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Warped product semi-invariant submanifolds of a locally Product Riemannian manifold

by
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Abstract

In this paper, we study warped product semi-invariant submanifolds of a locally product Riemannian manifold. We show that the geometry of such warped products is quiet different from the geometry of warped product CR-submanifolds of Kaehler manifolds [5]. We give a characterization, some examples and establish an inequality for warped product semi-invariant submanifolds. The equality case is also considered.

Key Words: Almost product Riemannian manifold, semi-invariant submanifold, warped product
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1 Introduction

In [3], R. L. Bishop and B. O'Neill introduced a class of warped product manifolds as follows: Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds, $f : M_1 \to (0, \infty)$ and $\pi : M_1 \times M_2 \to M_1$, $\eta : M_1 \times M_2 \to M_2$ the projection maps given by $\pi(p, q) = p$ and $\eta(p, q) = q$ for every $(p, q) \in M_1 \times M_2$. The warped product $M = M_1 \times M_2$ is the manifold $M_1 \times M_2$ equipped with the Riemannian structure such that

$$g(X, Y) = g_1(\pi_* X, \pi_* Y) + (fo\pi)^2 g_2(\eta_* X, \eta_* Y)$$

for every X and Y on M and $*$ is symbol for the tangent map.The function f is called the warping function of the warped product manifold. In particular, if the warping function is constant, then the manifold M is said to be trivial. Let X, Y be vector fields on M_1 and V, W vector fields on M_2, then from Lemma 7.3 of [3], we have

$$\nabla_X V = \nabla_V X = (\frac{Xf}{f})V \tag{1.1}$$

where ∇ is the Levi-Civita connection on M.

On the other hand, CR-submanifolds of Kaehler manifolds were introduced by A. Bejancu [2] as a generalization of totally real submanifolds and holomorphic manifolds. Later, the concept of CR-submanifold has been also considered in various manifolds. In [1] and [11], as analogous of CR-submanifolds, semi-invariant submanifolds of locally product Riemannian manifolds have been studied.

Recently, B.Y. Chen [5] (see also, [6], [7]) studied warped product CR-submanifolds and showed that there exist no warped product CR-submanifolds in the form $M_\perp \times_f M_T$ such that $M_\perp$ is a totally real submanifold and M_T is a holomorphic submanifold of a Kaehler manifold $\bar{M}$. Then he introduced CR-warped product submanifolds as follows: A submanifold M of a Kaehler manifold $\bar{M}$ is called CR-warped product if it is the warped product $M_T \times_f M_\perp$ of a holomorphic submanifold M_T and totally real submanifold $M_\perp$ of $\bar{M}$. He also established general sharp inequalities for CR-warped products in Kaehler manifolds. After the papers of Chen, CR-warped product submanifolds have been studied in many papers [4],[9],[12] and [13].

In this paper, we study warped product semi-invariant submanifolds of a locally product Riemannian manifold. We observe that the study is far from the lines drawn by the study of warped product CR-submanifolds of Kaehler manifolds. For instance, as mentioned before, it is known that there do not exist warped product CR-submanifolds of the form $M_\perp \times_f M_T$ in a Kaehler manifold $\bar{M}$ such that M_T is a holomorphic submanifold and $M_\perp$ is a totally real submanifold of $\bar{M}$. But, we show that there are examples of warped product semi-invariant submanifolds of the form $M_\perp \times_f M_T$ in a locally product manifold $\bar{M}$ such that M_T is an invariant submanifold and $M_\perp$ is an anti-invariant submanifold of $\bar{M}$. We also show that there do not exist warped product semi-invariant submanifolds of the $M_T \times_f M_\perp$ in a locally product manifold $\bar{M}$ such that M_T is an invariant submanifold and $M_\perp$ is an anti-invariant submanifold of $\bar{M}$, certainly this is not true for CR-submanifolds of Kaehler manifolds. In fact, this kind of CR-submanifolds are called CR-warped products [5] and there are many examples of CR-warped products. We also give a simple characterization for warped product semi-invariant submanifolds of the form $M_\perp \times_f M_T$ in a locally product Riemannian manifold and establish an inequality for squared norm of the second fundamental form in terms of the warping function for an arbitrary warped product semi-invariant submanifolds of the form $M_\perp \times_f M_T$ in a locally product manifold $\bar{M}$ such that M_T is an invariant submanifold and $M_\perp$ is an anti-invariant submanifold of $\bar{M}$.

2 Preliminaries

Let $\bar{M}$ be an $(m+n)$-dimensional manifold with a tensor of type $(1,1)$ such that

$$F^2 = I, \tag{2.1}$$

where I denotes the identity transformation. Then we say that $\bar{M}$ is an almost product manifold with almost product structure F. We put

$$P = \frac{1}{2}(I+F),\ Q = \frac{1}{2}(I-F), \tag{2.2}$$

then we have

$$P+Q = I, P^2 = P,\ Q^2 = Q,\ PQ = QP = 0 \tag{2.3}$$

and

$$F = P - Q. \tag{2.4}$$

If an almost product manifold $\bar{M}$ admits a Riemannian metric g such that

$$g(FX, FY) = g(X, Y) \tag{2.5}$$

for any vector fields X and Y on $\bar{M}$, then $\bar{M}$ is called an almost product Riemannian manifold [15]. Let $\bar{\nabla}$ denotes the Levi Civita connection on $\bar{M}$ with respect to g. In particular, if $\bar{\nabla}_X F = 0$, $X \in \Gamma(T\bar{M})$, where $\Gamma(T\bar{M})$ denotes the set of all vector fields of $\bar{M}$, then $\bar{M}$ is called a locally product Riemannian manifold.

Let M be a Riemannian manifold isometrically immersed in $\bar{M}$ and denote by the same symbol g the Riemannian metric induced on M. Let $\Gamma(TM)$ be the Lie algebra of vector fields in M and $\Gamma(TM^{\perp})$ the set of all vector fields normal to M. Denote by ∇ the Levi-Civita connection of M. Then the Gauss and Weingarten formulas are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.6}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N \tag{2.7}$$

for any $X, Y \in \Gamma(TM)$ and any $N \in \Gamma(TM^{\perp})$, where $\nabla^{\perp}$ is the connection in the normal bundle $TM^{\perp}$, h is the second fundamental form of M and A_N is the Weingarten endomorphism associated with N. The second fundamental form and the shape operator A are related by

$$g(A_N X, Y) = g(h(X,Y), N). \tag{2.8}$$

The submanifold M is said to be invariant (or $F-$ invariant) if $FX \in \Gamma(TM)$, for any $X \in \Gamma(TM)$. On the other hand, M is said to be anti -invariant if $FX \in \Gamma(TM^{\perp})$, for any $X \in \Gamma(TM)$ [15]. Let $\bar{M}$ be a locally product Riemannian manifold and M be a submanifold of $\bar{M}$. The submanifold M is called a semi-invariant submanifold [1] if there exists a differentiable distribution $D\ :\ p \rightarrow\ D_p \subset T_pM$

such that D is invariant with respect to F and the complementary distribution $D^{\perp}$ is anti-invariant with respect to F. Thus, we have

$$TM = D \oplus D^{\perp}.$$

Denoting the orthogonal complementary subbundle to $F(D^{\perp})$ in $TM^{\perp}$ by $\mathcal{V}$, then we have

$$TM^{\perp} = F(D^{\perp}) \oplus \mathcal{V}.$$

3 Warped products $M_T \times_f M_{\perp}$ in locally product Riemannian manifolds

In this section, we study semi-invariant submanifolds in a locally product Riemannian manifold $\bar{M}$ which are warped products of the form $M_T \times_f M_{\perp}$, where M_T is an invariant submanifold and $M_{\perp}$ is an anti-invariant submanifold $\bar{M}$.

Theorem 3.1. *Let $\bar{M}$ be a locally product Riemannian manifold. Then there do not exist warped product semi-invariant submanifold in the form $M_T \times_f M_{\perp}$ in $\bar{M}$ such that M_T is an invariant submanifold and $M_{\perp}$ is an anti-invariant submanifold of $\bar{M}$.*

Proof: Let $M = M_T \times_f M_{\perp}$ be a warped product submanifold of a locally product Riemannian manifold $\bar{M}$ such that M_T is an invariant submanifold and $M_{\perp}$ is an anti-invariant submanifold of $\bar{M}$. From (1.1) we have $g(\nabla_X Z, W) = g(\nabla_Z X, W)$ for $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_{\perp})$. Using (2.6) we get

$$g(\nabla_X Z, W) = g(\bar{\nabla}_Z X, W).$$

Then, from (2.5) we obtain $g(\nabla_X Z, W) = g(\bar{\nabla}_Z FX, FW)$. Thus, using(1.1) and (2.6), we have

$$X(lnf)g(Z, W) = g(h(Z, FX), FW) \tag{3.1}$$

for any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_{\perp})$. On the other hand, from (2.6) we have $g(h(Z, FX), FW) = g(\bar{\nabla}_{FX} Z, FW)$. Then (2.5) implies that

$$g(h(Z, FX), FW) = g(\bar{\nabla}_{FX} FZ, W).$$

Thus, using (2.7) we derive

$$g(h(Z, FX), FW) = -g(A_{FZ} FX, W).$$

Then, from (2.8) we arrive at

$$g(h(Z, FX), FW) = -g(h(FX, W), FZ). \tag{3.2}$$

From (3.1) we also have

$$g(h(FX, W), FZ) = X(lnf)g(Z, W) \tag{3.3}$$

for $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\perp)$. Thus, from (3.1), (3.2) and (3.3) we conclude that

$$2X(lnf)g(Z,W) = 0.$$

Since $M_\perp$ is a Riemannian manifold, we get $X(lnf) = 0$ which shows that f is constant on M_T. This implies that M is a usual product manifold. Thus proof is complete. □

Remark. 1. We note that this result shows that the geometry of warped product submanifolds of a locally product Riemannian manifold is very different from the geometry of warped product submanifolds of Kaehler manifolds. Indeed, it is known that there are many examples of warped product CR-submanifolds in the form $M_T \times_f M_\perp$ in a Kaehler manifold such that M_T is a holomorphic submanifold and $M_\perp$ is a totally real submanifold of a Kaehler manifold $\bar{M}$ (For examples of warped product CR-submanifolds of Kaehler manifolds, See:[5], [6], [14]).

4. Warped products $M_\perp \times_f M_T$ in locally product Riemannian manifolds

Theorem 3.1 shows us that there are no warped product semi-invariant submanifolds in the form $M_T \times_f M_\perp$ such that M_T is an invariant submanifold and $M_\perp$ is an anti-invariant submanifold of a locally product Riemannian manifold $\bar{M}$. In this section, we study semi-invariant submanifolds in a locally product Riemannian manifold $\bar{M}$ which are warped products of the form $M_\perp \times_f M_T$, where M_T is an invariant submanifold and $M_\perp$ is an anti-invariant submanifold of $\bar{M}$. First, we are going to give two examples.

Example 4.1. Let us consider the almost product manifold R^4 with coordinates (x_1, x_2, y_1, y_2) and product structure

$$F(\frac{\partial}{\partial x_i}) = -\frac{\partial}{\partial x_i},\ F(\frac{\partial}{\partial y_i}) = \frac{\partial}{\partial y_i}.$$

Then consider a submanifold M given by

$$x_1 = t\cos\theta \quad , \quad x_2 = t\sin\theta,$$
$$y_1 = t\cos\varphi \quad , \quad y_2 = t\sin\varphi.$$

Then, the tangent bundle of M is spanned by

$$\begin{aligned} Z_1 &= \cos\theta\,\partial x_1 + \sin\theta\,\partial x_2 + \cos\varphi\,\partial y_1 + \sin\varphi\,\partial y_2, \\ Z_2 &= -t\sin\theta\,\partial x_1 + t\cos\theta\,\partial x_2, \\ Z_3 &= -t\sin\varphi\,\partial y_1 + t\cos\varphi\,\partial y_2. \end{aligned}$$

It follows that $FZ_2 = -Z_2, FZ_3 = Z_3$ and FZ_1 is orthogonal to TM. Thus M is a semi-invariant submanifold of R^4 with $D = span\{Z_2, Z_3\}$ and $D^\perp = span\{Z_1\}$.

It is easy to check that D is integrable. We denote the integral manifolds of D and $D^{\perp}$ by M_T and $M_{\perp}$, respectively. Then, the metric tensor of M is

$$g = 2\,dt^2 + t^2(d\,\varphi^2 + d\,\theta^2) = g_{M_{\perp}} + t^2 g_{M_T}.$$

Thus M is a warped product semi-invariant submanifold of R^4 with warping function $f = t$.

Example 4.2. Let us consider the almost product manifold R^4 with coordinates (x_1, x_2, x_3, y_1) and product structure

$$F(\frac{\partial}{\partial\,x_i}) = -\frac{\partial}{\partial\,x_i},\; F(\frac{\partial}{\partial\,y_1}) = \frac{\partial}{\partial\,y_1}.$$

Consider a submanifold M given by

$$x_1 = -sin\,\theta\,cos\,\varphi \quad , \quad x_2 = cos\,\theta\,cos\,\varphi$$
$$x_3 = sin\,\varphi \quad , \quad y_1 = \varphi.$$

Then, the tangent bundle of M is spanned by

$$\begin{aligned} Z_1 &= -cos\,\theta\,cos\,\varphi\,\partial\,x_1 - sin\,\theta\,cos\,\varphi\,\partial\,x_2, \\ Z_2 &= sin\,\theta\,sin\,\varphi\,\partial\,x_1 - cos\,\theta\,sin\,\varphi\,\partial\,x_2 \\ &\quad + cos\,\varphi\,\partial\,x_3 + \partial\,y_1. \end{aligned}$$

It is easy to see that $FZ_1 = -Z_1$ and FZ_2 is orthogonal to TM. Thus M is a semi-invariant submanifold of R^4 with $D = span\{Z_1\}$ and $D^{\perp} = span\{Z_2\}$. The integrability of D and $D^{\perp}$ are obvious. We denote the integral manifolds of them by M_T and $M_{\perp}$, respectively. Then the metric tensor is

$$g = 2d\varphi^2 + cos^2\,\varphi d\,\theta^2 = g_{M_{\perp}} + cos^2\,\varphi\,g_{M_T}.$$

Thus M is a warped product semi-invariant submanifold of R^4 with warping function $f = cos\,\varphi$.

Remark 2. We note that this two examples show that it is easy to find examples of warped product semi-invariant submanifolds in the form $M_{\perp} \times_f M_T$. This is also interesting, because we know that there do not exist warped product CR-submanifolds of the form $M_{\perp} \times_f M_T$ in a Kaehler manifold $\bar{M}$ such that M_T is a holomorphic submanifold and $M_{\perp}$ is a totally real submanifold of $\bar{M}$ [5].

From now on, we will consider warped product semi-invariant submanifolds in the form $M_{\perp} \times_f M_T$, where $M_{\perp}$ is an anti invariant submanifold and M_T is an invariant submanifold of a locally product Riemannian manifold$\bar{M}$. We call such warped product semi-invariant submanifolds *semi-invariant warped products.*

Next, we are going to give a simple characterization for a semi-invariant warped product $M = M_{\perp} \times_f M_T$ of a locally product Riemannian manifold

$\bar{M}$. First, we need the following lemmas.

Lemma 4.1. *Let $M = M_\perp \times_f M_T$ be a semi-invariant warped product of a locally product Riemannian manifold $\bar{M}$. Then we have*

$$g(h(X,Z),FW) = 0 \tag{4.1}$$

and

$$g(h(X,Y),FZ) = -Z(lnf)g(FY,X) \tag{4.2}$$

for $X, Y \in \Gamma(D)$ and $Z, W \in \Gamma(D^\perp)$.

Proof: From (2.6), we have $g(h(X,Z),FW) = g(\bar{\nabla}_Z X, FW)$ for $X \in \Gamma(D)$ and $Z, W \in \Gamma(D^\perp)$. Using (2.5) and (2.6), we get

$$g(h(X,Z),FW) = g(\bar{\nabla}_Z FX, W).$$

Thus, (2.6) implies that

$$g(h(X,Z),FW) = g(\nabla_Z FX, W).$$

Then, from (1.1), we derive

$$g(h(X,Z),FW) = Z(lnf)g(FX,W) = 0$$

due to D and $D^\perp$ are orthogonal. In similar way, from (2.6), we get

$$g(h(X,Y),FZ) = g(\bar{\nabla}_X Y, FZ)$$

for $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D^\perp)$. Hence, we have

$$g(h(X,Y),FZ) = -g(Y, \bar{\nabla}_X FZ).$$

Using (2.5) and (2.6) we obtain

$$g(h(X,Y),FZ) = -g(FY, \nabla_X Z).$$

Thus, from (1.1) we have

$$g(h(X,Y),FZ) = -Z(lnf)g(FY,X).$$

□

Lemma 4.2. *Let M be a semi-invariant submanifold of a locally product Riemannian manifold $\bar{M}$. Then, the distribution $D^\perp$ defines a totally geodesic foliation on M if and only if*

$$A_{FW}Z \in \Gamma(D^\perp) \tag{4.3}$$

for $Z, W \in \Gamma(D^\perp)$.

Proof: For $Z, W \in \Gamma(D^{\perp})$ and $X \in \Gamma(D)$, from (2.5), (2.6) and (2.7) we have $g(\nabla_Z W, X) = -g(A_{FW}Z, FX)$ which proves the assertion. □

Lemma 4.3. *Let M be a semi-invariant submanifold of a locally Riemannian product manifold $\bar{M}$. Then we have*

$$g([X,Y],Z) = g(A_{FZ}X, FY) - g(A_{FZ}Y, FX) \tag{4.4}$$

for $Z \in \Gamma(D^{\perp})$ and $X, Y \in \Gamma(D)$.

Proof: From (2.5) and (2.6) we have

$$g([X,Y],Z) = g(\bar{\nabla}_X FY, FZ) - g(\bar{\nabla}_Y FX, FZ)$$

for $Z, W \in \Gamma(D^{\perp})$ and $X \in \Gamma(D)$. Then, using (2.6) and (2.8) we obtain (4.4). □

Now, we are ready to prove a characterization theorem for a semi-invariant local warped product $M_{\perp} \times_f M_T$ of a locally product Riemannian manifold $\bar{M}$. But we first recall that we have the following result of S. Hiepko [10], (cf.[8], Remark 2.1): *Let D_1 be a vector subbundle in the tangent bundle of a Riemannian manifold M and D_2 be its normal bundle. Suppose that the two distributions are involutive. We denote the integral manifolds of D_1 and D_2 by M_1 and M_2, respectively. Then M is locally isometric to warped product $M_1 \times_f M_2$ if the integral manifold M_1 is totally geodesic and the integral manifold M_2 is an extrinsic sphere, i.e, M_2 is a totally umbilical submanifold with parallel mean curvature vector.*

Thus, we can give the following definition

Definition 4.1. Let M be a semi-invariant submanifold of a locally product Riemannian manifold $\bar{M}$. Then, we say that M is a semi-invariant local warped product submanifold of $\bar{M}$ if $D^{\perp}$ defines a totally geodesic foliation on M and D defines a spherical foliation on M, i.e., each leaf of D is totally umbilical with parallel mean curvature vector field in M.

Theorem 4.1. *Let M be a semi-invariant submanifold of a locally Riemannian product manifold $\bar{M}$. Then, M is a semi-invariant local warped product submanifold of $\bar{M}$, if and only if, for $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$,*

$$A_{FZ}X = -Z(\mu)FX, \tag{4.5}$$

for some function on M satisfying $Y(\mu) = 0$, $Y \in \Gamma(D)$.

Proof: If $M = M_\perp \times_f M_T$ is a semi-invariant warped product of a locally product Riemannian manifold $\bar{M}$. Then , from (4.2), (2.8) and (2.5) we have

$$g(A_{FZ}X, Y) = -Z(lnf)g(Y, FX)$$

for $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D^\perp)$. Also from (4.1) we get

$$g(A_{FZ}X, W) = 0$$

for $X \in \Gamma(D)$ and $Z, W \in \Gamma(D^\perp)$. Thus, we conclude that

$$A_{FZ}X = -Z(lnf)FX.$$

Since f is a function on $M_\perp$, it can be considered a function on M such that $X(f) = 0$ for $X \in \Gamma(D)$. Conversely, suppose that M is a semi-invariant submanifold of a locally Riemannian product manifold $\bar{M}$ satisfying

$$A_{FZ}X = -Z(\mu)FX, \forall X \in \Gamma(D), Z \in \Gamma(D^\perp),$$

for some function on M satisfying $Y(\mu) = 0$, $Y \in \Gamma(D)$. Then, it shows that $A_{FZ}X \in \Gamma(D)$. Since A is self-adjoint we obtain $g(A_{FZ}W, X) = 0$ for $X \in \Gamma(D)$ and $Z, W \in \Gamma(D^\perp)$. Hence, we get $A_{FZ}W \in \Gamma(D^\perp)$. Then, it follows from (4.3), $D^\perp$ is integrable and its leaves are totally geodesic in M. On the other hand, from(4.4) and (4.5) we get

$$\begin{aligned} g([X,Y], Z) &= -Z(\mu)g(FX, FY) + Z(\mu)g(FX, FY) \\ &= 0 \end{aligned}$$

for $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D^\perp)$. Thus D is integrable. Let M_T be a leaf of D. We denote the second fundamental form of M_T in M by h_1. Then, from (2.6) and (2.5), we have

$$\begin{aligned} g(\nabla_X Y, Z) &= g(\bar{\nabla}_X FY, FZ) \\ &= -g(FY, \bar{\nabla}_X FZ) \end{aligned}$$

for $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D^\perp)$. Using (2.7) we get

$$g(h_1(X,Y), Z) = g(FY, A_{FZ}X).$$

Then, from (4.5) and (2.5) we get

$$g(h_1(X,Y), Z) = -Z(\mu)g(X,Y).$$

Hence, M_T is a totally umbilical submanifold in M. Moreover, $Y(\mu) = 0$, $Y \in \Gamma(D)$ implies that M_T has parallel mean curvature vector. Then applying the result of Hiepko [10], we obtain that M is a locally warped product manifold in the form $M_\perp \times_f M_T$ such that M_T is an invariant submanifold and $M_\perp$ is an anti-invariant submanifold of $\bar{M}$. Thus, proof is complete. □

In the rest of this section, we are going to obtain an inequality for the squared norm of the second fundamental form in terms of the warping function for semi-invariant warped products of locally product Riemannian manifolds. We recall that M is called mixed totally geodesic if h satisfies $h(X,Z)=0$, for $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$.

Theorem 4.2. *Let $M = M_{\perp} \times_f M_T$ be a semi-invariant warped product of a locally product Riemannian manifold $\bar{M}$. Then, we have*

1. *The squared norm of the second fundamental form of M satisfies*
$$\| h \|^2 \geq p \| \nabla(lnf) \|^2, \tag{4.6}$$
where $\nabla(lnf)$ is the gradient of lnf and p is the dimension of M_T.
2. *If the equality sign of (4.6) holds identically. Then, $M_{\perp}$ is totally geodesic submanifold of $\bar{M}$ and M is mixed totally geodesic. Moreover, M is never a minimal submanifold of $\bar{M}$.*

Proof: Let $e_1, ..., e_{m+n}$ be an orthonormal frame of a locally product Riemannian manifold $\bar{M}$ such that $e_1, ..., e_p$ in M_T, $e_{p+1}, ..., e_m$ in $M_{\perp}$, $e_{m+1} = F(e_{p+1}), ..., e_{m+q}$ in $F(TM_{\perp})$ and $e_{m+1+q}, ..., e_{m+n}$ in $\mathcal{V}$. Since
$$\begin{aligned} \| h \|^2 &= \| h(D,D) \|^2 + 2 \| h(D, D^{\perp} \|^2 \\ &\quad + \| h(D^{\perp}, D^{\perp} \|^2, \end{aligned}$$
using (4.1) and (4.2), we get
$$\begin{aligned} \| h \|^2 &= p \| \nabla(lnf) \|^2 \\ &\quad + \sum_{i,j=1}^{p} \sum_{k=m+1+q}^{m+n} g(h(e_i, e_j), e_k)^2 \\ &\quad +2 \sum_{s=p+1}^{m} \sum_{i=1}^{p} \sum_{k=m+1+q}^{m+n} g(h(e_s, e_i), e_k)^2 \\ &\quad + \| h(D^{\perp}, D^{\perp}) \|^2 . \end{aligned} \tag{4.7}$$
Thus, we obtain (4.6). Now, suppose that the equality case holds. Then from (4.7) we have
$$h(D^{\perp}, D^{\perp}) = 0, \; h(X,Y) \subset F(D^{\perp}) \tag{4.8}$$
and
$$h(X,Z) \subset F(D^{\perp}). \tag{4.9}$$
for $X, Y \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_{\perp}$. Since $M_{\perp}$ is totally geodesic in M,(4.8) implies that $M_{\perp}$ is a totally geodesic submanifold of $\bar{M}$. On the other hand,from (4.1) and (4.9), it follows that $h(X,Z)=0$ for $X \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_{\perp})$.

This implies that M is mixed totally geodesic in $\bar{M}$. Now, suppose that M is minimal, then from(4.2), we obtain $\| \nabla lnf \| = 0$, which is a contradiction. Thus, proof is complete. □

Remark 3. It is known that a mixed totally geodesic CR-warped product of a Kaehler manifold is a CR-product [5], Lemma 4.1.(5). But this is not true for a semi-invariant warped product. Indeed, the submanifold presented in Example 4.1 is a mixed totally geodesic submanifold and it is not usual product.

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