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Does the Jordan Form of a Matrix have the Greatest Number of Zeros?

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Abstract

Does the Jordan form of a matrix have more zeros than any other matrix in its orbit? We show that the answer is *yes* for semisimple and nilpotent matrices, but *no* in the general case.

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What is the simplest form of a matrix? There are at least two candidates:
i) the form that is closest to a diagonal matrix (Jordan form);
ii) the form with the most zeros (sparsest form).
We seek to determine if these two are the same.

Let $A \in M_n(k)$ be a square matrix with entries in the algebraically closed field k. The *orbit* of A is $\mathcal{O}(A) = \{ UAU^{-1} \mid U \in GL_n(k) \}$. It is clear that all matrices in $\mathcal{O}(A)$ have the same rank. The *Jordan form* $J(A)$ of A (see, e.g., [2, Ch. XV]) is the matrix in $\mathcal{O}(A)$ that it is the closest to a diagonal matrix (recall that $J(A)$ is unique up to a permutation of the blocks on the diagonal). The matrix having the largest number of zero entries among all the matrices in the orbit of A is the *sparsest* matrix in the orbit (the sparsest matrix needs not be unique).

We now ask the following question: is $J(A)$ the sparsest matrix in $\mathcal{O}(A)$?

We consider first the case when A is semisimple (which means that $J(A)$ is a diagonal matrix), or nilpotent (which means that $J(A)$ is a matrix having possibly nonzero entries only immediately above the main diagonal). In both of these cases, $J(A)$ has at most one nonzero entry in each column and in each row. Therefore, if we denote by r the number of nonzero entries in $J(A)$, we have clearly that $r = rank(J(A)) = rank(A)$. Now any matrix having more than $n^2 - r$ zeros will have at least $n - r + 1$ zero columns, therefore its rank will be less than r, and so it cannot belong to $\mathcal{O}(A)$. In conclusion, if A is semisimple or nilpotent, then $J(A)$ is the sparsest matrix in $\mathcal{O}(A)$.

Another characterization of $J(A)$ is the following: $A = S + N$, where S is a semisimple matrix, N is a nilpotent matrix, and $SN = NS$. Then $J(A) = J(S) + J(N)$ (see [1, 4.2, p.17]). This might suggest that since $J(S)$ and $J(N)$ are the sparsest matrices in the orbits of S and N, respectively, then $J(A)$ might be the sparsest matrix in $\mathcal{O}(A)$ for a general A. The following example shows that this is not the case.

Let $A \in M_{2n}(\mathbf{C})$ be a matrix whose minimal polynomial and characteristic polynomial are both equal to $(x^n - 1)^2$. Then the rational form of A has $2n + 1$ nonzero entries, and its Jordan form has $3n$ nonzero entries. So if $n > 1$, the Jordan form has fewer zeros than the rational form.

For example: take n=2 and consider the 4×4 matrix

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

in rational form. It has 5 nonzero entries but its Jordan form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

has 6 nonzero entries.

References

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