. 15) a primit din partea lui *G. Dospinescu* şi *F. Bozgan* o soluţie care se bazează pe aceeaşi idee (vezi [5], pag.68). În această notă ne propunem să generalizăm rezultatul din propoziţia 1.1., iar drept consecinţă vom obţine, printre altele, o nouă demonstraţie pentru teorema lui *Jarnik*, precum şi variante de reciprocă pentru teorema lui *Cauchy*.

1)C. N. „Iancu de Hunedoara“, Hunedoara, România, marinescuds@yahoo.com

2)C. N. „Decebal“, Deva, România, mihaimonea@yahoo.com

3)C. N. „Avram Iancu“, Brad, România, opincariumihai@yahoo.com

4)C. N. „Emanoil Gojdu“, Hunedoara, România, maricu_stroe@yahoo.com

2. Rezultatul principal

Vom enunţa şi demonstra rezultatul principal al acestei note în teorema 2.1. Menţionăm că pe tot parcursul acestui text vom considera $I \subset \mathbb{R}$, ca fiind un interval nedegenerat de numere reale.

Teorema 2.1. *Fie $f, g : I \to \mathbb{R}$ două funcţii derivabile astfel încât $g'(x) \neq 0$, pentru orice $x \in I$. Se consideră mulţimea*

$$A = \left\{ \frac{f(b) - f(a)}{g(b) - g(a)} \,\middle|\, a, b \in I,\ a < b \right\}.$$

Atunci:

a) Mulţimea A este interval ;

b) Avem $A \subseteq \frac{f'}{g'}(I) \subseteq \overline{A}$.

Demonstraţie. Deoarece g' este nenulă, g este injectivă, deci elementele mulţimii A sunt corect definite.

Pentru punctul a), considerăm $u, v \in A$, $u < v$ şi demonstrăm că pentru orice $w \in (u, v)$ avem $w \in A$. Dacă $u \in A$ atunci există $a, b \in I$, $a < b$ astfel încât

$$u = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

De asemenea, există $c, d \in I, c < d$, astfel încât

$$v = \frac{f(d) - f(c)}{g(d) - g(c)}.$$

Construim funcţia

$$h : [0,1] \to \mathbb{R}, h(t) = \frac{f(tb + (1-t)d) - f(ta + (1-t)c)}{g(tb + (1-t)d) - g(ta + (1-t)c)}.$$

Funcţia este corect definită deoarece $g(tb + (1-t)d) = g(ta + (1-t)c)$ ar conduce la $tb + (1-t)d = ta + (1-t)c$ pe baza injectivităţii lui g şi apoi la $t(b-a) + (1-t)(d-c) = 0$ ceea ce este imposibil.

Observăm că funcţia h este continuă şi în plus avem $h(0) = v$, $h(1) = u$. Atunci $h(1) < w < h(0)$, deci există $s \in (0,1)$ astfel încât $h(s) = w$, prin urmare

$$w = \frac{f(sb + (1-s)d) - f(sa + (1-s)c)}{g(sb + (1-s)d) - g(sa + (1-s)c)}.$$

Cum $sb + (1-s)d,\ sa + (1-s)c \in I$ şi $sa + (1-s)c < sb + (1-s)d$, deducem că $w \in A$, deci A este interval.

Pentru punctul b) fie $y \in A$. Atunci există $a, b \in I$, $a < b$, astfel încât

$$y = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Din teorema de medie a lui *Cauchy* obţinem existenţa unui punct $c \in (a, b)$ astfel încât $y = \frac{f'(c)}{g'(c)}$, deci $y \in \frac{f'}{g'}(I)$, adică $A \subseteq \frac{f'}{g'}(I)$.

Pentru a demonstra a doua incluziune, fie $z \in \frac{f'}{g'}(I)$. Atunci există $c \in I$ astfel încât $z = \frac{f'(c)}{g'(c)}$. Deoarece I este interval, deducem că există un şir de numere $(c_n)_{n \in \mathbb{N}}$ din I astfel încât $c_n \neq c$ pentru orice $n \in \mathbb{N}$ şi $c_n \to c$. Atunci

$$f'(c) = \lim_{n \to \infty} \frac{f(c_n) - f(c)}{c_n - c}$$

şi

$$g'(c) = \lim_{n \to \infty} \frac{g(c_n) - g(c)}{c_n - c},$$

de unde obţinem

$$\frac{f'(c)}{g'(c)} = \lim_{n \to \infty} \frac{f(c_n) - f(c)}{g(c_n) - g(c)}$$

şi concluzia $z \in \overline{A}$, adică $\frac{f'}{g'}(I) \subseteq \overline{A}$.

Cu aceasta demonstraţia este încheiată. □

3. Consecinţe

În acest paragraf vom prezenta câteva rezultate care decurg din propoziţia 2.1. Prima consecinţă o reprezintă teorema lui *Jarnik*, căreia îi vom da o nouă şi foarte scurtă demonstraţie.

Propoziţia 3.1. *(Teorema lui Jarnik). Fie $f, g : I \to \mathbb{R}$ două funcţii derivabile astfel încât $g'(x) \neq 0$, pentru orice $x \in I$. Atunci funcţia $\frac{f'}{g'}$ are proprietatea lui Darboux pe I.*

Demonstraţie. Fie $J \subset I$ un interval oarecare. Atunci, conform 2.1.b) rezultă că $\frac{f'}{g'}(J)$ este tot interval. Prin urmare $\frac{f'}{g'}$ are proprietatea lui *Darboux*. □

În continuare vom enunţa şi demonstra o propoziţie pe care o putem considera ca o întărire a teoremei lui *Cauchy* şi evident, ca un caz particular, se obţime o întărire a teoremei lui *Lagrange*.

Propoziţia 3.2. *Fie $f, g : I \to \mathbb{R}$ două funcţii derivabile cu $g'(x) \neq 0$, pentru orice $x \in I$. Fie $a.b \in I, a < b$, pentru care există $d \in (a, b)$ cu proprietatea*

$$\frac{f(b) - f(d)}{g(b) - g(d)} \neq \frac{f(d) - f(a)}{g(d) - g(a)}.$$

Atunci există o vecinătate V a numărului $\frac{f(b)-f(a)}{g(b)-g(a)}$ astfel încât pentru orice $\lambda \in V$ există $c \in I$ pentru care

$$\frac{f'(c)}{g'(c)} = \lambda.$$

Demonstraţie. Avem identitatea

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{g(d)-g(a)}{g(b)-g(a)}\frac{f(d)-f(a)}{g(d)-g(a)}+\frac{g(b)-g(d)}{g(b)-g(a)}\frac{f(b)-f(d)}{g(b)-g(d)}.$$

Deoarece g' este nenulă, atunci funcţia g este strict monotonă şi atunci vom avea $\frac{g(d)-g(a)}{g(b)-g(a)}>0$ şi $\frac{g(b)-g(d)}{g(b)-g(a)}>0$.

Dacă notăm $\frac{f(d)-f(a)}{g(d)-g(a)}=A$, $\frac{f(b)-f(d)}{g(b)-g(d)}=B$ şi $\frac{g(d)-g(a)}{g(b)-g(a)}=t$, atunci

$$\frac{f(b)-f(a)}{g(b)-g(a)}=tA+(1-t)B.$$

Presupunând că $A<B$, vom obţine

$$A<\frac{f(b)-f(a)}{g(b)-g(a)}<B.$$

Fie $V=(A,B)$; vom vedea că aceasta este vecinătatea căutată. Pentru aceasta considerăm mulţimea

$$J=\left\{\frac{f(u)-f(v)}{g(u)-g(v)}\,|u,v\in[a,b],v<u\right\}.$$

Din 2.1, obţinem că J este interval. Cum $A,B\in J$ atunci $V=(A,B)\subset J$. Tot din 2.1 avem

$$J\subset\frac{f'}{g'}\left([a.b]\right).$$

Cu aceasta demonstraţia este încheiată deoarece pentru orice $\lambda\in V$ avem

$$\lambda\in\frac{f'}{g'}\left([a.b]\right)$$

care este echivalent cu cerinţa din enunţ. □

Propoziţia următoare prezintă un rezultat extrem de util pe care îl vom exploata ulterior cu unele consecinţe.

Propoziţia 3.3. *Fie $f,g:I\to\mathbb{R}$ două funcţii derivabile astfel încât $g'(x)\neq 0$, pentru orice $x\in I$. Dacă există $c\in I$ astfel încât*

$$\frac{f'(c)}{g'(c)}\neq\frac{f(b)-f(a)}{g(b)-g(a)},$$

pentru orice $a,b\in I$, $a<b$, atunci c este punct de extrem pentru $\frac{f'}{g'}$.

Demonstraţie. Dacă considerăm mulţimea

$$A=\left\{\frac{f(b)-f(a)}{g(b)-g(a)}\,|a,b\in I,a<b\right\},$$

atunci $\frac{f'(c)}{g'(c)} \notin A$. Aplicând 2.1.b, deducem că $\frac{f'(c)}{g'(c)} \in \overline{A}$. Dar A este interval, de unde deducem că $\frac{f'(c)}{g'(c)}$ este una dintre extremităţile lui $\overline{A}$, deci c este punct de extrem pentru $\frac{f'}{g'}$. □

Consecinţa 3.4. (*Bogdan Enescu*) *Fie $f:\mathbb{R}\to\mathbb{R}$ o funcţie de două ori derivabilă pe $\mathbb{R}$ pentru care există un $c\in\mathbb{R}$ astfel încât*

$$\frac{f(b)-f(a)}{b-a} \neq f'(c),$$

pentru orice $a,b\in\mathbb{R}$, $a\neq b$. Atunci $f''(c)=0$.

Demonstraţie. Aplicăm 3.3. pentru funcţia $g:\mathbb{R}\to\mathbb{R}$, $g(x)=x$ şi deducem că punctul c este punct de extrem pentru funcţia f', de unde obţinem $f''(c)=0$ conform teoremei lui *Fermat*. □

În [4], *J. Tong* şi *P. Braza* prezintă următorul rezultat:

Fie $f:I\to\mathbb{R}$ o funcţie derivabilă şi $c\in I$ care nu este punct de extrem pentru f'. Atunci există $a,b\in I$ astfel încât

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$

Autorii îl numesc „*forma slabă a reciprocei teoremei lui Lagrange*". Propoziţia 3.5 va generaliza acest rezultat, motiv pentru care o putem numi „*forma slabă a reciprocei teoremei lui Cauchy*". O formă a acestui rezultat însoţită de o altă demonstraţie o regăsim în teorema 3.a din [2], autor *C. Mortici*.

Propoziţia 3.5. *Fie $f,g:I\to\mathbb{R}$ două funcţii derivabile astfel încât $g'(x)\neq 0$, pentru orice $x\in I$. Fie $c\in I$, care nu este punct de extrem pentru $\frac{f'}{g'}$. Atunci există $a,b\in I$, $a<b$, astfel încât*

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

Demonstraţie. Dacă am presupune prin reducere la absurd că

$$\frac{f'(c)}{g'(c)} \neq \frac{f(b)-f(a)}{g(b)-g(a)},$$

pentru orice $a,b\in I, a<b$, atunci c ar fi punct de extrem pentru $\frac{f'}{g'}$ conform lui 3.2, ceea ce contrazice ipoteza. □

Tot o versiune a reciprocei teoremei lui *Lagrange* este dată şi de rezultatul următor:

Fie $f:I\to\mathbb{R}$ o funcţie derivabilă şi c un punct interior lui I. Dacă f' este strict monotonă atunci există $a,b\in I$, $a<c<b$, astfel încât

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$

Generalizarea acestui rezultat este dată de propoziţia care urmează.

Propoziţia 3.6. *Fie $f, g : I \to \mathbb{R}$ două funcţii derivabile astfel încât $g'(x) \neq 0$, pentru orice $x \in I$ şi $\frac{f'}{g'}$ este strict monotonă. Fie c un punct interior lui I. Atunci există $a, b \in I$, $a < c < b$, astfel încât*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Demonstraţie. Dacă $\frac{f'}{g'}$ este strict monotonă pe I, atunci este injectivă. Deoarece c este punct interior al intervalului I, atunci nu este punct de extrem pentru $\frac{f'}{g'}$ şi există $a, b \in I$, $a < b$, astfel încât

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)},$$

conform consecinţei 3.4. Aplicând teorema lui *Cauchy* pe intervalul $[a, b]$, găsim un punct $d \in (a, b)$ astfel încât

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(d)}{g'(d)},$$

de unde

$$\frac{f'(c)}{g'(c)} = \frac{f'(d)}{g'(d)}.$$

Cum $\frac{f'}{g'}$ este injectivă obţinem $d = c$ adică $a < c < b$, ceea ce trebuia demonstrat. □

Tot în [4], *Tong* şi *Braza* prezintă şi o „*formă tare a teoremei lui Lagrange*“, adică:

Fie $f : I \to \mathbb{R}$ o funcţie derivabilă şi c un punct interior lui I care nu este punct de extrem local pentru f'. Dacă c nu este punct de acumulare al mulţimii

$$A = \left\{ x \in I \,\middle|\, f'(x) = f'(c) \right\},$$

atunci există $a, b \in I$, $a < c < b$, astfel încât

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

În finalul acestei note vom prezenta o generalizare pe care o vom numi „*forma tare a reciprocei teoremei lui Cauchy*“. O formulare a acestui rezultat se regăseşte de asemenea în [2], respectiv teorema 3.b, fără a fi însoţită de o soluţie. Noi vom prezenta însă două demonstraţii pentru acest rezultat.

Propoziţia 3.7. *Fie $f, g : I \to \mathbb{R}$ două funcţii derivabile astfel încât $g'(x) \neq 0$, pentru orice $x \in I$. Fie c un punct interior lui I, care nu este punct de extrem local pentru $\frac{f'}{g'}$. Dacă c nu este punct de acumulare pentru mulţimea*

$$A = \left\{ x \in I \,\middle|\, \frac{f'(x)}{g'(x)} = \frac{f'(c)}{g'(c)} \right\},$$

atunci există $a, b \in I$, $a < c < b$, astfel încât

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Demonstraţie. *Prima soluţie.* Dacă c nu este punct de extrem local pentru $\frac{f'}{g'}$, atunci există $a, b \in I$, $a < c < b$, astfel încât c nu este punct de extrem pe intervalul (a, b). Definim două şiruri $(a_n)_{n\in\mathbb{N}}$ şi $(b_n)_{n\in\mathbb{N}}$ astfel:

$$a_0 = a,\ a_{n+1} = \frac{a_n + c}{2},$$

respectiv

$$b_0 = b,\ b_{n+1} = \frac{b_n + c}{2}.$$

Evident, vom avea

$$a_0 \leqslant a_1 \leqslant ... \leqslant a_n \leqslant ... \leqslant c \leqslant ... \leqslant b_n \leqslant ... \leqslant b_1 \leqslant b_0$$

şi, deoarece

$$b_{n+1} - a_{n+1} = \frac{b_n - a_n}{2},$$

deducem că

$$\lim_{n\to\infty} (b_n - a_n) = 0,$$

de unde $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = c$.

Pentru fiecare $n \in \mathbb{N}$, avem $c \in (a_n, b_n)$. Aplicăm propoziţia 3.4. şi găsim două numere distincte $p_n, q_n \in (a_n, b_n)$, $p_n < q_n$, astfel încât

$$\frac{f'(c)}{g'(c)} = \frac{f(q_n) - f(p_n)}{g(q_n) - g(p_n)}.$$

Demonstraţia este încheiată dacă arătăm că există un $n_0 \in \mathbb{N}$, pentru care $c \in (p_{n_0}, q_{n_0})$.

Presupunem contrariul. Atunci , pentru orice $n \in \mathbb{N}$, avem $c \notin (p_n, q_n)$. Aplicând teorema lui *Cauchy* pe fiecare interval găsim un $c_n \in (p_n, q_n)$ astfel încât

$$\frac{f(q_n) - f(p_n)}{g(q_n) - g(p_n)} = \frac{f'(c_n)}{g'(c_n)}.$$

Obţinem atunci

$$\frac{f'(c_n)}{g'(c_n)} = \frac{f'(c)}{g'(c)},$$

pentru orice $n \in \mathbb{N}$. Dar $c_n \in (p_n, q_n) \subset (a_n, b_n)$, deci $\lim_{n\to\infty} c_n = c$, deci punctul c este punct de acumulare al mulţimii A ceea ce contrazice ipoteza şi încheie demonstraţia.

A doua soluţie. Deoarece c nu este punct de acumulare al mulţimii A atunci exită $u, v \in I$, $u < v$, astfel încât

$$\frac{f'(x)}{g'(x)} \neq \frac{f'(c)}{g'(c)},$$

pentru orice $x \in (u, v) - \{c\}$. Definim funcţia

$$h : (u, v) \to \mathbb{R}, h(x) = f(x) - \frac{f'(c)}{g'(c)} g(x),$$

care este derivabiă. Avem

$$h'(x) = f'(x) - \frac{f'(c)}{g'(c)} g'(x)$$

şi evident $h'(c) = 0$ şi $h'(x) \neq 0$ dacă $x \in (u, v) - \{c\}$. Deoarece c nu este punct de extrem pentru $\frac{f'}{g'}$ atunci funcţia h are semn diferit pe intervalele (u, c) şi respectiv (c, v). Deducem că punctul c este punct de extrem pentru h. Atunci există $a \in (u, c)$ şi $b \in (c, v)$, astfel încât $h(a) = h(b)$ care este echivalent cu

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)},$$

ceea ce încheie demonstraţia. □

Bibliografie

[1] D.M. Bătineţu-Giurgiu, I. Tomescu, *Olimpiadele Naţionale de matematică pentru liceu 1954-2003*, Ed. Enciclopedică, Bucureşti,2004.

[2] C. Mortici, *A converse of mean value theorem made easy,*Internat. J. Math., Ed. Sci. Tech., Vol 41, Issue 7.

[3] S. B. Nedler Jr., *A Proof of Darboux's Theorem*, American Mathematical Monthly, 117(2010), nr. 2, 174-175.

[4] J. Tong, P. Braza, *A converse of the mean value theorem*, American Mathematical Monthly, 104 (1997), nr. 10, 939-942.

[5] $* * *$, *Romanian Mathematical Competitions*, Ed. Theta, Bucureşti, 2008.

NOTE MATEMATICE

Improved Finsler-Hadwiger inequality revisited

CEZAR LUPU[1)] and VIRGIL NICULA[2)]

Abstract. In this note we give another generalized and sharpened version of the Finsler-Hadwiger inequality different from the one given by Wu and Debnath in [6]. Our approach is rather elementary and it doesn't use auxiliary trigonometric inequalities.

Keywords: inequality, Finsler-Hadwiger inequality, trigonometric inequality.

MSC : 51Mxx, 51Nxx, 51Axx, 26D15.

1. Introduction and Main result

Throughout this note we use the notation $\sum$ to express the cyclic sum, for example: $\sum f(A) = f(A) + f(B) + f(C)$.

In [6] it is proved the following generalized sharpened version of the Finsler-Hadwiger inequality, namely

Theorem 1.1. *Let a, b, c be the lengths of sides of triangle ABC and let F, R, d denote respectively its area, circumradius and the distance between circumcenter and the incenter. Then for real numbers $\lambda \geq 2$, the following inequality is true*

$$\sum a^\lambda \geq 2^\lambda 3^{1-\frac{\lambda}{2}} \left(3 + \frac{d^2}{R^2}\right)^{\frac{\lambda}{4}} F^{\frac{\lambda}{4}} + \sum |a-b|^\lambda.$$

This theorem is an improvment of the following generalized Finsler-Hadwiger inequality given by Wu in [4],

Theorem 1.2. *For any triangle ABC we have the inequality*

$$\sum a^\lambda \geq 2^\lambda 3^{1-\frac{\lambda}{4}} F^{\frac{\lambda}{2}} + \sum |a-b|^2.$$

For $\lambda = 2$, we obtain the celebrated Finsler-Hadwiger inequality (see [1]), namely

Theorem 1.3. *For any triangle ABC we have the inequality*

$$\sum a^2 \geq 4\sqrt{3}F + \sum (a-b)^2.$$

1) University of Bucharest, Faculty of Mathematics, Bucharest and University of Craiova, Faculty of Mathematics, Craiova, Romania, lupucezar@yahoo.com, lupucezar@gmail.com

2) Ion Creangă High School, Bucharest, Romania, levinicula@yahoo.com

In this paper, we prove other generalized and improved version of the Finsler-Hadwiger inequality. Our approach is rather elementary and different from the one of Wu and Debnath from [6].

The main result of this note is the following

Theorem 1.4. *Let a, b, c be the lengths of sides of triangle ABC and let F, R, r, s denote respectively its area, circumradius, inradius and the semiperimeter. Then for real numbers $\lambda \geq 2$, the following inequality is true*

$$\sum a^\lambda \geq 2^\lambda 3^{1-\frac{\lambda}{2}} (KP)^{\frac{\lambda}{2}} F^{\frac{\lambda}{2}} + \sum |a-b|^\lambda,$$

where $K = \dfrac{1}{s}\sqrt{4R^2+4Rr+3r^2}$ *and* $P = \sqrt{3 + \dfrac{4(R-2r)}{4R+r}}$.

2. Proof of the Main result

In the proof of Theorem 1.5, we shall use the following result from [2].

Lemma 2.1. *Let $x_1, x_2, \ldots, x_n$ be nonnegative numbers, and $p \geq 1$. Then*

$$\sum_{i=1}^n x_i^p \geq n^{1-p} \left(\sum_{i=1}^n x_i\right)^p,$$

with equality holding if and only if $x_1 = x_2 = \ldots = x_n$ or $p = 1$. Moreover,

$$(x_1 + x_2)^p \geq x_1^p + x_2^p,$$

with equality holding if and only if $x_1 = 0$ of $x_2 = 0$ or $p = 1$.

This lemma as a consequence of Holder's inequality and it is used also to prove other inequalities in [4] and [6]. Now, we give the

Proof of Theorem 1.4. By Lemma 2.1 we have

$$\sum a^\lambda = \sum((b-c)^2+(c+a-b)(a+b-c)]^{\frac{\lambda}{2}} \geq \sum(|b-c|^\lambda+(c+a-b)^{\frac{\lambda}{2}}(a+b-c)^{\frac{\lambda}{2}}).$$

Further again, by Lemma 2.1, we have

$$\sum(c+a-b)^{\frac{\lambda}{2}}(a+b-c)^{\frac{\lambda}{2}} \geq 3^{1-\frac{\lambda}{2}} \left(\sum(b+c-a)(c+a-b)\right)^{\frac{\lambda}{2}}.$$

Thus, we obtain

$$\sum a^\lambda \geq \sum |b-c|^\lambda + 3^{1-\frac{\lambda}{2}} \left(\sum(b+c-a)(c+a-b)\right)^{\frac{\lambda}{2}}.$$

Now, we prove that

$$\sum(b+c-a)(c+a-b) \geq 4F \cdot K \cdot P,$$

where K and P were defined in the previous section. By the well-known equalities in a triangle, $ab + bc + ca = s^2 + 4Rr + r^2$ and $a^2 + b^2 + c^2 =$

$= 2(s^2 - r^2 - 4Rr)$, we deduce that $\sum(b+c-a)(c+a-b) = 4r(4R+r)$ and we have to prove that

$$4r(4R+r) \geq 4F \cdot K \cdot P$$

which is succesively equivalent to

$$4R + r \geq \sqrt{4R^2 + 4Rr + 3r^2} \cdot \sqrt{3 + \frac{4(R-2r)}{4R+r}},$$

$$(4R^2 + 4Rr + 3r^2)(16R - 5r) \leq (4R+r)^3,$$

$$64R^3 + 44R^2r + 28Rr^2 - 15r^3 \leq 64R^3 + 48R^2r + 12Rr^2 + r^3$$

and finally equivalent to

$$4r(R-2r)^2 \geq 0.$$

Finally, we obtain

$$\sum a^\lambda \geq 2^\lambda 3^{1-\frac{\lambda}{2}} (KP)^{\frac{\lambda}{2}} F^{\frac{\lambda}{2}} + \sum |a-b|^\lambda,$$

where $K = \dfrac{1}{s}\sqrt{4R^2 + 4Rr + 3r^2}$ and $P = \sqrt{3 + \dfrac{4(R-2r)}{4R+r}}$. □

By the well-known *Gerretsen*'s inequality, $s^2 \leq 4R^2 + 4Rr + 3r^2$ (see [7]), we obtain a generalization of Theorem 5 from [8], namely

Theorem 2.2. *For any trianagle ABC, we have the inequality*

$$\sum a^\lambda \geq 2^\lambda 3^{1-\frac{\lambda}{2}} (P)^{\frac{\lambda}{2}} F^{\frac{\lambda}{2}} + \sum |a-b|^\lambda.$$

For $\lambda = 2$, Theorem 1.4 and Theorem 2.2 leads to the following refinements of the *Finsler-Hadwiger* inequality

Corollary 2.3. *For any triangle ABC, we have the inequality*

$$\sum a^2 \geq 4F \cdot \frac{1}{s}\sqrt{4R^2 + 4Rr + 3r^2} \cdot \sqrt{3 + \frac{4(R-2r)}{4R+r}} + \sum (a-b)^2.$$

Corollary 2.4. *For any triangle ABC, we have the inequality*

$$\sum a^2 \geq 4F \cdot \sqrt{3 + \frac{4(R-2r)}{4R+r}} + \sum (a-b)^2.$$

Corollary 2.4 appears also in [8]. The proof was based on a variant of a particular case of *Schur*'s algebraic inequality.

References

[1] P. Finsler, H. Hadwiger, *Einige Relationen im Dreick*, Comm. Mat. Helv., **10**(1937), 316–326.

[2] G. Hardy, J. E. Littlewood, G. Polya, *Inequalities*, 2nd edition, Cambridge University Press, 1952.

[3] D.S. Mitrinovic, J. E. Pecaric, V. Volonec, *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, Dordrecht, Netherlands, 1989.

[4] S. Wu, *Generalization and sharpness of Finsler-Hadwiger's inequality and its applications*, Math. Inequal. Appl. (MIA), **9**(2006), 421–426.

[5] S. Wu, *A new sharpened version of the Neuberg-Pedoe inequality*, Appl. Math. Lett., **21**(2008), 558–562.

[6] S. Wu, L. Debnath, *Parametrized Garfunkel-Bankoff inequality and improved Finsler-Hadwiger*, Appl. Math. Lett., **23**(2010), 331–336.

[7] S. Wu, M. Bencze, *An equivalent form of the fundamental trinagle inequality and its applications*, JIPAM, **10**(2009), article 16, 6pp.

[8] C. Lupu, C. Pohoaţă, *Sharpening the Hadwiger-Finsler inequality*, Crux Math., **34**(2008), 97–101.

[9] C. Lupu, V. Nicula, *Another look at some geometric inequalities*, in preparation.

On positivity of bivariate polynomials

EUGEN J. IONAŞCU[1)]

Abstract. We give two solutions of a problem that appeared in the AMM which has connections to a wide range of similar questions in real algebraic geometry.

Keywords: inequalities, real algebraic geometry
MSC : 26C05.

1. Introduction

In a recent issue of Amer. Math. Monthly [4], *A. Yakub* proposed the following problem which we considered interesting because of its connections with the topic in real algebraic geometry of finding decompositions as sums of squares of polynomials for non-negative polynomials in several variables.

Problem 1.1. *Let a, b and c be positive real numbers with $a+b+c=1$. Prove that*

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{b}\geq\frac{25}{1+48abc}.$$

One way to approach this problem is to reduce it to the following:

$$\begin{cases} a>0,\ b>0,\ c>0 \\ a+b+c=1 \end{cases} \Rightarrow f(a,b,c):=$$

$$=[(a+b+c)^3+48abc](ab+bc+ac)-25abc(a+b+c)^2\geq 0.$$

We observe that f is a homogeneous polynomial in a, b and c. Expanding we obtain

$$f(a,b,c)=10ab^2c^2+10a^2bc^2+10a^2b^2c+a^4b+a^4c+b^4a+b^4c+c^4b+c^4a+$$
$$+a^3b^2+3c^2a^3+3a^2b^3+3a^2c^3+3b^3c^2+3b^2c^3-18a^3bc-18ab^3c-18abc^3.$$

Let us observe that the coefficients of the expected monomials are non-negative except three of them. Related with this one may try to solve this problem using the following theorem of *Pólya* ([2]):

Theorem 1.1. *For $n\in\mathbb{N}$ let $P\in R\left[x_1,x_2,\ldots,x_n\right]$ be a homogeneous polynomial in variables $x_1,x_2,\ldots,x_n$ and*

$$\Delta=\{(x_1,x_2,\ldots,x_n)\mid x_1\geq 0,x_2\geq 0,\ldots,x_n\geq 0,\ \ \textit{and}\ \ x_1+x_2+\ldots+x_n=1\}.$$

Then f is (strictly) positive on Δ_n if and only if there exist m such that $(x_1+x_2+\ldots+x_n)^m P\left(x_1,\ldots,x_n\right)$ is a polynomial whose coefficients are all positive real numbers.

[1)] Columbus State University, Columbus, GA 31907, ionascu_eugen@colstate.edu

So one may try to prove that for every $\varepsilon > 0$ there exists m, such that the polynomial $(a+b+c)^{m_\varepsilon}\left[f(a,b,c)+\varepsilon(a+b+c)^5\right]$ has only positive coefficients. The value of m_ε can be found from [3] where *Reznick* and *Powers* show that

$$N > \frac{d(d-1)}{2}\frac{L}{\lambda} - d$$

where $L = \max\{P(x) \mid x \in \Delta_n\}$ and $\lambda = \min\{P(x) \mid x \in \Delta_n\}$. Using the well-known inequalities $abc \le \left(\frac{a+b+c}{3}\right)^3 = \frac{1}{27}$ and $a^2+b^2+c^2 \ge ab+bc+ac$ we obtain $abc \le \frac{1}{27}$ and $ab+bc+ac \le \frac{1}{3}$. This gives an upper bound for $P(a,b,c) = f(a,b,c) + \varepsilon(a+b+c)^5$ on Δ_3 to be $L = \frac{25}{27} + \varepsilon$. Hence if $\varepsilon < \dfrac{2}{27}$ then m_ε could be taken to be the smallest integer greater than $\frac{10}{\varepsilon} - 5$. However for $\varepsilon = \frac{1}{10}$ we calculated using Maple that it is enough to take $m_\varepsilon \ge 7$.

Another important result along these lines is *E. Artin*'s theorem in [1], which is the celebrated solution of *Hilbert*'s 17th problem: *a polynomial is non-negative if and only if it is the sum of squares of rational functions.* One knows then that such a representation exists for $f\left(x^2,y^2,z^2\right)$ (the restriction $x^2+y^2+z^2=1$ can be ignored at this point) and so if one can can find this representation it proves the problem.

2. A different approach

Proof of Problem 1.1. Without loss of generality we may assume that $a \ge b \ge c$. Then we can substitute $a-b=x$, $b-c=y$ and $c=z$. The new variables are then satisfying $x \ge 0$, $y \ge 0$, $z > 0$ and $x+2y+3z=1$. If we substitute $a=x+y+z$, $b=y+z$ and $c=z$ into f and define $g(x,y,z) := = f(a,b,c)$. The advantage of doing this is that we obtain a polynomial with only two negative coefficients whose corresponding monomials one contains y and one does not:

$$g(x,y,z) = 7x^3y^2 + 18x^2y^3 + x^4y + 20xy^4 + 16x^2y^2z - 3x^2yz^2 + 32xy^3z +$$
$$+15xy^2z^2 + 2xyz^3 + 2x^4z - 4x^3z^2 + 2x^2z^3 + 8y^5 + 16y^4z + 10y^3z^2 + 2y^2z^3.$$

Observe that $g(x,0,z) = 2x^2z(x-z)^2 \ge 0$ under our assumptions on x, y, z. Let us define $h(x,y,z)$ by $g(x,y,z) = g(x,0,z) + yh(x,y,z)$. Then $h(x,0,z) = x(x+2z)(x-z)^2 \ge 0$. Hence

$$g(x,y,z) = \big(7x^3 + 8y^3 + 2z^3 + 20xy^2 + 18x^2y + 15xz^2 +$$
$$+16x^2z + 10yz^2 + 16y^2z + 32xyz\big)\, y^2 + yh(x,0,z) + g(x,0,z) \ge 0. \qquad \square$$

We observe that the equality $g(x,y,z) = 0$ happens only if $y=0$ and $x=0$, corresponding to $a=b=c=\dfrac{1}{3}$, or if $y=0$ and $x=z$ which corresponds to $a=\dfrac{1}{2}$ and $b=c=\dfrac{1}{4}$.

One interesting question at this point is whether or not the next case of four variables is true: *does* $a, b, c, d > 0$, $a + b + c + d = 1$ *imply*

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \geq \frac{791}{26 + 6000abcd}\,?$$

References

[1] E. Artin, *Über die Zerlegung definiter Funktionen in Quadrate*, Hamb. Abh. 5, p. 100-115 (1927), The collected papers of Emil Artin, p 273-288 (1965).

[2] G. Polya, *Über positive Darstenllung von Polynomen Vierteljsche*, Naturforsch. Ges. Zurich 73 (1928) p. 141- 145, in Collecled Papers 2 (1974), MIT Press, p. 309-313.

[3] V. Powers and B. Reznick, *A new bound for Polya's Theorem with applications to polynomials positive on polyhedra*, J. Pure Appl. Algebra, 164(2001), no. 1-2, p. 221-229.

[4] A. Yakub, *Problem 11199*, American Mathematical Montly, Vol. , no. 1, 2006, p. 80 .

Another trace inequality for unitary matrices

Cezar Lupu[1)]

Abstract. In this short note we give another interresting trace inequality for unitary matrices different from the one given in [6]. Our inequality relies on a *Cauchy-Schwarz* 3-vectors type inequality to which we present three proofs. In fact, the main result is that for any unitary matrices U, V the following inequality holds: $|m(U)|^2 + |m(V)|^2 + |m(UV)|^2 \leq \leq 1 + 2|m(U)||m(V)||m(UV)|$, where $m(X)$ is the arithmetic mean of the elements from the principal diagonal of the matrix X.

Keywords: Cauchy-Schwarz inequality, Bessel inequality, Gram inequality, unitary matrix, trace, inner product

MSC : 15A03, 15A42, 15A45.

1. Introduction and main result

In this short note we prove another trace inequality for real unitary matrices which is a *Cauchy-Schwarz*-type inequality relating the average of the eigenvalues of each of two unitary matrices to that of their product. As it has been already highlighted in [2], these kind of trace inequalities are in connection with Connes embedding problem from C^*-algebras. The main idea is to study the set of unitary matrices

$$\{[\mathrm{tr}(U_i^* U_j)]_{ij}, U_1, U_2, \ldots, U_n \in M_m(\mathbb{C}),\ m \geq 1\},$$

which seems to be unknown even for $n = 3$. In this case, we deal with the set of triples $(\mathrm{tr}(U), \mathrm{tr}(V), \mathrm{tr}(UV))$ where U, V are unitary. As remarked in [2], the only connection between these traces seems to be the inequality established by *Wang* and *Zhang* in [5] and [6], namely

$$\sqrt{1 - |\,\mathrm{tr}(UV)|^2} \leq \sqrt{1 - |\,\mathrm{tr}(U)|^2} + \sqrt{1 - |\,\mathrm{tr}(V)|^2}.$$

It is well-known that the trace of a n-square matrix X is equal with the sum of the eigenvalues of X. We denote by $m(X)$ the algebraic mean of the eigenvalues of X. We will show the following

Theorem 1.1. *For any unitary complex matrices U, V the following inequality holds true*

$$|m(U)|^2 + |m(V)|^2 + |m(UV)|^2 \leq 1 + 2|m(U)||m(V)||m(UV)|.$$

In what follows we shall prove an inequality in a linear space $\mathcal{V}$ endowed with the inner product $(\cdot, \cdot)$. This is basically a *Cauchy-Schwarz*-type inequality for three vectors.

[1)] University of Bucharest, Faculty of Mathematics, Bucharest and University of Craiova, Faculty of Mathematics, Craiova, Romania, lupucezar@yahoo.com, lupucezar@gmail.com

Lemma 1.2. *For any vectors $u, v, w \in V$ the following inequality holds*

$$||u||^2|(v,w)|^2 + ||v||^2|(w,u)|^2 + ||w||^2|(u,v)|^2 \leq$$
$$\leq ||u||^2||v||^2||w||^2 + 2|(u,v)||(w,v)||(w,u)|. \tag{1}$$

First proof. For $u, v, w \in \mathcal{V}$ we consider the *Gram* matrix

$$G(u,v,w) = \begin{pmatrix} ||u||^2 & (u,v) & (u,w) \\ (v,u) & ||v||^2 & (v,w) \\ (w,u) & (w,v) & ||w||^2 \end{pmatrix}.$$

It is a well-known fact that $\det G(u,v,w) \geq 0$, so by computing the determinant by *Sarrus* or triangle rule, we have

$$0 \leq \det G(u,v,w) = ||u||^2||v||^2||w||^2 + 2|(u,v)||(v,w)||(w,u)| -$$
$$-(||u||^2|(v,w)|^2 + ||v||^2|(w,u)|^2 + ||w||^2|(u,v)|^2),$$

which is equivalent to the inequality (1).

Second proof. By *Cauchy-Schwarz* inequality, for any real number λ, we infer

$$||u - \lambda v||^2||w||^2 \geq |(u - \lambda v, w)|^2.$$

This is succesively equivalent to

$$(||u||^2 - 2\lambda|(u,v)| + \lambda^2||v||^2)||w||^2 \geq |(w,u)|^2 - 2\lambda|(w,u)||(v,w)| + \lambda^2|(v,w)|^2,$$
$$\left(||v||^2||w||^2 - |(v,w)|^2\right)\lambda^2 + 2\lambda\left(|(u,v)|||w||^2 - |(w,u)||(v,w)|\right) +$$
$$+||u||^2||w||^2 - |(u,w)|^2 \geq 0.$$

Denote

$$p(\lambda) = \left(||v||^2||w||^2 - |(v,w)|^2\right)\lambda^2 + 2\lambda(|(u,v)|||w||^2 -$$
$$-|(w,u)||(v,w)|) + ||u||^2||w||^2 - |(u,w)|^2.$$

By the *Cauchy-Schwarz* inequality we have that the coefficient of λ^2 is positive and since $p(\lambda) \geq 0, \forall \lambda \in \mathbb{R}$ we conclude that the discriminant of p must be negative. The discriminant of p is given by

$$\Delta_\lambda = 4(|(u,v)|||w||^2 - |(w,u)||(v,w)|)^2 - 4(||v||^2||w||^2 -$$
$$-|(v,w)|^2)(||u||^2||w||^2 - |(u,w)|^2).$$

It follows that

$$(|(u,v)|||w||^2 - |(w,u)||(v,w)|)^2 \leq (||v||^2||w||^2 - |(v,w)|^2)(||u||^2||w||^2 - |(u,w)|^2)$$

which is equivalent to

$$|(u,v)|^2||w||^2 - 2|(u,v)||(v,w)||(w,u)|||w||^2 \leq$$
$$\leq ||u||^2||v||^2||w||^4 - |(w,u)|^2||v||^2||w||^2.$$

Without loss of generality, we assume that $||w||^2 \neq 0$, so the inequality follows immediately.

Third proof. Consider the following matrix

$$D(u,v) = \begin{pmatrix} (w,w) & (w,u) \\ (v,w) & (u,v) \end{pmatrix}.$$

We will prove that

$$(\det D(u,v))^2 \leq |\det D(u,u)| \cdot |\det D(v,v)|. \tag{2}$$

Denote the polynomial

$$f(t) = D(u,u)t^2 - 2D(u,v)t + D(v,v).$$

A simple calculation shows that

$$f(t) = \det D(ut-v, ut-v) = \det \begin{pmatrix} (w,w) & (tu-v,w) \\ (w,ut-v) & (ut-v,ut-v) \end{pmatrix} =$$

$$= ||w||^2||ut-v||^2 - |(ut-v,w)|^2 \geq 0$$

by the *Cauchy-Schwarz* inequality. Thus $f(t) \geq 0, \forall t \in \mathbb{R}$ and this means that the discriminant must be negative,

$$\Delta_t = (\det D(u,v))^2 - |\det D(u,u)| \cdot |\det D(v,v)| \leq 0$$

which is equivalent to (2). The inequality (2) is equivalent to

$$(||w||^2|(u,v)| - |(w,u)||(w,v)|)^2 \leq (||w||^2||u||^2 - |(w,u)|^2)(||w||^2||v||^2 - |(v,w)|^2).$$

By expanding this inequality we obtain (1). □

Remark 1.3. *By definition the angle of two vectors in the inner product space is given by the formula*

$$\cos\theta_{u,v} = \frac{(u,v)}{||u||||v||}.$$

In this sense, we can rewrite the inequality (1) as

$$\cos^2\theta_{u,v} + \cos^2\theta_{v,w} + \cos^2\theta_{w,u} \leq 1 + 2|\cos\theta_{u,v}||\cos\theta_{v,w}||\cos\theta_{w,u}|.$$

Proof of Theorem 1.1. For any complex matrices A, B of size n, we denote the *Frobenius* inner product by $(A,B) := \operatorname{tr}(B^*A)$, where $*$ means the conjugate of the transpose. Puttting

$$w = \frac{1}{\sqrt{n}}I_n, \quad u = \frac{1}{\sqrt{n}}U, \quad v = \frac{1}{\sqrt{n}}V^*,$$

where I_n is the identity matrix and U, V are any unitary $n \times n$ matrices. Clearly $||u|| = ||v|| = ||w|| = 1$ and

$$(u,v) = \frac{1}{n}\operatorname{tr}(UV^*) = \frac{1}{n}\operatorname{tr}(UV),$$

$$(v,w) = \frac{1}{n}\operatorname{tr}(I_n^*U) = \frac{1}{n}\operatorname{tr}(U),$$

$$(w,u) = \frac{1}{n}\operatorname{tr}(V).$$

By applying Lemma 1.2, we employ

$$\frac{1}{n^2}(|\operatorname{tr}(UV)|^2+|\operatorname{tr}(U)|^2+|\operatorname{tr}(V)|^2)\le 1+\frac{2}{n^3}(|\operatorname{tr}(UV)||\operatorname{tr}(U)||\operatorname{tr}(V)|$$

equivalent to

$$|m(U)|^2+|m(V)|^2+|m(UV)|^2\le 1+2|m(U)||m(V)||m(UV)|. \qquad \square$$

Lemma 1.2 can be also extended in the framework of Probability Theory. If X, Y are two random variables, then we can define their inner product as $(X,Y)=E(XY)$, where $E(M)$ is the expectation of M. In this case, by Lemma 1.2 we deduce that for X, Y, Z random variables we have

$$\sum_{cyc} E(X^2)|E(YZ)|^2\le \prod E(X^2)+2\prod |E(XY)|.$$

Moreover, if we put $\mu=E(X)$ and $\nu=E(Y)$, then we have

$$|\operatorname{Cov}(X,Y)|^2=|E((X-\mu)(E-\nu))|^2=|(X-\mu,X-\nu)|^2.$$

By Lemma 1.2, we have

$$\sum_{cyc}|\operatorname{Cov}(X,Y)|^2\operatorname{Var}(Z)\le\prod\operatorname{Var}(X)+2\prod|\operatorname{Cov}(XY)|.$$

Other generalizations of Lemma 1.2 exist in the context of Operator Theory where one can discuss inner products as positive functionals. More specific, given a *Hilbert* space $L^2(\gamma)$, where γ is a measure, the inner product $(\cdot,\cdot)$ gives rise to a positive functional φ by $\varphi(g)=(g,1)$. For all $f\in L^2(\gamma)$ every positive functional φ gives a corresponding iner product

$$(f,g)_\varphi=\varphi(g^*f),$$

where g^* is the pointwise conjugate of g.

Now, Lemma 1.2 will become

$$\sum_{cyc}|\varphi(g^*f)|^2\varphi(h^*h)\le\prod\varphi(f^*f)+2\prod|\varphi(g^*f)|,$$

for all $f,g,h\in L^2(\gamma)$. The extension of the above inequality to C^*-algebras as well as other extensions, generalizations and applications will be given in [3] and [4].

Acknoledgements. The author is grateful to the Department of Mathematics of the Central European University, Budapest for hospitality during February–May 2010 when this note has been completed. The author is also indebted to *Călin Popescu* for many fruitful discussions regarding this topic. Many thanks go also to „Dinu Patriciu“ Foundation for support.

References

[1] W. Fulton, *Eigenvalues, invariant factors, highest weights and Schubert calculus*, Bull. Amer. Math. Soc. **37**(2000), 209–249.

[2] K. Juschenko, S. Popovych, *Algebraic reformulation of Connes emebedding problem and the free group algebra*, preprint at
`http : //www.math.chalmers.se/ stanislp/files/cep.pdf`.

[3] C. Lupu, T. Lupu, *Another look at the equivalence of Cauchy-Schwarz, Bessel, Gram inequalities and applications*, in preparation.

[4] C. Lupu, C.P. Niculescu, *An extension of a Cauchy-Schwarz type inequality for positive maps on C*-algebras*, in preparation.

[5] B. Y. Wang, F. Zhang, *A trace inequality for unitary matrices*, Amer. Math. Mon., **101**(1994), 453–455.

[6] F. Zhang, *Matrix Theory: Basic Results and Techniques*, Springer Verlag, 2001.

PROBLEMS

Authors should submit proposed problems to `office@rms.unibuc.ro` or to `gmaproblems@gmail.com`. Files shold be in PDF or DVI format. Once a problem is accepted and considered for publication, the author will be asked to submit the TeX file also, if avalaible. The referee process will usually take between several weeks and two months. Solutions may also be submitted to the same e-mail adress. For this issue, solutions should arrive before **March 15, 2011**.

PROPOSED PROBLEMS

309. Consider p prime and a is a rational number with $\sqrt[p]{a} \notin \mathbb{Q}$. Define the sequence of polynomials by $f_1 = X^p - a$ and $f_{n+1} = f_n^p - a$ for all $n \geq 1$. Show that all terms of the sequence f_n are irreductible polynomials.

Proposed by Marius Cavachi, Ovidius University of Constanţa, Romania.

310. If $n \geq 4$, $1 \leq \delta < \Delta \leq n-1$, $n, \delta, \Delta \in \mathbb{N}$, consider the function

$$f(x_\delta, x_{\delta+1}, \ldots, x_\Delta) = \sum_{\delta \leq i < j \leq \Delta} \left(\frac{1}{\sqrt{i}} - \frac{1}{\sqrt{j}} \right)^2 x_i x_j$$

and the domain

$$D = \{(x_\delta, x_{\delta+1}, \ldots, x_\Delta) : x_i \in \mathbb{N} \,\text{for}\, \delta \leq i \leq \Delta, \sum_{i=\delta}^{\Delta} x_i = n\}.$$

Show that if $(x_\delta, \ldots, x_\Delta) \in D$ then

$$f(x_\delta, \ldots, x_\Delta) \leq \left(\frac{1}{\sqrt{\delta}} - \frac{1}{\sqrt{\Delta}} \right)^2 \alpha(n),$$

where $\alpha(n) = \frac{n^2}{4}$ for n even and $\alpha(n) = (n^2-1)/4$ for n odd. When does equality hold?

Proposed by Ioan Tomescu, University of Bucharest, Romania.

311. Show that for any matrix $A \in M_2(\mathbb{R})$ there exist $X, Y \in \mathcal{M}_2(\mathbb{R})$ with $XY = YX$ such that $A = X^{2n+1} + Y^{2n+1}$ for all $n \geq 1$.

Proposed by Vlad Matei, student, University of Bucharest, Romania.

312. Let p_n be the n-th prime number. Show that the sequence $(x_n)_{n \geq 1}$ defined by

$$x_n = \left\{ \frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_n} \right\} - \{\log \log n\}.$$

is divergent. Here $\{x\}$ denotes the fractional part of the real number x.

Proposed by Cezar Lupu, Polytechnic University of Bucharest, Romania and Cristinel Mortici, Valahia University, Târgovişte, Romania.

313. Does there exist a set M of points in the Euclidian plane such that the distance between any two of them is larger than 1 and such that there is a point in M between any two distinct parallel lines in the plane? Justify your answer.

Proposed by Marius Cavachi, Ovidius University of Constanţa, Romania.

314. Let $f : [0,1] \to \mathbb{R}$ be a C^2 real-valued function on $[0,1]$ which is convex on $[0,1]$. Show that

$$\int_0^1 f(x)\mathrm{d}x \le \frac{1}{4}\left(f(0) + f\left(\frac{1}{3}\right) + f\left(\frac{2}{3}\right) + f(1)\right).$$

Proposed by Tudorel Lupu, Decebal High School of Constanţa, Romania.

315. Let $f : [0;2\pi] \to \mathbb{R}$ such that $\int_0^{2\pi} f(x)\cos kx\,\mathrm{d}x = 1$, for all $k = \overline{1,n}$, where $n \ge 2$ is a fixed positive integer. Find the minimum of $\int_0^{2\pi} f^2(x)\,\mathrm{d}x$, over all such functions f.

Proposed by Vlad Matei, student, University of Bucharest, Romania.

316. Let $f : [0,1] \to [0,\infty)$ and $g : [0,1] \to [0,1]$ be two integrable functions. Prove that for any positive integers $p, q, s, t, p \ne q$, the following inequality holds

$$\int_0^1 f^p(x)g^s(x)\mathrm{d}x \cdot \int_0^1 f^q(x)g^t(x)\mathrm{d}x \le \int_0^1 f^{p+q}(x) \cdot g^{\frac{sp-qt}{p-q}}(x)\mathrm{d}x.$$

Proposed by Andrei Deneanu, student, University of Cambridge, UK and Cezar Lupu, Polytechnic University of Bucharest, Romania.

317. For integer $n \ge 2$, determine the dimension of

$$V = \operatorname{span}\left\{\frac{P(x)}{1 - x^{\deg P+1}}\ ;\ P(x) \in \mathbb{R}[x],\ 0 \le \deg P < n-1, \deg P + 1 \mid n\right\}$$

as a subspace of the linear space $\mathbb{R}(x)$ over $\mathbb{R}$.

Proposed by Dan Schwarz, Bucharest, Romania.

318. Let f be a polynomial with integer coefficients, $\deg(f) \geq 1$ and k a positive integer. Show that there are infinitely many positive integers n such that $f(n)$ can be written in the form $f(n) = d_1 d_2 \dots d_k d_{k+1}$, where $1 \leq d_1 < d_2 < \dots < d_k < n$.

Proposed by Marian Tetiva, Gheorghe Roşca Codreanu National College, Bârlad, Romania.

319. Let μ be the Möbius function defined by $\mu(1) = 1$, $\mu(p_1 p_2 \dots p_k) = = (-1)^k$, for all distinct primes $p_1, p_2, \dots, p_k$ and $\mu(n) = 0$ for any other positive integer n and let $q \geq 1$ and $n \geq 2$. Define

$$N_j = \frac{1}{j} \sum_{\frac{d}{j}} \mu\left(\frac{n}{d}\right) q^d,$$

for all positive integers $j \geq 1$. Show that

$$\sum_{x_1+2x_2+\ldots+nx_n=n, x_i \geq 0} \binom{N_1}{x_1}\binom{N_2}{x_2}\cdots\binom{N_n}{x_n} = q^n - q^{n-1}.$$

Proposed by Gabriel Dospinescu, École Polytechnique, Paris, France, and Marian Tetiva, Gheorghe Roşca Codreanu National College, Bârlad, Romania.

320. For $n > 1$, does there exist a quadratic polynomial $f \in \mathbb{Q}[X]$ such that $f^{2^n} + 1$ is reductible over $\mathbb{Q}$?

Proposed by Gabriel Dospinescu, École Polytechnique, Paris, France and Marian Tetiva, Gheorghe Roşca Codreanu National College, Bârlad, Romania.

321. Find the probability that, by choosing a positive integer n, the numbers $n\sqrt{2}$ and $n\sqrt{3}$ have even integral parts both.

Proposed by Radu Gologan, The Simion Stoilow Institute of Mathematics of the Romanian Academy, Bucharest.

322. Let K be an algebrically closed field and let $P \in K[X_1, \dots, X_n]$, $P = aX_1^{i_1} \cdots X_n^{i_n} + bX_1^{j_1} \cdots X_n^{j_n} + cX_1^{k_1} \cdots X_n^{k_n}$, where $abc \neq 0$. Assume that $X_t \nmid P$ for all t and the points of coordinates $(i_1, \dots, i_n)$, $(j_1, \dots, j_n)$ and $(k_1, \dots, k_n)$ are non-colinear (in $\mathbb{R}^n$).

Prove that P is reducible if and only if $\operatorname{char} K = p$ and $p \mid i_t, j_t, k_t$ for all t for some prime p.

Proposed by Constantin Nicolae Beli, The Simion Stoilow Institute of Mathematics of the Romanian Academy, Bucharest.

SOLUTIONS

285. Prove that for all sequences x_k of positive numbers

$$\sum_{k=1}^{\infty} \frac{x_k}{1+(x_1+x_2+\ldots+x_{k-1})^2} > \frac{\pi}{2}.$$

In addition, if $x_1+x_2+\ldots+x_n=1$, then

$$\sum_{k=1}^{\infty} \frac{x_k}{1+(x_1+x_2+\ldots+x_{k-1})^2} > \frac{\pi}{4}.$$

Moreover these are the best constants.

(Correction)

Proposed by Radu Gologan, Institute of Mathematics of the Romanian Academy, Bucharest.

Solution by Marian Tetiva, Gheorghe Roşca Codreanu High School, Bârlad, Romania. To prove the inequality we denote with f the function defined by $f(x)=1/(1+x^2)$ for all $x\in[0,\infty)$, which, surely, is decreasing on $[0,\infty)$. We have then

$$\sum_{k=1}^{\infty} \frac{x_k}{1+(x_1+x_2+\cdots+x_{k-1})^2} = \sum_{k=1}^{\infty} x_k f(x_1+\cdots+x_{k-1}) >$$

$$> \sum_{k=1}^{\infty} \int_{x_1+\cdots+x_{k-1}}^{x_1+\cdots+x_k} f(x)\mathrm{d}x = \int_0^{\infty} f(x)\mathrm{d}x = \frac{\pi}{2}.$$

($x_1+\cdots+x_{k-1}$ is considered to be 0 for $k=1$).

We note that

$$\sum_{k=1}^{\infty} \frac{x_k}{1+(x_1+x_2+\cdots+x_{k-1})^2} - \frac{\pi}{2} =$$

$$= \sum_{k=1}^{\infty} x_k f(x_1+\cdots+x_{k-1}) - \sum_{k=1}^{\infty} \int_{x_1+\cdots+x_{k-1}}^{x_1+\cdots+x_k} f(x)\mathrm{d}x <$$

$$< \sum_{k=1}^{\infty} x_k(f(x_1+\cdots+x_{k-1}) - f(x_1+\cdots+x_k)) \le \sum_{k=1}^{\infty} x_k^2$$

(we used again the monotony of function f, and for the last inequality we used the fact that $f(u)-f(v)\le v-u$ for all $0\le u\le v$ - it is proven easily). It is sufficifient that, given $\varepsilon>0$, to choose $x_k=\frac{a}{k}$, for which $\sum_{k=1}^{\infty} x_k=\infty$

holds and $\sum\limits_{k=1}^{\infty} x_k^2 = \frac{a^2\pi^2}{6} < \varepsilon$ if we pick a so that $0 < a < \sqrt{\frac{6\varepsilon}{\pi}}$; for this choice of x_k, and the inequalities above, we get

$$\sum_{k=1}^{\infty} \frac{x_k}{1+(x_1+x_2+\cdots+x_{k-1})^2} < \frac{\pi}{2} + \varepsilon.$$

Concerning the second part, it seems that there is also some doubt. The inequality follows, but not that $\frac{\pi}{4}$ is the best constant. For example, take $n = 2$, so if we have $x_1 + x_2 = 1$, then

$$\sum_{k=1}^{\infty} \frac{x_k}{1+(x_1+\cdots+x_{k-1})^2} > x_1 + \frac{x_2}{1+x_1^2} > \frac{4}{5} > \frac{\pi}{4}.$$

Indeed, we have (with the notation $x_1 = x$) to prove the inequality

$$x + \frac{1-x}{1+x^2} > \frac{4}{5}$$

for all $x \in (0,1)$. This follows using $0 < 4x(1-x) \leq 1$ and $0 < x < 1$; multiplied term by term, these give $4x^2 - 4x^3 < 1 \Rightarrow 4x^3 - 4x^2 + 1 > 0$. The more we would have $5x^3 - 4x^2 + 1 > 0$, for all $x \in (0,1)$, which is equivalent with the inequality we wanted proved; this shows that $\pi/4$ is the best lower bound for the left member(my intuition tells me that this happens for all n). The correction we make is the following.

Prove that for any sequence (x_k) of pozitive numbers with $\sum\limits_{k=1}^{\infty} x_k = 1$, we have

$$\sum_{k=1}^{\infty} \frac{x_k}{1+(x_1+\cdots+x_{k-1})^2} > \frac{\pi}{4}$$

the constant $\frac{\pi}{4}$ being the best lower bound. The proof is similar to the one above, we use in addition the fact that $\int\limits_0^1 f(x)\mathrm{d}x = \frac{\pi}{4}$ (we use the same notation for f). We have, using the monotony of f,

$$\sum_{k=1}^{\infty} \frac{x_k}{1+(x_1+x_2+\cdots+x_{k-1})^2} = \sum_{k=1}^{\infty} x_k f(x_1+\cdots+x_{k-1}) >$$

$$> \sum_{k=1}^{\infty} \int\limits_{x_1+\cdots+x_{k-1}}^{x_1+\cdots+x_k} f(x)\mathrm{d}x = \int\limits_0^1 f(x)\mathrm{d}x = \frac{\pi}{4}.$$

Then

$$\sum_{k=1}^{\infty} \frac{x_k}{1+(x_1+x_2+\cdots+x_{k-1})^2} - \frac{\pi}{4} =$$

$$= \sum_{k=1}^{\infty} x_k f(x_1 + \cdots + x_{k-1}) - \sum_{k=1}^{\infty} \int\limits_{x_1+\cdots+x_{k-1}}^{x_1+\cdots+x_k} f(x)\mathrm{d}x <$$

$$< \sum_{k=1}^{\infty} x_k \left(f(x_1 + \cdots + x_{k-1}) - f\left(x_1 + \cdots + x_k\right)\right) \le \sum_{k=1}^{\infty} x_k^2$$

and we can make this difference small enough (smaller then $\varepsilon > 0$ given) if we choose

$$x_k = \frac{1}{p}\left(1 - \frac{1}{p}\right)^{k-1},$$

with p large enough. Indeed, we would have $\sum\limits_{k=1}^{\infty} x_k = 1$ şi

$$\sum_{k=1}^{\infty} x_k^2 = \frac{1}{2p-1} < \varepsilon$$

if we pick $p > (1+\varepsilon)/(2\varepsilon)$ (and $p > 1$, of course). With these the proof ends. □

Also solved by the author.

286. Let $f : (0, \infty) \to (0, \infty)$ be a decreasing and differentiable function on $(0, \infty)$ and F be its antiderivative. Assume that the following hold:

i) the sequence $\left(\dfrac{f(n+1)}{f(n)}\right)_{n\ge1}$ has limit 1;

ii) the sequence $(F_n)_{n\ge1})$ converges to 0;

iii) the function $\frac{f'}{f}$ is increasing on the interval $(0, \infty)$.

Show that:

a) the sequence $(x_n)_{n\ge1}$ defined by

$$x_n = f(1) + f(2) + \ldots + f(n)$$

converges to x, where x is a real number.

b) the sequence $(u_n)_{n\ge1}$ defined by

$$u_n = \frac{x - x_n}{F(n)}$$

is strictly monoton and convergent. Find its limit.

Proposed by Marian Tetiva, Gheorghe Roşca Codreanu High School, Bârlad, Romania.

Solution by the author. a) We will prove that actually the sequence $(t_n)_{n\ge1}$ defined by

$$t_n = x_n - F(n) = f(1) + f(2) + \ldots + f(n) - F(n), \quad \forall\, n \ge 1,$$

is convergent. To prove we make us of the inequalities

$$f(n+1) < F(n+1) - F(n) < f(n), \quad \forall\, n \ge 1$$

(these can be proven using Lagrange's theorem for the function F and the monotony of f). We have

$$t_{n+1} - t_n = f(n+1) - (F(n+1) - F(n)) < 0, \quad \forall\, n \geq 1,$$

according to the first inequality, so the sequence $(t_n)_{n\geq 1}$ is decreasing. On the other hand summing up the inequalities

$$\begin{array}{c} F(2) - F(1) < f(1) \\ F(3) - F(2) < f(2) \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ F(n) - F(n-1) < f(n-1) \\ F(n+1) - F(n) < f(n) \end{array}$$

we obtain $F(n+1) - F(1) < f(1) + f(2) + \ldots + f(n) \Leftrightarrow t_n > F(n+1) - -F(n) - F(1) > -F(1)$, $\forall n \geq 1$, so the sequence $(t_n)_{n\geq 1}$ is lower bounded (we also used the montony of function F: since $F' = f > 0$, F is increasing on $(0, \infty)$), therefore convergent.

To wrapp up, since the limit of the sequence $(F(n))_{n\geq 1}$ is 0, we have that $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} (t_n + F(n)) = \lim\limits_{n\to\infty} t_n$, thus it exists and it is finite (we denote it by x).

b) For the second part we will begin by proving the inequalities

$$x_n - \frac{F(n)f(n+1)}{F(n+1) - F(n)} < x < x_n - \frac{F(n+1)f(n)}{F(n+1) - F(n)}$$

hold for all natural numbers $n \geq 1$.

Let us note that since F is increasing, the sequence $(F_n)_{n\geq 1}$ is also increasing; because it tends to 0 it follows that all it's terms are negative $F(n) < 0$ for all $n \geq 1$ natural number. Next

$$\lim_{n\to\infty} \frac{f(n+1)}{F(n+1) - F(n)} = \lim_{n\to\infty} \frac{f(n)}{F(n+1) - F(n)} = 1;$$

indeed, for all n there is $c_n \in (n, n{+}1)$ such that $F(n{+}1){-}F(n) = f(c_n)$ (from Lagrange's theorem), so, taking into account the monotony of the function f, we obtain $f(n+1) < F(n+1) - F(n) < f(n)$, even further

$$\frac{f(n+1)}{f(n)} < \frac{f(n+1)}{F(n+1) - F(n)} < 1, \quad \forall\, n \geq 1$$

and

$$1 < \frac{f(n)}{F(n+1) - F(n)} < \frac{f(n)}{f(n+1)}, \quad \forall n \geq 1.$$

Using the hypothesis i) the proof of the limits is done. This relations show that the sequences $(y_n)_{n\geq 1}$ and $(z_n)_{n\geq 1}$ defined as

$$y_n = x_n - \frac{F(n)f(n+1)}{F(n+1) - F(n)}, \quad z_n = x_n - \frac{F(n+1)f(n)}{F(n+1) - F(n)}, \quad \forall\, n \geq 1$$

have the limit x (making use of hypothesis ii)); if we could show that $(y_n)_{n\geq 1}$ is increasing, and $(z_n)_{n\geq 1}$ is decreasing, the inequalities $y_n < x < z_n$, $\forall\, n \geq 1$ would follow.

Firstly, let us show, that the sequence $(y_n)_{n\geq 1}$ is increasing (which proves the left inequality). We have that

$$y_{n+1} > y_n \Leftrightarrow x_{n+1} - \frac{F(n+1)f(n+2)}{F(n+2)-F(n+1)} > x_n - \frac{F(n)f(n+1)}{F(n+1)-F(n)}$$

inequality which is equivalent further with

$$f(n+1) + \frac{F(n)f(n+1)}{F(n+1)-F(n)} > \frac{F(n+1)f(n+2)}{F(n+2)-F(n+1)}$$

and we rewrite as

$$\frac{F(n+1)f(n+1)}{F(n+1)-F(n)} > \frac{F(n+1)f(n+2)}{F(n+2)-F(n+1)}.$$

Since we showed that $F(n+1) < 0$, it should suffice to prove

$$\frac{f(n+1)}{F(n+1)-F(n)} < \frac{f(n+2)}{F(n+2)-F(n+1)}$$

holds for all natural numbers $n \geq 1$. This would result if we knew the monotony of the function

$$g : (0;\infty) \to (0;\infty), \quad g(x) = \frac{f(x+1)}{F(x+1)-F(x)}, \quad \forall\, x > 0$$

morever if it is increasing. We note that

$$g'(x) = \frac{f'(x+1)(F(x+1)-F(x)) - f(x+1)(F'(x+1)-F(x))}{(F(x+1)-F(x))^2} =$$

$$= \frac{f'(x+1)(F(x+1)-F(x)) - F'(x+1)(f(x+1)-f(x))}{(F(x+1)-F(x))^2}$$

for all $x > 0$, which is equivalent to

$$\frac{f'(x+1)}{F'(x+1)} > \frac{f(x+1)-f(x)}{F(x+1)-F(x)}, \quad \forall\, x > 0.$$

This we can show using applying *Cauchy*'s theorem to the functions f and F; according to this, there is $\alpha \in (x, x+1)$ such that

$$\frac{f(x+1)-f(x)}{F(x+1)-F(x)} = \frac{f'(\alpha)}{F'(\alpha)}.$$

To end this assertion we make use of the hypothesis iii): since $\alpha < x+1$ and the function $\frac{f'}{f} = \frac{f'}{F'}$ is incresing, we have

$$\frac{f(x+1)-f(x)}{F(x+1)-F(x)} = \frac{f'(\alpha)}{F'(\alpha)} < \frac{f'(x+1)}{F'(x+1)}$$

which is exactly what we needed to show.

Secondly we must prove $(z_n)_{n\geq 1}$ is decreasing; we have:

$$z_{n+1} > z_n \Leftrightarrow x_{n+1} - \frac{F(n+2)f(n+1)}{F(n+2)-F(n+1)} < x_n - \frac{F(n+1)f(n)}{F(n+1)-F(n)} \Leftrightarrow$$

$$\Leftrightarrow f(n+1) - \frac{F(n+2)f(n+1)}{F(n+2)-F(n+1)} < -\frac{F(n+1)f(n)}{F(n+1)-F(n)} \Leftrightarrow$$

$$\Leftrightarrow -\frac{F(n+1)f(n+1)}{F(n+2)-F(n+1)} < -\frac{F(n+1)f(n)}{F(n+1)-F(n)}$$

we divide by $-F(n+1) > 0$ and it remains to show

$$\frac{f(n+1)}{F(n+2)-F(n+1)} < \frac{f(n)}{F(n+1)-F(n)},$$

which would be obtained that the function

$$h : (0;\infty) \to (0;\infty), \quad h(x) = \frac{f(x)}{F(x+1)-F(x)}, \quad \forall\, x > 0$$

is decreasing on $(0,\infty)$. Proceeding as above we have that

$$h'(x) < 0 \Leftrightarrow f'(x)(F(x+1)-F(x)) < f(x)(F'(x+1)-F(x)) \Leftrightarrow$$

$$\Leftrightarrow \frac{f'(x)}{F'(x)} < \frac{f(x+1)-f(x)}{F(x+1)-F(x)}, \;\; \forall x > 0.$$

Similarly this inequality like above (using *Cauchy* and the hypothesis iii):

$$\frac{f(x+1)-f(x)}{F(x+1)-F(x)} = \frac{f'(\alpha)}{F'(\alpha)} > \frac{f'(x)}{F'(x)}.$$

Thus the inequalities

$$x_n - \frac{F(n)f(n+1)}{F(n+1)-F(n)} < x < x_n - \frac{F(n+1)f(n)}{F(n+1)-F(n)}$$

are completely proved. We can rewrite them as

$$-\frac{f(n)}{F(n+1)-F(n)} \cdot \frac{F(n+1)}{F(n)} < \frac{x-x_n}{F(n)} < -\frac{f(n+1)}{F(n+1)-F(n)}.$$

As we have shown, we have

$$\lim_{n\to\infty} \frac{f(n+1)}{F(n+1)-F(n)} = \lim_{n\to\infty} \frac{f(n)}{F(n+1)-F(n)} = 1$$

and it follows

$$\lim_{n\to\infty} \frac{F(n+2)-F(n+1)}{F(n+1)-F(n)} =$$

$$= \lim_{n\to\infty} \frac{F(n+2)-F(n+1)}{f(n+1)} \cdot \frac{f(n)}{F(n+1)-F(n)} \cdot \frac{f(n+1)}{f(n)} = 1$$

so according to *Cesaro-Stolz*'s theorem (case $\frac{0}{0}$), we have that $\lim\limits_{n\to\infty} \frac{F(n+1)}{F(n)} = 1$.

It remains to prove the monotony of this sequence. Summing up all the information, we can conclude that the sequence $\left(\frac{x-x_n}{F(n)}\right)_{n\geq 1}$ is convergent and morever $\lim\limits_{n\to\infty} \frac{x-x_n}{F(n)} = -1$.

It remains to prove the monotony of this sequence. For this let us rewrite the inequality $x > x_n - \frac{F(n)f(n+1)}{F(n+1)-F(n)}$ as

$$(F(n+1)-F(n))x > (F(n+1)-F(n))x_n - F(n)f(n+1)$$

and further (since $f(n+1) = x_{n+1} - x_n$),

$$(F(n+1)-F(n))x > F(n+1)x_n - F(n)x_{n+1} \Leftrightarrow$$

$$\Leftrightarrow F(n+1)(x-x_n) > F(n)(x-x_{n+1}).$$

We divide it by $F(n)F(n+1) > 0$ to obtain

$$\frac{x-x_n}{F(n)} > \frac{x-x_{n+1}}{F(n+1)}, \quad \forall\, n \geq 1,$$

which is exactly the fact that the sequence asked is decreasing.

287. Let $f : [a,b] \to \mathbb{R}$ be a continuous function such that

$$\int_a^t f(x)\mathrm{d}x \int_b^t f(x)\mathrm{d}x \neq 0,\ t \in (a,b).$$

Show that there exists $c \in (a,b)$ such that

$$(c-a)f(c)\left(\int_c^b f(x)\mathrm{d}x - \int_a^c f(x)\mathrm{d}x\right) = \frac{1}{2}\int_a^c f(x)\mathrm{d}x \int_c^b f(x)\mathrm{d}x.$$

`Proposed by Cezar Lupu, student University of Bucharest, Romania`, and `Tudorel Lupu, Decebal High School, Constanţa, Romania`.

Solution by the authors. We have the following.

Lemma. *Let $g : [a,b] \to \mathbb{R}$ be a differentiable function such that $g'(a) = g'(b)$. Then there exists $c \in (a,b)$ such that*

$$g'(c) = \frac{g(c)-g(a)}{c-a}.$$

A proof of the above lemma can be found in [1]. Now, returning to our solution, let us define the following auxiliary function $\phi : [a,b] \to \mathbb{R}$ given by

$$\phi(t) = \left(\int_a^t f(x)\mathrm{d}x \int_t^b f(x)\mathrm{d}x\right)^2, \forall t \in [a,b].$$

A short computation of the derivatives shows that

$$\phi'(t) = 2\int_a^t f(x)dx \int_t^b f(x)\mathrm{d}x f(t)\left(\int_t^b f(x)\mathrm{d}x - \int_a^t f(x)\mathrm{d}x\right)$$

and it is quite clear that $\phi'(a) = \phi'(b)$. So, applying the lemma for ϕ, there exists $c \in (a, b)$ such that

$$2\int_a^c f(x)\mathrm{d}x \int_c^b f(x)\mathrm{d}x f(c)\left(\int_c^b f(x)\mathrm{d}x - \int_a^c f(x)\mathrm{d}x\right)(c-a) =$$

$$= \left(\int_a^c f(x)\mathrm{d}x \cdot \int_c^b f(x)\mathrm{d}x\right)^2.$$

Simplyfing by $\int_a^c f(x)\mathrm{d}x \cdot \int_c^b f(x)dx \neq 0$, we obtain our conclusion. □

References

[1] T.M. Flett, *A mean value theorem*, The Mathematical Gazette 42(1958), 38–39.

Second solution by Marian Tetiva, Gh. Roşca Codreanu High School, Bârlad, Romania. Let F the primitive of f which is null in a, explicitily the function defined by

$$F(t) = \int_a^t f(x)\mathrm{d}x$$

for all $t \in [a, b]$.

We know that F is differentiable on $[a, b]$ with $F'(t) = f(t), \forall\, t \in [a, b]$ and that

$$\lim_{t\to a} \frac{F(t)}{t-a} = f(a),$$

from which

$$\lim_{t\to\infty} \frac{F(t)}{\sqrt{t-a}} = 0.$$

We denote with $A = \int_a^b f(x)\mathrm{d}x$, so we have that

$$\int_t^b f(x)\mathrm{d}x = A - \int_a^t f(x)\mathrm{d}x = A - F(t),$$

for all $t \in [a, b]$ and we consider the function G defined on $[a, b]$ by

$$G(t) = \frac{F(t)(A - F(t))}{\sqrt{t-a}}$$

for $t \in (a, b]$ and $G(a) = 0$. It follows easily that G is continuous on $[a, b]$ (the continuity in a is obtained from the considerations above) and differentiable on (a, b) with

$$G'(t) = \frac{(t-a)f(t)(A - F(t) - F(t)) - \frac{1}{2}F(t)(A - F(t))}{(t-a)\sqrt{t-a}}$$

for all $t \in (a, b)$.

Morever $G(a) = G(b) = 0$; *Rolle*'s theorem gives us that there is $c \in (a, b)$ such that $G'(c) = 0$, which is equivalent to

$$(c-a)f(c)(A - F(c) - F(c)) = \frac{1}{2}F(c)(A - F(c));$$

this being identical to the identity asked by the problem.□

Remark. The condition given in the hypothesis is superfluous.

Also solved by Marius Olteanu (Râmnicu Vâlcea) and Nicuşor Minculete (Braşov).

288. Let $p, q \geq 2$ integers such that $\gcd(p, q) = 1$. Show that the number $\log_p q$ is transcedental.

Proposed by Adrian Troie, Sfântul Sava National College, Bucharest, Romania.

Solution by the editors. It is easy to see that $\log_p q$ is irrational because otherwise, if $\log_p q = \frac{a}{b}$ with $a, b \geq 1$, then one has $p^a = q^b$ which is in contradiction with the hypothesis $\gcd(p, q) = 1$. Now, we shall use *Gelfond*'s criterion:

Lemma. *If $\alpha > 0$, $\alpha \neq 1$, is algebraic and β is algebraic and irrational, then α^β is transcedental.*

Assume, by contradiction that $\log_p q$ is algebraic. Using the fact that is irrational, it follows from the lemma that $p^{\log_p q} = p$ is transcendental, contradiction. Thus, the number $\log_p q$ is transcedental. □

Also solved by Marius Olteanu (Rîmnicu Vâlcea).

289. Find all curves in the plane such that the angle between the vector radius of the curve and the tangent of the curve in a point is $\alpha \in \left(0, \frac{\pi}{2}\right)$.

Proposed by Adrian Corduneanu, Iaşi, Romania.

Solution by the author. We have that $\alpha = \beta + \gamma$ and we denote with $\mathrm{tg}\alpha = a = \text{const} > 0$. We refer to the point $P(x, y)$ on the curve $(y = y(x))$; $\mathrm{tg}\beta = \frac{y}{x}$ $(x \neq 0)$, $\mathrm{tg}\gamma = y'(x)$, $\mathrm{tg}\gamma = -y'(x)$ and it follows

$$a = \frac{\frac{y}{x} - y'(x)}{1 + \frac{y}{x} \cdot y'(x)} \Rightarrow \quad y' = \frac{-ax + y}{x + ay} \tag{4}$$

This equation can be easily integrated by the substitution $y = xz$ ($z =$ the new unknown function), but this leads to an equation of the type $F(x, z) = $const, from which we cannot gather any information about the curve searched. We will aply the parametric method, letting $x = x(t)$ and $y = y(t)$, obtaining the homogeneus liniar system

$$\dot{x} = x + ay \quad \dot{y} = -ax + y$$

whose matrix $A = \begin{pmatrix} 1 & a \\ -a & 1 \end{pmatrix}$ has eigenvalues are $\lambda_1 = 1 + \mathrm{i}a$, $\lambda_2 = 1 - \mathrm{i}a$. Replacing $x = \alpha_1 \mathrm{e}^{\lambda_1 t}$, $y = \alpha_2 \mathrm{e}^{\lambda_2 t}$ we obtain the solution

$$x = -\frac{\mathrm{i}}{a}\mathrm{e}^{(1+\mathrm{i}a)t} \quad y = \frac{1}{a}\mathrm{e}^{(1+\mathrm{i}a)t}.$$

It follows taking real and imaginary in one of this equalities, that a fundamental system of solutions for the parametric equation is

$$\begin{cases} x = \mathrm{e}^t \sin at \\ y = \mathrm{e}^t \cos at \end{cases} \quad \begin{cases} x = -\mathrm{e}^t \cos at \\ y = \mathrm{e}^t \sin at \end{cases}$$

so the general solution is

$$\begin{cases} x = \mathrm{e}^t (C_1 \sin at - C_2 \cos at) \\ y = \mathrm{e}^t (C_1 \cos at + C_2 \sin at) \end{cases}$$

with C_1, C_2 arbitrary real constants. In a certain classification for the parametric equation we say that the origin $(0, 0)$ is a focal point. We note that $x^2 + y^2 = (C_1^2 + C_2^2)\mathrm{e}^{2t} \to \infty$ as $t \to \infty$, it follows that it is an unstable focal point. Along the line $-ax + y = 0$, in the point of intersections, the tangent at the curve is parallel with Ox ($\dot{y} = 0$), while along the line $x + ay = 0$, the tangent at the curve, is parallel to Oy ($\dot{x} = 0$). Trough each point of the plane, passes a unique such curve.

Remark. Marius Olteanu informed us that in the book *G. Milu, I.P. Iambor* – „Curbe Plane“, Ed. Tehnică, Bucureşti, 1989, such curves are called the logarithm spirals. □

290. Study the convergence of the sequence defined by

$$x_1 = a, 1 + x_n = (1 + x_n)^{x_{n+1}}, n \geq 1.$$

`Proposed by Radu Gologan, The Simion Stoilow Institute of Mathematics of the Romanian Academy, Bucharest`

Solution by the author. The following proof will contain and the fact that the sequence is correctly defined.Let $f : (0,\infty) \to (0,\infty)$ given by $f(x) = \frac{\ln(1+x)}{x}$. It can be proved without any difficulty that is decreasing and bijective.

Let $g : (0,\infty) \to (0,\infty)$ defined by $g(x) = \ln(1+x)$. The recurrence can be rewritten as $x_{n+1} = f^{-1} \circ g(x_n)$, so the sequences $(x_{2n})_n$, $(x_{2n+1})_n$ are monotone sequences, of different monotonies. Since $x_n > 1$ it follows $x_{n+1} < 1$ and thus the boundness of the sequence is proven.

Take $\lim\limits_{n\to\infty} x_{2n} = l$ and $\lim\limits_{n\to\infty} x_{2n+1} = l'$. Taking limits in both sides of recurrence we obtain $1 + l = (1 + l')^l$ and $1 + l' = (1 + l)^{l'}$, and a simple argument leads to $ll' = 1$ and afterwards to $l = l' = 1$.

Thus $\lim\limits_{n\to\infty} x_n = 1$. □

291. For $n, k \in \mathbb{N}$, $n \geq 1$, determine the dimension of the linear space of polynomials in n variables over some field K, of degree at most k, as a subspace of $K[x_1, x_2, \ldots, x_n]$.

`Proposed by Dan Schwarz, Bucharest, Romania.`

Solution by the author. Let us denote by $\dim(n,k)$ the dimension sought after. We claim the answer is $\boxed{\dim(n,k) = \binom{n+k}{n}}$. The proof goes by double induction.

For $n = 1$ it is clear that $\dim(1,k) = k + 1 = \binom{1+k}{1}$.

For $k = 0$ it is trivial that $\dim(n,0) = 1 = \binom{n}{n}$.

Now, the following recurrence relation holds

$$\dim(n,k) = \sum_{i=0}^{k} \dim(n-1,i),$$

by looking at a polynomial in n variables, of degree at most k, as a polynomial in the n^{th} variable, with coefficients in $K[x_1, \ldots, x_{n-1}]$.

Assume $\dim(n-1,k) = \binom{n+k-1}{n-1}$, and also $\dim(n,k-1) = \binom{n+k-1}{n}$, as induction hypothesis; therefore having

$$\dim(n,k) = \dim(n-1,k) + \sum_{i=0}^{k-1} \dim(n-1,i) = \dim(n-1,k) + \dim(n,k-1) =$$

$$= \binom{n+k-1}{n-1} + \binom{n+k-1}{n} = \binom{n+k}{n},$$

thus proving our claim. □

Remarks. If we denote by $N(n,k)$ the number of distinct monomials in n variables, of degree k, we get the relation

$$N(n,k) = \dim(n,k) - \dim(n,k-1) = \dim(n-1,k) = \binom{n+k-1}{n-1}.$$

In particular, $N(n,n) = \dim(n-1,n) = \dim(n,n-1)$ (and as a byproduct, $\dim(n,n) = 2\dim(n,n-1)$, hence even).

Another way to put it is that, when $K = \mathbb{Z}_2$, the total number of such polynomials in n variables, of degree at most k, is $2^{\binom{n+k}{n}}$, the total number of the polynomials of degree exactly k is $2^{\binom{n+k}{n}} - 2^{\binom{n+k-1}{n}}$, while the number of the homogeneous polynomials of degree k is $2^{\binom{n+k-1}{n-1}} - 1$.

Solution by Marian Tetiva, Gh. Roşca Codreanu High School, Bârlad, Romania. The monomials which form a base of the subspaces of polynomials of degree at most k from $K[x_1, x_2, \ldots, x_n]$ are the monomials of degree at most k from the expansion of the product of formal series

$$\prod_{i=1}^{n}(1 + x_i + x_i^2 + \cdots).$$

Therefore the dimension required is the sum of the coefficients of x at power at most k from the expansion of

$$(1 + x + x^2 + \cdots)^n.$$

However this expansion is equal to

$$(1 + x + x^2 + \cdots)^n = \sum_{j\geq 0}\binom{n+j-1}{j}x^j$$

so the dimension of the subspace $K[x_1, x_2, \ldots, x_n]$ formed by polynomials of degree at most k eis

$$\sum_{j=0}^{k}\binom{n+j-1}{j} = \binom{n+k}{k}.$$

For example, the monomials $1, x_1, x_2, \ldots, x_n$ (in number of $n+1 = \binom{n+1}{1}$) form a base of the subspace of polynomials of degree ≤ 1, and the same monomials togheter with $x_1^2, x_2^2, \ldots, x_n^2$ and with $x_1x_2, x_1x_3, \ldots, x_{n-1}x_n$ (in total $n+1+n+n(n-1)/2 = (n+1)(n+2)/2 = \binom{n+2}{2}$) form a base of the of the subspace of polynomials of degree at most 2.

Remark. The equality

$$(1 + x + x^2 + \cdots)^n = \sum_{j\geq 0}\binom{n+j-1}{j}x^j$$

can be proved, for example, by induction. The step of induction is made using the identity

$$\sum_{j=0}^{k}\binom{n+j-1}{j} = \binom{n+k}{k}$$

which is done easily if we note that we can transform it in a telescopic sum; indeed:

$$\binom{n+j-1}{j} = \binom{n+j}{j} - \binom{n+j-1}{j-1}$$

(so, fundamentally, both identities are based on the recurrence formula of binomial coefficients). Another proof of the first identity is based on the definition of binomials with repetition (look up, for example, problem 1. 23 from „Probleme de combinatorică şi teoria grafurilor" by *Ioan Tomescu*); of course, in essence, the same proof. $\square$

292. For $n \geq 1$ positive integer, $I \subseteq \mathbb{R}$ is an interval and define the function $f : I \to \mathbb{R}$.

a) Assume that f is $n-1$ times differentiable on I and the derivative $f^{(n-1)}$ is increasing on I. Show that

$$f(b) - \binom{n}{1} f\left(\frac{(n-1)b+a}{n}\right) + \binom{n}{2} f\left(\frac{(n-2)b+2a}{n}\right) - \ldots + (-1)^n f(a) \geq 0,$$

for all $a \leq b, a, b \in I$. (For n even the inequality holds for all $a, b \in I$);

b) Assume that f is n times differentiable and $f^{(n)}$ is continuous on I. If the following inequality

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} f\left(\frac{(n-k)b+ka}{n}\right) \geq 0$$

holds for all $a \leq b, a, b \in I$, then $f^{(n)}$ is nonnegative on I.

Proposed by Marian Tetiva, Gh. Roşca Codreanu High School, Bârlad, Romania.

Solution by the author. a) The proof goes by induction on n. For $n = 1$ (we make the convention that the derivative of order 0 of f is f), the hypothesis tells us that f is increasing on I and the inequality to be proved is $f(b) - f(a) \geq 0$, for all $b \geq a$, obvious.

Assume that we have proved the inequality stated in the problem for all functions $n-2$ times differentiable on I with the derivative of order $n-2$ increasing, and let $f : I \to \mathbb{R}$ a function $n-1$ times differentiable, with $f^{(n-1)}$ increasing on I; this means, that we can apply the hypothesis to f' the derivative of f, the hypothesis of induction.

Let $u : I \cap [a, \infty) \to \mathbb{R}$ defined by

$$u(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f\left(\frac{(n-k)x+ka}{n}\right),$$

for all $x \geq a$; u is certainly differentiable on I and morever

$$u'(x) = \sum_{k=0}^{n-1} (-1)^k \frac{n-k}{n} \binom{n}{k} f'\left(\frac{(n-k)x+ka}{n}\right) =$$

$$= \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} f'\left(\frac{(n-k)x+ka}{n}\right) \geq 0\,,\ \ \forall x \in I\,,\ \ x \geq a.$$

To obtain the inequality stated, we applied the hypothesis of induction to the derivative of f; the numbers a and b from the hypothesis must be replaced with $\frac{x+(n-1)a}{n}$ and respectively with x(we have $\frac{x+(n-1)a}{n} \leq x \Leftrightarrow a \leq x$). Thus we obtain that the derivative of u is nonnegative pe $I \cap [a, \infty)$, so u is increasing on this interval; so $u(x) \geq u(a) = 0$, $\forall x \in I$, $x \geq a$, which is exactly the inequality we wanted to prove(with x instead of b) and the first part ends. We note that the inequality for n even, holds no matter of the choice for a, b since it is symmetric.

b) We first prove:

Claim. *Let $I \subset \mathbb{R}$ an interval and $f : I \to \mathbb{R}$ a function n times differentiable on I: then, for all $a < b$ from I, there is a $\zeta \in (a, b)$ such that*

$$\frac{\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k} f\left(\frac{(n-k)b+ka}{n}\right)}{(b-a)^n} = \frac{1}{n^n} f^{(n)}(\zeta).$$

Proof of the claim. We will do it with induction over $n \geq 1$. For $n = 1$ the statement of the claim is nothing but Lagrange's theorem let us assume it true for functions $n-1$ ($n \geq 2$) times differentiable on I and let us prove it for f n times differentiable on I.

Let $a, b \in I$, $a < b$ and define $g, h : [a, b] \to \mathbb{R}$ as:

$$g(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f\left(\frac{(n-k)x+ka}{n}\right) \text{ and } h(x) = (x-a)^n,\ \forall\, x \in [a, b].$$

It is noted easily that f and g are both differentiable on $[a, b]$ and that h does not cancel over (a, b); thus, they fulfill the conditions of Cauchy's theorem on the interval $[a, b]$. According to it, there is an $\xi \in (a, b)$ such that

$$\frac{g(b)-g(a)}{h(b)-h(a)} = \frac{g'(\xi)}{h'(\xi)}$$

which means

$$\frac{\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k} f\left(\frac{(n-k)b+ka}{n}\right)}{(b-a)^n} = \frac{\displaystyle\sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} f'\left(\frac{(n-k)\xi+ka}{n}\right)}{n(\xi-a)^{n-1}}.$$

We can rewrite as

$$\frac{\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k} f\left(\frac{(n-k)b+ka}{n}\right)}{(b-a)^n} = \frac{\displaystyle\sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} f'\left(\frac{(n-k)\xi+ka}{n}\right)}{\frac{n^n}{(n-1)^{n-1}}\left(\xi - \frac{\xi+(n-1)a}{n}\right)^{n-1}}.$$

If we denote with $R_n(f;a;b)$ the expression in the statement of the lemma we obtained the existence of $\xi \in (a;b)$ such that

$$R_n(f;a;b) = \frac{(n-1)^{n-1}}{n^n} R_{n-1}\left(f';a;\frac{\xi + a(n-1)}{n}\right).$$

We use the hypothesis of induction for the derivative of f which is $n-1$ times differentiable therefore there is $\zeta \in \left(a, \frac{\xi+a(n-1)}{n}\right)$ (which implies $\zeta \in (a,b)$) such that

$$R_{n-1}\left(f';a;\frac{\xi + a(n-1)}{n}\right) = \frac{1}{(n-1)^{n-1}}(f')^{(n-1)}(\zeta) = \frac{1}{(n-1)^{n-1}} f^{(n)}(\zeta)$$

and the conclusion follows.

Now to finish the proof of part b). According to the hypothesis and the claim, we can find for any $a < b$ from I, a $\zeta \in (a,b)$ which satisfies:

$$\frac{1}{n^n} f^{(n)}(\zeta) = \frac{\displaystyle\sum_{k=0}^{n}(-1)^k \binom{n}{k} f\left(\frac{(n-k)b + ka}{n}\right)}{(b-a)^n} \geq 0$$

thus it we also have $f^{(n)}(\zeta) \geq 0$. In other words, the set of points in which $f^{(n)}$ takes nonnegative values is dense in the interval I; that this means that any point $t \in I$ is the limit of a sequence $(t_p)_{p\geq 1}$ of points from this set ($f(t_p) \geq 0$, $\forall p \geq 1$). Using the continuity of the n-th derivative of f we obtain

$$f^{(n)}(t) = \lim_{p\to\infty} f^{(n)}(t_p) \geq 0$$

since all the terms of the sequence are nonnegative; however t was chosen arbitrarily in I, the problem is solved. □

293. Prove that for any continuous function $f : [-1,1] \to \mathbb{R}$ the following inequality is valid:

$$\int_{-1}^{1} f(x)\mathrm{d}x \geq \frac{5}{2}\left(\int_{-1}^{1} x^2 f(x)\mathrm{d}x\right)^2 + \frac{3}{2}\left(\int_{-1}^{1} x f(x)\mathrm{d}x\right)^2.$$

Proposed by Cezar Lupu, student, University of Bucharest and Tudorel Lupu, Decebal High School, Constanţa, Romania.

First solution given by the authors. In what follows we shall prove an inequality in a linear space $\mathcal{V}$ endowed with the inner product $(\cdot,\cdot)$. This is basically a *Cauchy-Schwarz*-type inequality for three vectors.

Lemma. *For any vectors $u, v, w \in \mathcal{V}$ the following inequality holds*

$$||u||^2|(v,w)|^2 + ||v||^2|(w,u)|^2 + ||w||^2|(u,v)|^2 \leq ||u||^2||v||^2||w||^2 +$$

$$+2|(u,v)||(w,v)||(w,u)|. \tag{1}$$

Proof of the lemma. For $u, v, w \in \mathcal{V}$ we consider the *Gram* matrix

$$G(u,v,w) = \begin{pmatrix} ||u||^2 & (u,v) & (u,w) \\ (v,u) & ||v||^2 & (v,w) \\ (w,u) & (w,v) & ||w||^2 \end{pmatrix}.$$

It is a well-known fact that $\det G(u,v,w) \geq 0$, so by computing the determinant by *Sarrus* or triangle rule, we have

$$0 \leq \det G(u,v,w) = ||u||^2||v||^2||w||^2 + 2|(u,v)||(v,w)||(w,u)| -$$
$$-(||u||^2|(v,w)|^2 + ||v||^2|(w,u)|^2 + ||w||^2|(u,v)|^2),$$

which is equivalent to the inequality (1). □

Consider the inner product of two continuous functions $f, g : [a,b] \to \mathbb{R}$ given by $(f,g) := \int\limits_a^b f(x)g(x)\mathrm{d}x$.

In our case, let $a = -1$ and $b = 1$ and let us define the continuous functions $g(x) = x$ and $h(x) = x^2$ where $g, h : [-1,1] \to \mathbb{R}$. By the lemma, we have

$$\int\limits_{-1}^{1} f^2(x)\mathrm{d}x \int\limits_{-1}^{1} x^2\mathrm{d}x \int\limits_{-1}^{1} x^4\mathrm{d}x \geq \left(\int\limits_{-1}^{1} x^4\mathrm{d}x\right)\left(\int\limits_{-1}^{1} xf(x)\mathrm{d}x\right)^2 +$$
$$+\left(\int\limits_{-1}^{1} x^5\mathrm{d}x\right)^2 \int\limits_{-1}^{1} f^2(x)\mathrm{d}x + \left(\int\limits_{-1}^{1} x^2f(x)\mathrm{d}x\right)^2 \int\limits_{-1}^{1} x^4\mathrm{d}x$$

which is equivalent to

$$\frac{4}{15}\int\limits_{-1}^{1} f^2(x)\mathrm{d}x \geq \frac{2}{5}\left(\int\limits_{-1}^{1} xf(x)\mathrm{d}x\right)^2 + \frac{2}{3}\left(\int\limits_{-1}^{1} x^2f(x)\mathrm{d}x\right)^2,$$

and thus we finally obtain the desired inequality.□

Second solution by the authors. Our inequality remains true in $L^2[-1,1]$. Indeed, it can be rewritten as

$$||f||^2_{L^2[-1,1]} \geq \left|\left(f; \sqrt{\frac{5}{2}}x^2\right)\right|^2_{L^2[-1,1]} + \left|\left(f; \sqrt{\frac{3}{2}}x\right)\right|^2_{L^2[-1,1]},$$

where $(\cdot,\cdot)_{L^2[-1,1]}$ is the scalar product on $L^2[-1,1]$ and $||\cdot||_{L^2[-1,1]}$ is the norm on $L^2[-1,1]$. Denote $e_1(x) = \sqrt{\frac{5}{2}}x^2$ and $e_2(x) = \sqrt{\frac{3}{2}}x$, where $e_1, e_2 :$

$[-1,1] \to \mathbb{R}$. Now, it is easy to see that e_1, e_2 are orthonormal in $L^2[-1,1]$ and thus by *Bessel*'s inequality the conclusion follows.□

Third solution by Marius Olteanu, Râmnicu Vâlcea, Romania. From *Cauchy-Buniakowski-Schwarz*(denoted by C.B.S) inequality in integral form applied for the function $h : [0,1] \to \mathbb{R}$, Riemann integrable and $u : [0,1] \to \mathbb{R}$, $u(x) = x^2$, the following holds:

$$\int_0^1 h^2(x)\mathrm{d}x \cdot \int_0^1 x^4 \mathrm{d}x \geq \left(\int_0^1 x^2 h(x)\mathrm{d}x \right)^2 \Rightarrow \int_0^1 h^2(x)\mathrm{d}x \geq 5 \left(\int_0^1 x^2 h(x)\mathrm{d}x \right)^2.$$

Similarly, considering the functions $h : [0,1] \to \mathbb{R}$ and $v : [0,1] \to \mathbb{R}$, $v(x) = x$, from the same C.B.S we have:

$$\int_0^1 h^2(x)\mathrm{d}x \cdot \int_0^1 x^2 \mathrm{d}x \geq \left(\int_0^1 x h(x)\mathrm{d}x \right)^2 \Rightarrow \int_0^1 h^2(x)\mathrm{d}x \geq 3 \left(\int_0^1 x h(x)\mathrm{d}x \right)^2.$$

Let, in the following sequel, the continuous functions f_1, f_2, defined on the interval $[-1,1]$ with real values, such that $f_1(x) = \frac{f(x)+f(-x)}{2}$, $f_2(x) = \frac{f(x)-f(-x)}{2}$. It is obvious that f_1 is an even function and that f_2 is an odd function, morever $f = f_1 + f_2$. Also the function $p : [-1,1] \to \mathbb{R}$, $p = f_1 f_2$ is odd and continuous on $[-1,1]$.

It is well known that for an odd function $r : [-1,1] \to \mathbb{R}$ that $\int_{-1}^{1} r(x)\,\mathrm{d}x = 0$.

Thus we have

$$\int_{-1}^1 f^2(x)\mathrm{d}x = \int_{-1}^1 (f_1(x) + f_2(x))^2 \mathrm{d}x = \int_{-1}^1 f_1^2(x)\mathrm{d}x + \int_{-1}^1 f_2^2(x)\mathrm{d}x +$$

$$+2\int_{-1}^1 p(x)\mathrm{d}x = \int_{-1}^1 f_1^2(x)\mathrm{d}x + \int_{-1}^1 f_2^2(x)\mathrm{d}x,$$

since p is odd.

It can be proved without any difficulty, that if s is an odd or even function, s^2 is certainly an even function. We deduce that both f_1^2 and f_2^2 are both even. Another general result we make use of is the following: For an even function $F : [-1,1] \to \mathbb{R}$, *Riemann* integrable we have that $\int_{-1}^{1} F(x)\,\mathrm{d}x = 2\int_{0}^{1} F(x)\,\mathrm{d}x$.

With these we note that

$$\int_{-1}^{1} f_1^2(x)\,\mathrm{d}x = 2\int_{0}^{1} f_1^2(x)\,\mathrm{d}x$$

and

$$\int_{-1}^{1} f_2^2(x)\,\mathrm{d}x = 2\int_{0}^{1} f_2^2(x)\mathrm{d}x; \quad \int_{-1}^{1} f^2(x)\,\mathrm{d}x = 2\int_{0}^{1} f_1^2(x)\,\mathrm{d}x + 2\int_{0}^{1} f_2^2(x)\,\mathrm{d}x.$$

In the rightside

$$\left(\int_{-1}^{1} x^2 f(x)\mathrm{d}x\right)^2 + \left(\int_{-1}^{1} x f(x)\mathrm{d}x\right)^2 =$$

$$= \left(\int_{-1}^{1} x^2(f_1(x)+f_2(x))\mathrm{d}x\right)^2 + \left(\int_{-1}^{1} x(f_1(x)+f_2(x))\mathrm{d}x\right)^2 =$$

$$= \left(\int_{-1}^{1} x^2 f_1(x)\mathrm{d}x + \int_{-1}^{1} x^2 f_2(x)\mathrm{d}x\right)^2 + \left(\int_{-1}^{1} x f_1(x)\mathrm{d}x + \int_{-1}^{1} x f_2(x)\mathrm{d}x\right)^2.$$

If we look at the functions $K : [-1,1] \to \mathbb{R}$, $K(x) = xf_1(x)$ is an odd function, and the function $L : [-1,1] \to \mathbb{R}$, $L(x) = x^2 f_2(x)$ is also odd, and thus

$$\int_{-1}^{1} x f_1(x)\mathrm{d}x = \int_{-1}^{1} x^2 f_2(x)\mathrm{d}x = 0.$$

So we can modify the right term to become

$$\frac{5}{2}\left(\int_{-1}^{1} x^2 f(x)\mathrm{d}x\right)^2 + \frac{3}{2}\left(\int_{-1}^{1} x f(x)\mathrm{d}x\right)^2 =$$

$$= \frac{5}{2}\left(\int_{-1}^{1} x^2 f_1(x)\mathrm{d}x\right)^2 + \frac{3}{2}\left(\int_{-1}^{1} x f_2(x)\mathrm{d}x\right)^2.$$

Denoting by $M(x) = x^2 f_1(x)$, $M : [-1,1] \to \mathbb{R}$ and $N(x) = xf_2(x)$, $N : [-1,1] \to \mathbb{R}$ they are both even functions and based on the result stated above the following are true:

$$\left(\int_{-1}^{1} x^2 f_1(x)\mathrm{d}x\right)^2 = 4\left(\int_{0}^{1} x^2 f_1(x)\mathrm{d}x\right)^2$$

and

$$\left(\int_{-1}^{1} xf_2(x)\mathrm{d}x\right)^2 = 4\left(\int_{0}^{1} xf_2(x)\mathrm{d}x\right)^2;$$

thus

$$\frac{5}{2}\left(\int_{-1}^{1} x^2 f_1(x)\,\mathrm{d}x\right)^2 + \frac{3}{2}\left(\int_{-1}^{1} xf_2(x)\mathrm{d}x\right)^2 =$$

$$= 10\left(\int_{0}^{1} x^2 f_1(x)\,\mathrm{d}x\right)^2 + 6\left(\int_{0}^{1} xf_2(x)\,\mathrm{d}x\right)^2.$$

Finally we return to the inequalities from the beggining. If we take firstly $h(x) = f_1(x)$, we obtain:

$$\int_{0}^{1} f_1^2(x)\,\mathrm{d}x \geq 5\left(\int_{0}^{1} x^2 f_1(x)\,\mathrm{d}x\right)^2.$$

Secondly take $h(x) = f_2(x)$, so $\int\limits_0^1 f_2^2(x)\,\mathrm{d}x \geq 3\left(\int\limits_0^1 xf_2(x)\,\mathrm{d}x\right)^2$.

We summ them up to obtain

$$2\int_{0}^{1} f_1^2(x)\,\mathrm{d}x + 2\int_{0}^{1} f_2^2(x)\,\mathrm{d}x \geq 10\left(\int_{0}^{1} x^2 f_1(x)\,\mathrm{d}x\right)^2 + 6\left(\int_{0}^{1} xf_2(x)\,\mathrm{d}x\right)^2.$$

From the identities obtained trough the solution, the inequality stated in the problem holds.

Remark. It can be proven, following the same steps of proof that the more general inequality holds

$$\int_{-1}^{1} f^{2p}(x)\,\mathrm{d}x \geq \frac{4p+1}{2}\left(\int_{-1}^{1} x^{2p}\cdot f^p(x)\,\mathrm{d}x\right)^2 + \frac{2p+1}{2}\left(\int_{-1}^{1} x^p\cdot f^p(x)\,\mathrm{d}x\right)^2,$$

where $p \in \mathbb{N}^*$ is odd and $f : [-1, 1] \to \mathbb{R}$ is a continuous function.□

294. Let $\mathcal{S}(O, R)$ be the circumscribed sphere of the tetrahedron $[ABCD]$ and r is inscribed radius of the sphere. If x, y, z, t are the normal coordinates of O, then

$$x + y + z + t \leq 4\sqrt{R^2 - 8r^2}.$$

Proposed by Marius Olteanu, Râmnicu Vâlcea, Romania.

Solution by the author. We denote by R_X the radius of the circumscribed circle opposed to the vertex $X \in \{A, B, C, D\}$. Let $x = OO_D$, $y = OO_C$, $z = OO_B$, $t = OO_A$, where O_X is the center of the circumscribed circle denoted above. Then

$x = \sqrt{R^2 - R_D^2}$; $y = \sqrt{R^2 - R_C^2}$; $z = \sqrt{R^2 - R_B^2}$; $t = \sqrt{R^2 - R_A^2}$.

If $x_i > 0$, $y_i > 0$, $x_i \geq y_i$, $i = \overline{1,n}$, $n \in \mathbb{N}^*$, it is known that

$$\sum_{i=0}^{n} \sqrt{x_i^2 - y_i^2} \leq \sqrt{\left(\sum_{i=0}^{n} x_i\right)^2 - \left(\sum_{i=0}^{n} y_i\right)^2} \quad \text{([1], page 9)}$$

We substitute in this $n = 4$, $x_1 = x_2 = x_3 = x_4 = R$; $y_1 = R_D$, $y_2 = R_C$, $y_3 = R_B$, $y_4 = R_A$. Thus we obtain:

$$x + y + z + t \leq \sqrt{16R^2 - (R_A + R_B + R_C + R_D)^2}. \tag{1}$$

According to relation (21), page 101 from [2] the following holds:

$$R_A + R_B + R_C + R_D \geq 8\sqrt{2}r. \tag{2}$$

To summ up, from (1) and (2) we obtain

$$x + y + z + t \leq \sqrt{16R^2 - 128r^2} = 4\sqrt{R^2 - 8r^2}.$$

The equality is attained if and only if $ABCD$ is a regular tetrahedron.

Remark. It is known that in the case of tetrahedrons $ABCD$ with the center of the circumscribed sphere located in it's interior or are right tetrahedrons, according to Kazarinoff's result we have:

$$PA + PB + PC + PD \geq 2\sqrt{2}(x + y + z + t)$$

where $P \in \text{int}(ABCD)$ and x, y, z, t are the normal coordinates of P. In our problem for $P \equiv O$, it follows

$$x + y + z + t \leq R\sqrt{2}. \quad (3).$$

The inequality proposed against this one has the advantage that it holds in any tetrahedron. Also for the tetrahedrons in which (3) holds, we cannot say that the inequality stated in the problem is weaker than it, because assuming the contrary we would have $4\sqrt{R^2 - 8r^2} \geq R\sqrt{2} \Leftrightarrow 14R^2 \geq 128r^2$, an inequality not necessarly true (All we know is that from *Euler*'s inequality, $R \geq 3r$ thus $14R^2 \geq 126r^2$) □

References

[1] Marcel Chiriţă, *Asupra unei inegalităţi*, G. M.- B nr. 1/2007

[2] Marius Olteanu, *Noi rafinări ale inegalităţii lui Durrande în tetraedru*, G.M.A 2/2008