Bull. Math. Soc. Sci. Math. Roumanie
Tome 49(97) No. 3, 2006, 209–221

On the holomorphic flag curvature of complex Finsler spaces

by

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Abstract

This paper is about the holomorphic flag curvature of complex Finsler metrics. For a complex Finsler space (M, F), the holomorphic flag curvature with respect to the Chern complex linear connection on the pull-back tangent bundle is defined. By means of holomorphic curvature, a special approach is devoted to obtain characterization of the holomorphic flag curvature. Finally we deal with holomorphic flag curvature of the generalized Einstein complex Finsler spaces.

Key Words: Complex Finsler spaces, holomorphic flag curvature.
2000 Mathematics Subject Classification: Primary 53B40, Secondary 53C60.

1 Introduction

In the previous papers, [Al1, Al2], we started to study the holomorphic curvature of complex Finsler spaces with respect to the Chern complex linear connection, briefly Chern $(c.l.c)$, on the holomorphic pull-back tangent bundle $\pi^*(T'M)$. Our goal was to determine the conditions in which a complex Finsler metric has constant holomorphic curvature. We solved this problem for a special class of complex Finsler spaces, called by us generalized Einstein, briefly $(g.E.)$.

In complex Finsler geometry, it is systematically used the concept of holomorphic curvature in direction η, briefly holomorphic curvature, [A-P]. But, the holomorphic curvature is not an analogue of the flag curvature from real Finsler geometry. This problem set up the subject of the present paper. Following the ideas from the real case, [BCS], we shall introduce the holomorphic flag curvature of complex Finsler spaces, with respect to Chern $(c.l.c)$ on $\pi^*(T'M)$, (Definition 3.1). We shall see that the holomorphic curvature is a particular form of the holomorphic flag curvature and some special properties of holomorphic flag curvature are obtained, (Propositions 3.1 - 3.3). Moreover, we determine the link between the holomorphic flag curvature and holomorphic curvature of the complex Finsler spaces (Proposition 3.5). A special approach is dedicated to the holomorphic flag curvature of the $(g.E.)$ complex Finsler spaces. As a consequences of the $(g.E.)$ condition we find:

- the holomorphic flag curvature of a $(g.E.)$ complex Finsler space is proportional with holomorphic curvature of same space, (Proposition 4.1);

- some inequalities between two kinds of holomorphic curvatures, (Proposition 4.2);

- a necessarily and sufficient condition that the holomorphic flag curvature to coincide with holomorphic curvature, (Theorem 4.2).

Moreover, we applies the obtained results to some examples.

2 Preliminaries

In the present section we setting the basic notions which are needed; for more information see [A-P, Mu, Al1, Al2].

Let M be a complex manifold, $\dim_C M = n$, and $T'M$ the holomorphic tangent bundle in which as a complex manifold the local coordinates will be denoted by (z^k, η^k). The complexified tangent bundle of $T'M$ is decomposed in $T_C(T'M) = T'(T'M) \oplus T''(T'M)$.

Considering the restriction of the projection to $\widetilde{T'M} = T'M \setminus \{0\}$, for pulling the holomorphic tangent bundle $T'M$ back, we obtain a holomorphic tangent bundle $\pi' : \pi^*(T'M) \longrightarrow \widetilde{T'M}$, called *the pull-back tangent bundle* over the slit $\widetilde{T'M}$. We denote by $\left\{\frac{\partial}{\partial z^k}^*, \frac{\partial}{\partial \overline{z}^k}^*\right\}$, and by $\{dz^{*k}, d\overline{z}^{*k}\}$, the local frame and its dual.

Let $V(T'M) = \ker \pi_* \subset T'(T'M)$ be the vertical bundle, spanned locally by $\{\frac{\partial}{\partial \eta^k}\}$. A complex nonlinear connection, briefly $(c.n.c.)$, determines a supplementary complex subbundle to $V(T'M)$ in $T'(T'M)$, i.e. $T'(T'M) = H(T'M) \oplus V(T'M)$. The adapted frames is $\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}$, where $N_k^j(z,\eta)$ are the coefficients of the $(c.n.c.)$. Further on we shall use the abbreviations $\delta_i = \frac{\delta}{\delta z^i}$, $\dot{\partial}_i = \frac{\partial}{\partial \eta^i}$, $\delta_{\overline{i}} = \frac{\delta}{\delta \overline{z}^i}$, $\dot{\partial}_{\overline{i}} = \frac{\partial}{\partial \overline{\eta}^i}$, and theirs conjugates ([A-P], [Ai], [Mu]).

On $T'M$ let $g_{i\overline{j}} = \frac{\partial^2 L}{\partial \eta^i \partial \overline{\eta}^j}$ be the fundamental metric tensor of a complex Finsler space $(M, F^2 = L)$.

The isomorphism between $\pi^*(T'M)$ and $T'M$ induces an isomorphism of $\pi^*(T_C M)$ and $T_C M$. Thus, $g_{i\overline{j}}$ defines an Hermitian metric structure $\mathcal{G}(z,\eta) := g_{j\overline{k}} dz^{*j} \otimes d\overline{z}^{*k}$on $\pi^*(T_C M)$, with respect to the natural complex structure. Further, the Hermitian metric structure $\mathcal{G}$ on $\pi^*(T'M)$ induces a Hermitian inner product $h(\chi,\gamma) := Re\mathcal{G}(\chi,\overline{\gamma})$ and the angle $\cos(\chi\gamma) = \frac{Re\mathcal{G}(\chi,\overline{\gamma})}{||\chi||||\overline{\gamma}||}$, for any χ, γ the sections on $\pi^*(T'M)$, where $||\chi||^2 = ||\overline{\chi}||^2 = \mathcal{G}(\chi,\overline{\chi})$, see [Al1].

On the other hand, $H(T'M)$ and $\pi^*(T'M)$ are isomorphic. Therefore the structures on $\pi^*(T_C M)$ can be pulled-back to $H(T'M) \oplus \overline{H(T'M)}$. By this isomorphism the natural cobasis dz^{*j} is identified with dz^j. In view of this construction the pull-back tangent bundle $\pi^*(T'M)$ admits a unique complex linear connection ∇, called the Chern (*c.l.c.*), which is metric with respect to $\mathcal{G}$ and of $(1,0)-$ type, [Al1]:

$$\begin{aligned} \omega_j^i(z,\eta) &= L_{jk}^i(z,\eta)dz^k + C_{jk}^i(z,\eta)\delta\eta^k; \\ L_{jk}^i &= g^{\overline{m}i}\frac{\delta g_{j\overline{m}}}{\delta z^k}\ ;\ C_{jk}^i = g^{\overline{m}i}\frac{\partial g_{j\overline{m}}}{\partial\eta^k}. \end{aligned} \tag{1}$$

The Chern (*c.l.c.*) on $\pi^*(T'M)$ determines the Chern-Finsler (*c.n.c.*) on $T'M$, with the coefficients $\overset{CF}{N_k^i}= g^{\overline{m}i}\frac{\partial g_{j\overline{m}}}{\partial z^k}\eta^j$, and its local coefficients of torsion and curvature are

$$\begin{aligned} T_{jk}^i &:= L_{jk}^i - L_{kj}^i\ ; \\ R_{j\overline{h}k}^i &:= -\delta_{\overline{h}}L_{jk}^i - \delta_{\overline{h}}(\overset{CF}{N_k^l})C_{jl}^i\ ;\ \Xi_{j\overline{h}k}^i := -\delta_{\overline{h}}C_{jk}^i = \Xi_{k\overline{h}j}^i; \\ P_{j\overline{h}k}^i &:= -\dot{\partial}_{\overline{h}}L_{jk}^i - \dot{\partial}_{\overline{h}}(\overset{CF}{N_k^l})C_{jl}^i\ ;\ S_{j\overline{h}k}^i := -\dot{\partial}_{\overline{h}}C_{jk}^i = S_{k\overline{h}j}^i. \end{aligned} \tag{2}$$

The Riemann type tensor

$$\mathbf{R}(W,\overline{Z},X,\overline{Y}) := \mathcal{G}(R(X,\overline{Y})W,\overline{Z})$$

has the properties:

$$\begin{aligned} \mathbf{R}(W,\overline{Z},X,\overline{Y}) &= W^i\overline{Z}^jX^k\overline{Y}^hR_{i\overline{j}k\overline{h}};\ \ R_{\overline{j}i\overline{h}k} := R_{i\overline{h}k}^lg_{l\overline{j}}; \\ R_{i\overline{j}k\overline{h}} &= -R_{i\overline{j}\overline{h}k} = \overline{R_{\overline{j}i\overline{h}\overline{k}}} = R_{\overline{j}i\overline{h}k}; \\ \text{If }\ R_{j\overline{h}k}^i &= R_{k\overline{h}j}^i\ \text{ then }\ R_{i\overline{j}k\overline{h}} = R_{k\overline{j}i\overline{h}} = R_{k\overline{h}i\overline{j}}. \end{aligned} \tag{3}$$

According to [A-P] the complex Finsler space (M,F) is *strongly Kähler* iff $T_{jk}^i = 0$, *Kähler* iff $T_{jk}^i\eta^j = 0$ and *weakly Kähler* iff $g_{i\overline{l}}T_{jk}^i\eta^j\overline{\eta}^l = 0$. Note that for a complex Finsler metric which comes from a Hermitian metric on M, so-called *purely Hermitian metric* in [Mu], i.e. $g_{i\overline{j}} = g_{i\overline{j}}(z)$, the three nuances of Kähler spaces coincide, [S].

The holomorphic curvature of F in direction η, with respect to the Chern (*c.l.c.*), is ([A-P])

$$\mathcal{K}_F(z,\eta) := \frac{2R(\eta,\overline{\eta},\eta,\overline{\eta})}{\mathcal{G}^2(\eta,\overline{\eta})} = \frac{2\overline{\eta}^j\eta^kR_{\overline{j}k}}{L^2(z,\eta)}, \tag{4}$$

where η is viewed as local section of $\pi^*(T'M)$, *i.e.* $\eta := \eta^i\frac{\partial}{\partial z^i}^*$. Further on, we shall simply call it holomorphic curvature. It depends both on the position $z \in M$ and the direction η.

From [Al2], we have

Definition 2.1. *The complex Finsler space* (M, F) *is called generalized Einstein if* $R_{\bar{j}k}$ *is proportional to* $t_{k\bar{j}}$, *i.e. if there exists a real valuated function* $K(z, \eta)$, *such that*

$$R_{\bar{j}k} = K(z, \eta) t_{k\bar{j}}, \tag{5}$$

where $R_{\bar{j}k} := R_{i\bar{j}k\bar{h}} \eta^i \overline{\eta}^h = -g_{l\bar{j}} \overset{CF}{\delta_{\bar{h}}} (N^l_k) \overline{\eta}^h$, $t_{k\bar{j}} := L(z, \eta) g_{k\bar{j}} + \eta_k \overline{\eta}_j$, $\eta_k := \frac{\partial L}{\partial \eta^k}$, $\bar{\eta}_j := \frac{\partial L}{\partial \bar{\eta}^j}$.

The main properties of the $(g.E.)$ complex Finsler spaces are collected in:

Theorem 2.1. *Let* (M, F) *be a* $(g.E.)$ *complex Finsler space. Then*

i) $K(z, \eta) = \frac{1}{4} \mathcal{K}_F(z, \eta)$ *and it depends on* z *alone.*

ii) If (M, F) *is connected and weakly Kähler, of complex dimension* ≥ 2, *then it is a space with constant holomorphic curvature.*

iii) If the space is of nonzero constant holomorphic curvature, then F *is weakly Kähler.*

iv) If the space is Kähler of nonzero constant holomorphic curvature, then F *is purely Hermitian. Conversely, a purely Hermitian complex Finsler space, which is Kähler of constant holomorphic curvature, is* $(g.E.)$.

Note that for the particular case of the complex Finsler spaces which are Kähler of nonzero constant holomorphic curvature, the notions of $(g.E.)$ and purely Hermitian spaces coincide.

3 Holomorphic flag curvature

The holomorphic curvature is the corespondent of the holomorphic sectional curvature in complex Finsler geometry. Our goal is to find a notion which to generalize the holomorphic curvature of complex Finsler spaces. By analogy with the naming from the real case [BCS], we shall call it the holomorphic flag curvature and we shall introduce it with respect to the Chern $(c.l.c.)$.

We consider $z \in M$ and $\eta \in T'_z M$, $\eta \neq 0$. A flag is given by the tangent vector field η, called flagpole, and another transversal vector field χ. Let (η, χ) be the flag in which η and χ are viewed as local sections of $\pi^*(T'M)$, *i.e.* $\eta := \eta^i \frac{\partial}{\partial z^i}^*$ and $\chi := \chi^j \frac{\partial}{\partial z^j}^*$.

Definition 3.1. *The holomorphic flag curvature of the complex Finsler metric* F, *along the flag* (η, χ), *is given by*

$$\mathcal{K}_F(z, \eta, \chi) := \frac{R(\eta, \overline{\chi}, \eta, \overline{\chi}) + R(\chi, \overline{\eta}, \chi, \overline{\eta})}{\mathcal{G}(\eta, \overline{\eta}) \mathcal{G}(\chi, \overline{\chi})}, \tag{1}$$

where, $\mathcal{G}(\chi, \overline{\chi}) \neq 0$.

The holomorphic flag curvature $\mathcal{K}_F(z,\eta,\chi)$ depends both on the position $z \in M$ and the flag (η,χ). Following the same steps like in Proposition 2.5.2 from [A-P], p.107, we obtain that if $R^i_{j\bar{h}k} = R^i_{k\bar{h}j}$ then the holomorphic flag curvature completely determines the curvature tensor $R^i_{j\bar{h}k}$. Note that $\mathcal{K}_F(z,\eta,\chi)$ is real valued because $R(\eta,\overline{\chi},\eta,\overline{\gamma}) = \overline{R(\chi,\overline{\eta},\gamma,\overline{\eta})}$.

In particular, choosing $\eta \equiv \chi$, we obtain the holomorphic curvature $\mathcal{K}_F(z,\eta)$ from (4), successfully studied in [A-P]. So, we can say that the holomorphic flag curvature is a generalization of the holomorphic curvature.

Proposition 3.1. *i)* $\mathcal{K}_F(z,\eta,\chi)$ *is real valued;*
ii) $\mathcal{K}_F(z,\frac{\eta}{F},\chi) = \mathcal{K}_F(z,\eta,\chi)$;
iii) $\mathcal{K}_F(z,\alpha\eta,\beta\chi) = \mathcal{K}_F(z,\eta,\chi)$, *for any* $\alpha,\beta \in \mathbf{R}^*$.

In local coordinate, the holomorphic flag curvature of a complex Finsler metric F, along the flag (η,χ) is given by

$$\mathcal{K}_F(z,\eta,\chi) = \frac{\left(\eta^i\overline{\chi}^j\eta^k\overline{\chi}^h + \chi^i\overline{\eta}^j\chi^k\overline{\eta}^h\right)R_{i\bar{j}k\bar{h}}}{L(z,\eta)L(z,\chi)}, \tag{2}$$

where $L(z,\chi) := g_{i\bar{j}}\chi^i\overline{\chi}^j \neq 0$.

As we can see, the holomorphic flag curvature is defined under assumption that $L(z,\chi) \neq 0$. Therefore, it is natural for us to inquire what this assumption involves. The answer comes bellow.

Denoting by φ the angle between directions of η and χ, we can write $\cos\varphi = \frac{Re\mathcal{G}(\eta,\bar{\chi})}{||\eta||||\bar{\chi}||}$, which locally is

$$\cos\varphi = \frac{\eta_i\chi^i + \bar{\eta}_j\bar{\chi}^j}{2\sqrt{L(z,\eta)L(z,\chi)}}, \tag{3}$$

where $\eta_i := g_{i\bar{k}}\bar{\eta}^k$.

We note here that φ is a real valued function depending on z and η, i.e. $\varphi = \varphi(z,\eta)$.

Proposition 3.2. *If* $L(z,\chi) \neq 0$ *for* $\chi = \chi^i\frac{\partial}{\partial z^i}^*$ *sections in* $\pi^*(T'M)$, *then at least one of the relations* $\frac{\partial\chi^i}{\partial\eta^k} \neq 0$, $\frac{\partial\chi^i}{\partial\bar{\eta}^k} \neq 0$ *and* $\frac{\partial\varphi}{\partial\eta^k} \neq 0$ *is true.*

Proof: We suppose by absurd that $\frac{\partial\chi^i}{\partial\eta^k} = \frac{\partial\chi^i}{\partial\bar{\eta}^k} = \frac{\partial\varphi}{\partial\eta^k} = 0$. Writing (3) in the form $2\cos\varphi\sqrt{L(z,\eta)L(z,\chi)} = \eta_i\chi^i + \bar{\eta}_j\bar{\chi}^j$ and deriving this relation with respect to η, we obtain

$$\cos\varphi\sqrt{L(z,\eta)L(z,\chi)}\left(\frac{\eta_k}{L(z,\eta)} + \frac{C_{i\bar{j}k}\chi^i\bar{\chi}^j}{L(z,\chi)}\right) = C_{i\bar{j}k}\chi^i\bar{\eta}^j + g_{k\bar{j}}\bar{\chi}^j.$$

By contraction with η^k, it leads to $\cos\varphi\sqrt{L(z,\eta)L(z,\chi)} = \bar{\eta}_j\bar{\chi}^j$. But, taking into account (3), it results $\bar{\eta}_j\bar{\chi}^j = \eta_i\chi^i$. Deriving the last relation with respect to η, we obtain $C_{il}\chi^i = g_{l\bar{j}}\overline{\chi}^j$. By contraction with η^l, it gives $\bar{\eta}_j\bar{\chi}^j = 0$. Deriving the relation $\bar{\eta}_j\bar{\chi}^j = 0$ with respect to η, it leads to $g_{l\bar{j}}\bar{\chi}^j = 0$. So that $L(z, \chi) = 0$, and this is in contradiction with our assumption. The proof is complete. □

Proposition 3.3. *Let (M, F) be a complex Finsler space. If $L(z, \chi) \neq 0$, then at least one of following statements: i) the coefficients of χ and ii) the angular function φ are non $-$ homogeneous with respect to η is true.*

Proof: We suppose that the coefficients χ^i of the section χ and the angle φ are r $-$homogeneous functions with respect to η. By the same steps like in Proposition 3.2 we obtain $(2r+1)\bar{\eta}_j\bar{\chi}^j + (2r-1)\eta_i\chi^i = 0$. Deriving this relation with respect to η, we have $(2r+1)C_{il}\chi^i + (2r-1)g_{l\bar{j}}\bar{\chi}^j = 0$, which contracted with η^l, leads to $\bar{\eta}_j\bar{\chi}^j = 0$. Like above, it results $L(z, \chi) = 0$, i.e. the contradiction. □

We note that Propositions 3.2 and 3.3 give necessary conditions for the existence of holomorphic flag curvature.

We propose now to express the holomorphic flag curvature by means of the holomorphic curvature. For this, we consider the unitary flag (l, m), where $l := \frac{\eta}{F(z,\eta)}$ and $m := \frac{\chi}{F(z,\chi)}$. l and m are viewed as local sections of $\pi^*(T'M)$. By means of these, we construct the flags (l, S_{lm}) and (l, D_{lm}) of certain flagpole l and of diagonal transversal vectors $S_{lm} = l + m$ and $D_{lm} = l - m$, respectively. By conjugation, $S_{\bar{l}\bar{m}} = \bar{l} + \bar{m}$ and $D_{\bar{l}\bar{m}} = \bar{l} - \bar{m}$.

We denote by φ the angle between the directions of the unitary sections l and m. We have $\cos\varphi = \frac{Re\mathcal{G}(l,\bar{m})}{||l||||\bar{m}||} = Re\mathcal{G}(l, \bar{m})$.

Proposition 3.4. *i)* $\mathcal{G}(S_{lm}, S_{\bar{l}\bar{m}}) = 4\cos^2\frac{\varphi}{2}$;
ii) $\mathcal{G}(D_{lm}, D_{\bar{l}\bar{m}}) = 4\sin^2\frac{\varphi}{2}$.

Proof: We have
i) $\mathcal{G}(S_{lm}, S_{\bar{l}\bar{m}}) = \mathcal{G}(l+m, \bar{l}+\bar{m}) = \mathcal{G}(l, \bar{l}) + \mathcal{G}(l, \bar{m}) + \mathcal{G}(m, \bar{l}) + \mathcal{G}(m, \bar{m})$
$= 2 + 2Re\mathcal{G}(l, \bar{m}) = 2 + 2\cos\varphi = 4\cos^2\frac{\varphi}{2}$;
ii) $\mathcal{G}(D_{lm}, D_{\bar{l}\bar{m}}) = \mathcal{G}(l-m, \bar{l}-\bar{m}) = \mathcal{G}(l, \bar{l}) - \mathcal{G}(l, \bar{m}) - \mathcal{G}(m, \bar{l}) + \mathcal{G}(m, \bar{m})$
$= 2 - 2Re\mathcal{G}(l, \bar{m}) = 2 - 2\cos\varphi = 4\sin^2\frac{\varphi}{2}$. □

By above considerations, we shall prove the following Proposition:

Proposition 3.5. *Let (M, F) be a complex Finsler space. Then*

$$\mathcal{K}_F(z,\eta,\chi) = 2\mathcal{K}_F(z,\eta,S_{lm})\cos^2\frac{\varphi}{2} + 2\mathcal{K}_F(z,\eta,D_{lm})\sin^2\frac{\varphi}{2} - \mathcal{K}_F(z,\eta), \quad (4)$$

where $\mathcal{K}_F(z,\eta,S_{lm})$ and $\mathcal{K}_F(z,\eta,D_{lm})$ are the holomorphic flag curvatures along the flags (η, S_{lm}) and (η, D_{lm}), respectively, and φ is the angle between the directions of the unitary sections l and m.

Proof: Taking into account Proposition 3.1, *iii*) and relation (1), we obtain

$$\mathcal{K}_F(z,\eta,\chi) = \aleph_F(z,l,m) = R(l,\bar{m},l,\bar{m}) + R(m,\bar{l},m,\bar{l}). \tag{5}$$

On the other hand, decomposing

$$R(l,S_{\bar{l}\bar{m}},l,S_{\bar{l}\bar{m}})\ ,\ R(S_{lm},\bar{l},S_{lm},\bar{l})$$

$$\text{and } R(l,D_{\bar{l}\bar{m}},l,D_{\bar{l}\bar{m}}),\ R(D_{lm},\bar{l},D_{lm},\bar{l}),$$

direct calculus gives:

$$R(l,S_{\bar{l}\bar{m}},l,S_{\bar{l}\bar{m}}) + R(S_{lm},\bar{l},S_{lm},\bar{l}) + R(l,D_{\bar{l}\bar{m}},l,D_{\bar{l}\bar{m}}) + R(D_{lm},\bar{l},D_{lm},\bar{l})$$

$$= 4R(l,\bar{l},l,\bar{l}) + 2\left[R(l,\bar{m},l,\bar{m}) + R(m,\bar{l},m,\bar{l})\right]$$

$$= 2\mathcal{K}_F(z,l) + 2\mathcal{K}_F(z,l,m).$$

In view of Definition 3.1 and Proposition 3.1, the last relation becomes

$$4\mathcal{K}_F(z,l,S_{\bar{l}\bar{m}})\cos^2\tfrac{\varphi}{2} + 4\mathcal{K}_F(z,l,D_{\bar{l}\bar{m}})\sin^2\tfrac{\varphi}{2} = 2\mathcal{K}_F(z,\eta) + 2\mathcal{K}_F(z,\eta,\chi),$$

that is (4). □

If χ and η are colinear, i.e. $\chi = \alpha\eta$, $\alpha \in \mathbf{R}^*$, then $\mathcal{K}_F(z,\eta,\chi) = \mathcal{K}_F(z,\eta,\alpha\eta) = \mathcal{K}_F(z,\eta,\eta) = \mathcal{K}_F(z,\eta)$. Conversely, if $\mathcal{K}_F(z,\eta,\chi) \equiv \mathcal{K}_F(z,\eta)$ then, the relation (4), yields

$$\mathcal{K}_F(z,\eta) = \mathcal{K}_F(z,\eta,S_{lm})\cos^2\frac{\varphi}{2} + \mathcal{K}_F(z,\eta,D_{lm})\sin^2\frac{\varphi}{2}. \tag{6}$$

Moreover, by the relation (4), if the holomorphic flag curvature is identically vanishing along any flag then the holomorphic curvature is zero. Conversely, if $\mathcal{K}_F(z,\eta) = 0$ then

$$\mathcal{K}_F(z,\eta,\chi) = \mathcal{K}_F(z,\eta,S_{lm})\cos^2\frac{\varphi}{2} + \mathcal{K}_F(z,\eta,D_{lm})\sin^2\frac{\varphi}{2}. \tag{7}$$

Particularly, if $\varphi = \frac{\pi}{2}$ then, the relation (4), yields

$$\mathcal{K}_F(z,\eta,\chi) = \mathcal{K}_F(z,\eta,S_{lm}) + \mathcal{K}_F(z,\eta,D_{lm}) - \mathcal{K}_F(z,\eta), \tag{8}$$

When the holomorphic flag curvature is a constant, i.e. it has the same constant value for any choice of z and (η,χ), we obtain the following characterization of holomorphic curvature

Proposition 3.6. *Let (M,F) be a complex Finsler space of constant holomorphic flag curvature along any flag (η,χ), i.e. $\mathcal{K}_F(z,\eta,\chi)=c$, $c\in\mathbf{R}$. Then $\mathcal{K}_F(z,\eta)=c$.*

Proof: By (4) and by $\mathcal{K}_F(z,\eta,\chi)=c$, for any flag, we get

$$c=2c\cos^2\tfrac{\varphi}{2}+2c\sin^2\tfrac{\varphi}{2}-\mathcal{K}_F(z,\eta).$$

This relation leads to $\mathcal{K}_F(z,\eta)=c$. □

Some special results for the holomorphic flag curvature will be obtained subsequently, when we study a fruity particular case.

4 The holomorphic flag curvature of $(g.E.)$ spaces

We establish some inequalities between the holomorphic flag curvature and holomorphic curvature of a $(g.E.)$ complex Finsler space. For the beginning, let us to express the holomorphic flag curvature of a $(g.E.)$ complex Finsler space by means of the holomorphic curvature of the same space.

Proposition 4.1. *Let (M,F) be a $(g.E.)$ complex Finsler space. Then*

$$\mathcal{K}_F(z,\eta,\chi)=\frac{\mathcal{K}_F(z)}{L(z,\chi)}\left\{Re\left(C_{\bar{j}\bar{h}}\overline{\chi}^j\overline{\chi}^h\right)+\frac{Re\left[\left(\bar{\eta}_j\overline{\chi}^j\right)^2\right]}{L(z,\eta)}\right\},\tag{1}$$

where $\mathcal{K}_F(z)$ is the holomorphic curvature of (M,F) and $C_{\bar{j}\bar{h}}:=C_{i\bar{j}\bar{h}}\eta^i$.

Proof: Because (M,F) is a $(g.E.)$ complex Finsler space, by from Proposition 3, *iii*) and Proposition 4 from [Al2], we obtain:

$$R_{\bar{j}l\bar{h}k}\eta^l\eta^k=2K(z)\bar{\eta}_j\bar{\eta}_h+C_{\overline{jh}|k|\bar{m}}\eta^k\bar{\eta}^m=2K(z)\bar{\eta}_j\bar{\eta}_h+2K(z)L(z,\eta)C_{\overline{jh}},$$

$$R_{\bar{j}l\bar{h}k}\bar{\eta}^j\bar{\eta}^h=2K(z)\left(L(z,\eta)C_{kl}+\eta_l\eta_k\right).$$

Plugging into (2), it results:

$$\begin{aligned}\mathcal{K}_F(z,\eta,\overline{\chi})&=\tfrac{2K(z)}{L(z,\eta)L(z,\chi)}(\bar{\eta}_j\overline{\chi}^j\bar{\eta}_h\overline{\chi}^h+\eta_l\chi^l\eta_k\chi^k)+\tfrac{2K(z)}{L(z,\chi)}(C_{\overline{jh}}\overline{\chi}^j\overline{\chi}^h+C_{kl}\chi^l\chi^k)\\&=\tfrac{2K(z)}{L(z,\eta)L(z,\chi)}\left[\left(\bar{\eta}_j\overline{\chi}^j\right)^2+\left(\eta_l\chi^l\right)^2\right]+\tfrac{2K(z)}{L(z,\chi)}\left(C_{\overline{jh}}\overline{\chi}^j\overline{\chi}^h+C_{kl}\chi^l\chi^k\right)\\&=\tfrac{4K(z)}{L(z,\eta)L(z,\chi)}Re\left[\left(\bar{\eta}_j\overline{\chi}^j\right)^2\right]+\tfrac{4K(z)}{L(z,\chi)}Re\left(C_{\overline{jh}}\overline{\chi}^j\overline{\chi}^h\right).\end{aligned}$$

But, $K(z) = \frac{1}{4}\mathcal{K}_F(z)$, which together with the last relation gives (1). □

Note that, if $\mathcal{K}_F(z) = 0$, from (1), it results $\mathcal{K}_F(z, \eta, \chi) = 0$.

Example 4.1. *In [Al1] we consider the complex version of Antonelli-Shimada metric on a domain from $\widetilde{T'M}$, $\dim_C M = 2$, such that its metric tensor be nondegenerated*

$$L_{AS}(z, w; \eta, \theta) := e^{2\sigma(z,w)} \left(|\eta|^4 + |\theta|^4\right)^{\frac{1}{2}}, \text{ with } \eta, \theta \neq 0, \tag{2}$$

where $z := z^1$, $w := z^2, \eta := \eta^1$, $\theta := \eta^2$, $\sigma(z, w)$ is a real valued function and $|\eta^i|^2 := \eta^i \bar{\eta}^i$, $\eta^i \in \{\eta, \theta\}$. We show that the (2) metric is not (g.E.) and its holomorphic curvature is $\mathcal{K}_F(z, \eta) = -\frac{4}{L_{AS}} \frac{\partial^2 \sigma}{\partial z^k \partial \overline{z}^h} \eta^k \overline{\eta}^h$, where $z^i \in \{z, w\}, \eta^i \in \{\eta, \theta\}$. If $\frac{\partial^2 \sigma}{\partial z^k \partial \overline{z}^h} = 0$ then the (2) metric is not purely Hermitian or weakly Kähler, but it is (g.E.) with $\mathcal{K}_F(z) = \mathcal{K}_F(z, \eta, \chi) = 0$.

Because $Re\left[\left(\bar{\eta}_j \overline{\chi}^j\right)^2\right] = \left[Re\left(\bar{\eta}_j \overline{\chi}^j\right)\right]^2 - \left[Im\left(\bar{\eta}_j \overline{\chi}^j\right)\right]^2$ and taking into account (3) we deduce

$$Re\left[\left(\bar{\eta}_j \overline{\chi}^j\right)^2\right] = L(z, \eta) L(z, \chi) \cos^2 \varphi - \left[Im\left(\bar{\eta}_j \overline{\chi}^j\right)\right]^2. \tag{3}$$

This relation leads to

Lemma 4.1. *Let (M, F) be a (g.E.) complex Finsler space of nonzero holomorphic curvature. Then*

$$\begin{aligned} \left[Im\left(\bar{\eta}_j \overline{\chi}^j\right)\right]^2 &= L(z, \eta) L(z, \chi) \left(\cos^2 \varphi - \frac{\mathcal{K}_F(z, \eta, \chi)}{\mathcal{K}_F(z)}\right) \\ &\quad + L(z, \eta) Re\left(C_{\bar{j}\bar{h}} \overline{\chi}^j \overline{\chi}^h\right) \end{aligned} \tag{4}$$

Proof: Replacing (3) in (1) we obtain

$$\begin{aligned} \frac{\mathcal{K}_F(z,\eta,\chi)}{\mathcal{K}_F(z)} &= \frac{Re\left(C_{\bar{j}\bar{h}} \overline{\chi}^j \overline{\chi}^h\right)}{L(z,\chi)} + \frac{L(z,\eta) L(z,\chi) \cos^2 \varphi - \left[Im\left(\bar{\eta}_j \overline{\chi}^j\right)\right]^2}{L(z,\eta) L(z,\chi)} \\ &= \frac{Re\left(C_{\bar{j}\bar{h}} \overline{\chi}^j \overline{\chi}^h\right)}{L(z,\chi)} + \cos^2 \varphi - \frac{\left[Im\left(\bar{\eta}_j \overline{\chi}^j\right)\right]^2}{L(z,\eta) L(z,\chi)}. \end{aligned}$$

From here, immediately results (4) relation. □

But, $\left[Im\left(\bar{\eta}_j \overline{\chi}^j\right)\right]^2 \geq 0$ so the (3) relation say that

$$\frac{Re\left(C_{\bar{j}\bar{h}} \overline{\chi}^j \overline{\chi}^h\right)}{L(z, \chi)} + \cos^2 \varphi \geq \frac{\mathcal{K}_F(z, \eta, \chi)}{\mathcal{K}_F(z)}. \tag{5}$$

Proposition 4.2. *Let (M, F) be a (g.E.) complex Finsler space.*

i) If $\mathcal{K}_F(z) > 0$, then

$$\mathcal{K}_F(z,\eta,\chi) \leq \mathcal{K}_F(z)\left[\frac{Re\,(C_{\bar{j}\bar{h}}\overline{\chi}^j\overline{\chi}^h)}{L(z,\chi)} + 1\right]. \tag{6}$$

ii) If $\mathcal{K}_F(z) < 0$, then

$$\mathcal{K}_F(z)\left[\frac{Re\,(C_{\bar{j}\bar{h}}\overline{\chi}^j\overline{\chi}^h)}{L(z,\chi)} + 1\right] \leq \mathcal{K}_F(z,\eta,\chi). \tag{7}$$

Proof: From (5) we have

$$\tfrac{\mathcal{K}_F(z,\eta,\chi)}{\mathcal{K}_F(z)} \leq \tfrac{Re\left(C_{\bar{j}\bar{h}}\overline{\chi}^j\overline{\chi}^h\right)}{L(z,\chi)} + 1,$$

which leads to (6) and (7). □

Example 4.2. *In [Al2] we proved that if (M, F) is (g.E.) complex Finsler space with $\mathcal{K}_F = -4$ then F is Kobayashi metric. Therefore, the holomorphic flag curvature of Kobayashi metric is*

$$\mathcal{K}_{F_K}(z,\eta,\chi) = -\tfrac{4}{L(z,\chi)}\left\{Re\,(C_{\bar{j}\bar{h}}\overline{\chi}^j\overline{\chi}^h) + \tfrac{Re\left[\left(\bar{\eta}_j\overline{\chi}^j\right)^2\right]}{L(z,\eta)}\right\}$$

and

$$\mathcal{K}_{F_K}(z,\eta,\chi) \geq -4\left[\tfrac{Re\left(C_{\bar{j}\bar{h}}\overline{\chi}^j\overline{\chi}^h\right)}{L(z,\chi)} + 1\right].$$

Proposition 4.3. *Let (M, F) be a (g.E.) complex Finsler space. If $\mathcal{K}_F(z) = \mathcal{K}_F(z,\eta,\chi) \neq 0$ then*

$$\frac{Re\,(C_{\bar{j}\bar{h}}\overline{\chi}^j\overline{\chi}^h)}{L(z,\chi)} \geq \sin^2\varphi. \tag{8}$$

The proof follows from (5).

In the remainder of this section, we treat the particular class of the (g.E.) complex Finsler space which is Kähler with nonzero constant holomorphic curvature. Therefore, Proposition 4.1 is reduced to

Corollary 4.1. *Let (M, F) be a (g.E.) complex Finsler space, Kähler with $\mathcal{K}_F(z) = c$, $c \in \mathbf{R}^*$. Then*

$$\mathcal{K}_F(z,\eta,\chi) = \frac{\mathcal{K}_F(z)Re\left[\left(\bar{\eta}_j\overline{\chi}^j\right)^2\right]}{L(z,\eta)L(z,\chi)}. \tag{9}$$

Proof: From Theorem 2.1 *iv)* we have $C_{\bar{j}\bar{h}} = 0$. So that, the relation (1) is transformed in (9). □

Theorem 4.1. *Let (M,F) be a (g.E.) complex Finsler space, Kähler with $\mathcal{K}_F(z) = c,\ c \in \mathbf{R}^*$.*

i) If $c > 0$, then $\mathcal{K}_F(z,\eta,\chi) \leq c$;

ii) If $c < 0$, then $c \leq \mathcal{K}_F(z,\eta,\chi)$.

Proof: From the relations (3) and (9) follows

$$Im(\bar{\eta}_j\overline{\chi}^j) = \pm F(z,\eta)F(z,\chi)\sqrt{\cos^2\varphi - \frac{\mathcal{K}_F(z,\eta,\chi)}{c}}, \tag{10}$$

where $F(z,\chi) = \sqrt{L(z,\chi)}$ and φ is the angle between direction of η and χ.

The (10) relation can be written as

$$\left[Im\left(\bar{\eta}_j\overline{\chi}^j\right)\right]^2 = L(z,\eta)L(z,\chi)\left(\cos^2\varphi - \tfrac{\mathcal{K}_F(z,\eta,\chi)}{c}\right).$$

But, $\left[Im\left(\bar{\eta}_j\overline{\chi}^j\right)\right]^2 \geq 0$, so that $\cos^2\varphi - \frac{\mathcal{K}_F(z,\eta,\chi)}{c} \geq 0$. It results $\frac{\mathcal{K}_F(z,\eta,\chi)}{c} \leq 1$, which proved $i)$ and $ii)$. □

Theorem 4.2. *Let (M,F) be a (g.E.) complex Finsler space Kähler of nonzero holomorphic curvature. Then χ and η are collinear if and only if $\mathcal{K}_F(z,\eta,\chi) = \mathcal{K}_F(z)$.*

Proof: The necessity results from Proposition 3.5, and the converse from (8). □

Corollary 4.2. *Let (M,F) be a (g.E.) complex Finsler space, Kähler with $\mathcal{K}_F(z) = c,\ c \in \mathbf{R}^*$. If $Im(\bar{\eta}_j\overline{\chi}^j) = 0$. Then*

i) $\bar{\eta}_j\overline{\chi}^j$ is real valued;

ii) $\mathcal{K}_F(z,\eta,\chi) = c \cdot \cos^2\varphi$.

Corollary 4.3. *Let (M,F) be a (g.E.) complex Finsler space, Kähler with $\mathcal{K}_F(z) = c,\ c \in \mathbf{R}^*$ and $Re(\bar{\eta}_j\overline{\chi}^j) = 0$. Then $\bar{\eta}_j\overline{\chi}^j = -\eta_j\chi^j$ and $\varphi = \frac{\pi}{2}$;*

Moreover

if $c > 0$, then $\mathcal{K}_F(z,\eta,\chi) \leq 0$;

if $c < 0$, then $0 \leq \mathcal{K}_F(z,\eta,\chi)$.

Example 4.3. *We consider the complex Finsler metrics*

$$L := \frac{|\eta|^2 + \varepsilon(|z|^2|\eta|^2 - <z,\eta> \overline{<z,\eta>})}{(1+\varepsilon|z|^2)^2}, \tag{11}$$

where $|z|^2 := \sum_{k=1}^{n} z^k\overline{z}^k$, $<z,\eta> := \sum_{k=1}^{n} z^k\overline{\eta}^k$, defined over the disk $\Delta_r^n = \left\{z \in \mathbf{C}^n,\ |z| < r,\ r := \sqrt{\frac{1}{|\varepsilon|}}\right\}$ if $\varepsilon < 0$; on $\mathbf{C}^n$ if $\varepsilon = 0$; and on the complex projective space $P^n(\mathbf{C})$ if $\varepsilon > 0$. Particularly, for $\varepsilon = -1$ we obtain the Bergman *metric on the unit disk* $\Delta^n := \Delta^n_1$, *for* $\varepsilon = 0$ *the Euclidian metric on* $\mathbf{C}^n$, *and for* $\varepsilon = 1$ *the Fubini-Study metric on* $P^n(\mathbf{C})$.

The metrics (11) are $(g.E.)$, *Kähler with* $\mathcal{K}_F = 4\varepsilon$. *From Theorem 4.1 we obtain: if* $\varepsilon < 0$ *then* $4\varepsilon \leq \mathcal{K}_F(z,\eta,\chi)$ *and if* $\varepsilon > 0$ *then* $\mathcal{K}_F(z,\eta,\chi) \leq 4\varepsilon$.

The holomorphic flag curvature is important in the study of a corespondent of the holomorphic bisectional curvature from Hermitian geometry in the complex Finsler geometry. But this subject will be in a coming paper.

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Received: 31.10.2005.

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