

Generic initial ideal for complete intersections of embedding dimension three with strong Lefschetz property

by
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Abstract

We compute the generic initial ideal of a complete intersection of embedding dimension three with strong Lefschetz property and we show that it is an almost reverse lexicographic ideal. This enable us to give a proof for Moreno's conjecture in the case $n = 3$ and characteristic zero.

Key Words: Complete intersection, generic initial ideal, Lefschetz property, Moreno's conjecture.

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Introduction

Let $S = K[x_1, x_2, x_3]$ be the polynomial ring over a field K of characteristic zero. Let f_1, f_2, f_3 be a regular sequence of homogeneous polynomials of degrees d_1, d_2 and d_3 respectively. We consider the ideal $I = (f_1, f_2, f_3) \subset S$. Obviously, S/I is a complete intersection artinian K -algebra. One can easily check that the Hilbert series of S/I depends only on the numbers d_1, d_2 and d_3 . More precisely,

$$H(S/I, t) = (1 + t + \cdots + t^{d_1-1})(1 + t + \cdots + t^{d_2-1})(1 + t + \cdots + t^{d_3-1}).$$

[10, Lemma 2.9] gives an explicit form of $H(S/I, t)$.

We say that a homogeneous polynomial f of degree d is *semiregular* for S/I if the maps $(S/I)_t \xrightarrow{f} (S/I)_{t+d}$ are either injective, either surjective for all $t \geq 0$. We say that S/I has the *weak Lefschetz property* (WLP) if there exists a linear form $\ell \in S$, semiregular on S/I . We say that ℓ is a weak Lefschetz element for S/I . A theorem of Harima-Migliore-Nagel-Watanabe (see [4]) states that S/I

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has (WLP). We say that S/I has the *strong Lefschetz property* (SLP) if there exists a linear form $\ell \in S$ such that ℓ^b is semiregular on S/I for all integer $b \geq 1$. In this case, we say that ℓ is a strong Lefschetz element for S/I . Of course, (SLP) \Rightarrow (WLP) but it is not known if S/I has (SLP) for any regular sequence of homogeneous polynomials f_1, f_2, f_3 . This is known only in certain cases, for example, when f_1, f_2, f_3 is generic, see [8] or when $f_2 \in K[x_2, x_3]$ and $f_3 \in K[x_3]$, see [5] and [6].

We say that a property (P) holds for a *generic* sequence of homogeneous polynomials $f_1, f_2, \dots, f_n \in S = K[x_1, x_2, \dots, x_n]$ of given degree d_1, d_2, \dots, d_n if there exists a nonempty open Zariski subset $U \subset S_{d_1} \times S_{d_2} \times \dots \times S_{d_n}$ such that for every $(f_1, f_2, \dots, f_n) \in U$ the property (P) holds. For example, a generic sequence of homogeneous polynomials $f_1, f_2, \dots, f_n \in S$ is regular.

Now, we present some conjectures and the relations between them (see [8]).

Conjecture A.(Fröberg) If $f_1, f_2, \dots, f_r \in S = K[x_1, \dots, x_n]$ is a generic sequence of homogeneous polynomials of given degrees d_1, d_2, \dots, d_r and

$$I = (f_1, f_2, \dots, f_r)$$

then the Hilbert series of S/I is

$$H(S/I) = \left| \frac{\prod_{i=1}^r (1 - t^{d_i})}{(1 - t)^n} \right|,$$

where $|\sum_{j \geq 0} a_j t^j| = \sum_{j \geq 0} b_j t^j$, with $b_j = a_j$ if $a_i > 0$ for all $i \leq j$ and $b_j = 0$ otherwise.

Conjecture B. If $f_1, f_2, \dots, f_n \in S = K[x_1, \dots, x_n]$ is a generic sequence of homogeneous polynomials of given degrees d_1, d_2, \dots, d_n and $I = (f_1, \dots, f_n)$ then x_n, x_{n-1}, \dots, x_1 is a semi-regular sequence on $A = S/I$, i.e. x_i is semiregular on $A/(x_n, \dots, x_{i+1})$ for all $1 \leq i \leq n$.

Conjecture C. If $f_1, f_2, \dots, f_n \in S = K[x_1, \dots, x_n]$ is a generic sequence of homogeneous polynomials of given degrees d_1, d_2, \dots, d_n , $I = (f_1, \dots, f_n)$ and J is the initial ideal of I with respect to the revlex order, then x_n, x_{n-1}, \dots, x_1 is a semi-regular sequence on $A = S/(f_1, \dots, f_n)$.

Conjecture D.(Moreno) If $f_1, f_2, \dots, f_n \in S = K[x_1, \dots, x_n]$ is a generic sequence of homogeneous polynomials of given degrees d_1, d_2, \dots, d_n , $I = (f_1, \dots, f_n)$ and J is the initial ideal of I with respect to the revlex order, then J is an almost revlex ideal, i.e. if $u \in J$ is a minimal generator of J then every monomial of the same degree which precedes u must be in J as well.

Pardue proved in [8] that if conjecture A is true for some positive integer n then the conjecture B is true for the same n . Also, conjecture C is true for n if and only if B is true for n and if conjecture B is true for some r then A is true for $n < r$ and exactly for that r . Also, if conjecture D is true for some n then B,

and thus C , are true for the same n . Fröberg [3] and Anick [1] proved that A is true for $n \leq 3$ and so B and C are true for $n \leq 3$. Moreno [7] remarked that D is true for $n = 2$. Note that Conjecture A for $n = 3$ does not imply the Moreno's conjecture D for $n = 3$.

Let $J = \text{Gin}_{<}(I)$ be the generic initial ideal of I , with respect to the reverse lexicographic order. Our aim is to compute J for all regular sequences f_1, f_2, f_3 of homogeneous polynomials of given degree d_1, d_2, d_3 such that S/I has (SLP). We will do this in the sections 2 and 3. These computations shown us in particular, that J depends only on the numbers d_1, d_2, d_3 (this has been already proved by Popescu and Vladioiu in [10]) and more important, that J is an almost reverse lexicographic ideal (Theorem 1.1). As a consequence, conjecture Moreno (D) is true for $n = 3$ and $\text{char}(K) = 0$ (Theorem 1.2).

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1 Main results

Theorem 1.1. *If f_1, f_2, f_3 is a regular sequence of homogeneous polynomials of given degrees d_1, d_2, d_3 and $I = (f_1, f_2, f_3)$ such that S/I has the (SLP) then $J = \text{Gin}(I)$ is uniquely determined and is an almost reverse lexicographic ideal.*

Proof: The theorem is a direct consequence of the Propositions 2.3, 2.8, 3.3, 3.8, 3.13 and 3.17. \square

Theorem 1.2. *The conjecture Moreno (D) is true for $n = 3$ (and $\text{char}(K) = 0$).*

Proof: Notice that (SLP) is an open condition. Also, the condition that a sequence of homogeneous polynomial is regular is an open condition. It follows, using Theorem 1.1, that for a generic sequence f_1, f_2, f_3 of homogeneous polynomials of given degrees d_1, d_2, d_3 , $J = \text{Gin}(I)$ is almost revlex, where $I = (f_1, f_2, f_3)$. But the definition of the generic initial ideal implies to choose a generic change of variables, and therefore for a generic sequence f_1, f_2, f_3 of homogeneous polynomials of given degrees $\text{in}(I)$ is almost revlex, as required. \square

Remark 1.3. *In order to compute the generic initial ideal J we will use the fact that J is a strongly stable ideal, i.e. for any monomial $u \in J$ and any indices $j < i$, if $x_i|u$ then $x_j u/x_i \in J$. Also, a theorem of Wiebe (see [11]) states that S/I has (SLP) if and only if x_3 is a strong Lefschetz element for S/J . We need to consider several cases: I. $d_1 + d_2 \leq d_3 + 1$ with 2 subcases $d_1 = d_2 < d_3$, $d_1 < d_2 < d_3$ (section 2) and II. $d_1 + d_2 > d_3 + 1$ with 4 subcases: $d_1 = d_2 = d_3$, $d_1 = d_2 < d_3$, $d_1 < d_2 = d_3$, $d_1 < d_2 < d_3$ (section 3).*

The construction of J in all cases, follows the next procedure. For any non-negative integer k , we denote by J_k the set of monomials of degree k in J . We can easily compute the cardinality of each J_k from the Hilbert series of S/J . We denote $\text{Shad}(J_k) = \{x_i u : 1 \leq i \leq n, u \in J_k\}$. We begin with $J_0 = \emptyset$ and we pass from J_k to J_{k+1} noticing that $J_{k+1} = \text{Shad}(J_k) \cup$ eventually some new monomial(s) (exactly $|J_{k+1}| - |\text{Shad}(J_k)|$ new monomials). The fact that J is strongly stable and that x_3 is a strong Lefschetz element for S/J tell us what we need to add to $\text{Shad}(J_k)$ in order to obtain J_{k+1} . We continue this procedure until $k = d_1 + d_2 + d_3 - 2$ since $J_k = S_k$ (=the set of all monomials of degree k) for any $k \geq d_1 + d_2 + d_3 - 2$ and so we cannot add any new monomials in larger degrees. J is the ideal generated by all monomials added to $\text{Shad}(J_k)$ at some step k . We will present detailed this construction only in the subcase $d_1 = d_2 < d_3$ of the case $I, d_1 + d_2 \leq d_3 + 1$ (Proposition 2.3), the other cases being presented in sketch, but the reader can easily complete the proofs.

2 Case $d_1 + d_2 \leq d_3 + 1$

- Subcase $d_1 = d_2 < d_3$.

Proposition 2.1. Let $2 \leq d := d_1 = d_2 < d_3$ be positive integers such that $2d \leq d_3 + 1$. The Hilbert function of the standard graded complete intersection $A = K[x_1, x_2, x_3]/I$, where I is the ideal generated by f_1, f_2, f_3 , with f_i homogeneous polynomials of degree d_i , for all i , with $1 \leq i \leq 3$, has the form:

1. $H(A, k) = \binom{k+2}{k}$, for $k \leq d - 1$.
2. $H(A, k) = \binom{d+1}{2} + \sum_{i=1}^j (d - i)$, for $k = d - 1 + j$, where $0 \leq j \leq d - 1$.
3. $H(A, k) = d^2$, for $2d - 2 \leq k \leq d_3 - 1$.
4. $H(A, k) = H(A, 2d + d_3 - 3 - k)$ for $k \geq d_3$.

Proof: It follows from [10, Lemma 2.9(a)]. □

Corollary 2.2. In the conditions of Proposition 2.1, let $J = \text{Gin}(I)$ be the generic initial ideal of I with respect to the reverse lexicographic order. If we denote by J_k the set of monomials of J of degree k , then:

1. $J_k = \emptyset$, for $k \leq d - 1$.
2. $|J_k| = j(j + 1)$, for $k = d - 1 + j$, where $0 \leq j \leq d - 1$.
3. $|J_k| = d(d - 1) + 2dj + \frac{j(j-1)}{2}$, for $k = 2d - 2 + j$, where $0 \leq j \leq d_3 + 1 - 2d$.
4. $|J_k| = \frac{d_3(d_3+1)}{2} - d^2 + jd_3 + j(j + 1)$, for $k = d_3 - 1 + j$, where $0 \leq j \leq d - 1$.

5. $|J_k| = \frac{d_3(d_3-1)}{2} + d(d_3 - 1) + j(d_3 + 2d)$, for $k = d + d_3 - 2 + j$, where $0 \leq j \leq d$.
6. $J_k = S_k$, for $k \geq 2d + d_3 - 2$, where S_k is the set of monomials of degree k .

Proof: Using that $|J_k| = |S_k| - H(S/J, k)$, together with the general fact that $H(S/J, k) = H(S/I, k)$, the proof follows immediately from Proposition 2.1. \square

Proposition 2.3. Let $2 \leq d := d_1 = d_2 < d_3$ be positive integers such that $2d \leq d_3 + 1$. Let $f_1, f_2, f_3 \in K[x_1, x_2, x_3]$ be a regular sequence of homogeneous polynomials of degrees d_1, d_2, d_3 . If $I = (f_1, f_2, f_3)$, and $J = \text{Gin}(I)$, the generic initial ideal with respect to the reverse lexicographic order, and S/I has (SLP), then:

$$J = (x_1^d, x_1^{d-j-1}x_2^{2j+1} \text{ for } 0 \leq j \leq d-1, x_2^{2d-2j-2}x_3^{d_3-2d+2j+2}\{x_1, x_2\}^j$$

for $0 \leq j \leq d-2$, and

$$x_3^{d_3+2j-2}\{x_1, x_2\}^{d-j} \text{ for } 1 \leq j \leq d).$$

Proof: We have $|J_d| = 2$, hence $J_d = \{x_1^{d-1}\{x_1, x_2\}\}$, since J is a strongly stable ideal. Therefore:

$$\text{Shad}(J_d) = \{x_1^{d-1}\{x_1, x_2\}^2, x_1^{d-1}x_3\{x_1, x_2\}\},$$

Now we have two possibilities to analyze: $d = 2$ and $d \geq 3$. First, suppose $d \geq 3$.

Using the formulae from Corollary 2.2 we have $|J_{d+1}| - |\text{Shad}(J_d)| = 1$, so there is only one generator to add to the set $\text{Shad}(J_d)$ in order to obtain J_{d+1} . Since J is strongly stable, we have only two possibilities: $x_1^{d-2}x_2^3$ or $x_1^{d-1}x_3^2$. We cannot have $x_1^{d-1}x_3^2$, since otherwise the application

$$(S/J)_{d-1} \xrightarrow{\cdot x_3^2} (S/J)_{d+1},$$

with $|(S/J)_{d-1}| < |(S/J)_{d+1}|$ (see Proposition 2.1) would not be injective ($0 \neq x_1^{d-1} \in (S/J)_{d-1}$ and is mapped to 0), which is a contradiction to the fact that x_3 is a strong Lefschetz element for S/J . Hence:

$$J_{d+1} = \{x_1^{d-2}\{x_1, x_2\}^3, x_1^{d-1}x_3\{x_1, x_2\}\}.$$

We prove by induction on j , with $1 \leq j \leq d-2$, that:

$$\begin{aligned} J_{d+j} &= \text{Shad}(J_{d+j-1}) \cup \{x_1^{d-j-1}x_2^{2j+1}\} = \\ &= \{x_1^{d-j-1}\{x_1, x_2\}^{2j+1}, x_1^{d-j}x_3\{x_1, x_2\}^{2j-1}, \dots, x_1^{d-1}x_3^j\{x_1, x_2\}\}. \end{aligned}$$

The assertion was checked above for $j = 1$. Assume now that the statement is true for some $j < d - 2$. Then $Shad(J_{d+j})$ is the following set:

$$\{x_1^{d-j-1}\{x_1, x_2\}^{2j+2}, x_1^{d-j-1}x_3\{x_1, x_2\}^{2j+1}, \\ x_1^{d-j}x_3^2\{x_1, x_2\}^{2j-1}, \dots, x_1^{d-1}x_3^{j+1}\{x_1, x_2\}\}.$$

We have $|J_{d+j+1}| - |Shad(J_{d+j})| = 1$, so we must add only one generator to $Shad(J_{d+j})$ to get J_{d+j+1} . The ideal J , being strongly stable, allows only two possibilities, namely $x_1^{d-j-2}x_2^{2j+3}$ or $x_1^{d-2}x_2^2x_3^{j+1}$. The second one is not allowed because the application

$$(S/J)_d \xrightarrow{\cdot x_3^{j+1}} (S/J)_{d+j+1}$$

would not be injective ($0 \neq x_1^{d-2}x_2^2 \in (S/J)_d$ and is mapped to 0), x_3 being a strong-Lefschetz element for S/J . Therefore, we must add $x_1^{d-j-2}x_2^{2j+3}$ and our claim is proved. In particular, we obtain

$$J_{2d-2} = \{x_1\{x_1, x_2\}^{2d-3}, x_1^2x_3\{x_1, x_2\}^{2d-5}, \dots, x_1^{d-1}x_3^{d-2}\{x_1, x_2\}\}.$$

In order to compute J_k , with $2d - 2 \leq k \leq d_3$, we must consider two possibilities.

- 1. $d_3 = 2d - 1$.

Since $|J_{2d-1}| - |Shad(J_{2d-2})| = 2$, there are two generators to add to $Shad(J_{2d-2})$. We prove that these generators are $x_2^{2d-1}, x_2^{2d-2}x_3$. Assuming by contradiction that we have other generators, since J is strongly stable, it follows that there is at least one generator from the set

$$\{x_1x_2^{2d-4}x_3^2, x_1^2x_2^{2d-6}x_3^3, \dots, x_1^{d-2}x_2^2x_3^{d-1}\}.$$

Then, the application $(S/J)_{2d-3} \xrightarrow{\cdot x_3^2} (S/J)_{2d-1}$ would not be injective, a contradiction since x_3 is a strong Lefschetz element for S/J . Therefore

$$J_{2d-1} = J_{d_3} = \{\{x_1, x_2\}^{2d-1}, x_3\{x_1, x_2\}^{2d-2}, \dots, x_1^{d-1}x_3^{d-1}\{x_1, x_2\}\}.$$

- 2. $d_3 > 2d - 1$.

Since $|J_{2d-1}| - |Shad(J_{2d-2})| = 1$, there is only one generator to add to $Shad(J_{2d-2})$, which can be selected from the set

$$\{x_2^{2d-1}, x_1x_2^{2d-4}x_3^2, x_1^2x_2^{2d-6}x_3^3, \dots, x_1^{d-2}x_2^2x_3^{d-1}\}$$

because J is strongly stable. In a similar manner to what we have done above can be shown that, x_3 being a strong Lefschetz element, leaves us as unique possibility x_2^{2d-1} , therefore:

$$J_{2d-1} = \{\{x_1, x_2\}^{2d-1}, x_1x_3\{x_1, x_2\}^{2d-3}, \dots, x_1^{d-1}x_3^{d-1}\{x_1, x_2\}\}.$$

One can easily show, using induction on $1 \leq j \leq d_3 - 2d$, if case, that $|J_{2d-1+j}| = |\text{Shad}(J_{2d-2+j})|$ and J_{2d-1+j} is the set

$$\{\{x_1, x_2\}^{2d-1+j}, \dots, x_3^j \{x_1, x_2\}^{2d-1}, \\ x_3^{j+1} x_1 \{x_1, x_2\}^{2d-2}, \dots, x_3^{d+j-1} x_1^{d-1} \{x_1, x_2\}\}.$$

In particular, we obtain that J_{d_3-1} is the set

$$\{\{x_1, x_2\}^{d_3-1}, \dots, x_3^{d_3-2d} \{x_1, x_2\}^{2d-1}, \\ x_3^{d_3-2d+1} x_1 \{x_1, x_2\}^{2d-3}, \dots, x_3^{d_3-d-1} x_1^{d-1} \{x_1, x_2\}\}.$$

Since $|J_{d_3}| - |\text{Shad}(J_{d_3-1})| = 1$, the generator which has to be add to $\text{Shad}(J_{d_3-1})$ can be selected from the set

$$\{x_2^{2d-2} x_3^{d_3-2d+2}, x_1 x_2^{2d-4} x_3^{d_3-2d+3}, \dots, x_1^{d-2} x_2^2 x_3^{d_3-d}\}$$

such that J is strongly stable. The generator is $x_2^{2d-2} x_3^{d_3-2d+2}$, otherwise the application $(S/J)_{2d-3} \xrightarrow{x_3^{d_3-2d+3}} (S/J)_{d_3}$ is not injective, a contradiction, since x_3 is a strong Lefschetz element for S/J . Hence, we get that J_{d_3} is

$$\{\{x_1, x_2\}^{d_3}, \dots, x_3^{d_3-2d+2} \{x_1, x_2\}^{2d-2}, \\ x_1^2 x_3^{d_3-2d+3} \{x_1, x_2\}^{2d-5}, \dots, x_3^{d_3-d} x_1^{d-1} \{x_1, x_2\}\},$$

and one can check that is the same formula as in 1. ($d_3 = 2d - 1$).

Now, we show by induction on $1 \leq j \leq d - 2$ that

$$J_{d_3+j} = \text{Shad}(J_{d_3-1+j}) \cup \{x_2^{2d-2j-2} x_3^{d_3-2d+2j+2} \{x_1, x_2\}^j\}.$$

Indeed, for $j = 1$, $|J_{d_3+1}| - |\text{Shad}(J_{d_3})| = 2$ and the generators which must be added are $x_1 x_2^{2d-4} x_3^{d_3-2d+4}$, $x_2^{2d-3} x_3^{d_3-2d+4}$. If not, since J is strongly stable, then at least one of the generators belongs to the set

$$\{x_1^2 x_2^{2d-6} x_3^{d_3-2d+5}, \dots, x_1^{d-2} x_2^2 x_3^{d_3-d+1}\}$$

(for $d = 3$ this is the emptyset). but then the map $(S/J)_{2d-4} \xrightarrow{x_3^{d_3-2d+5}} (S/J)_{d_3+1}$ is not injective, contradiction.

Assume now that we proved the assertion for some $j < d - 2$. Then $|J_{d_3+j+1}| - |\text{Shad}(J_{d_3+j})| = j+2$ and the new generators are $x_2^{2d-2j-4} x_3^{d_3-2d+2j+4} \{x_1, x_2\}^{j+1}$. Indeed, if not, since J is strongly stable, then at least one of the generators belongs to the set $\{x_1^{j+2} x_2^{2d-2j-6} x_3^{d_3-2d+2j+5}, \dots, x_1^{d-2} x_2^2 x_3^{d_3-d+j+1}\}$ (for $d = 3$ this is the emptyset...) but then the map $(S/J)_{2d-j-4} \xrightarrow{x_3^{d_3-2d+2j+5}} (S/J)_{d_3+j+1}$ is not injective, contradiction, and we are done. Hence,

$$J_{d_3+d-2} = \{\{x_1, x_2\}^{d_3+d-2}, x_3 \{x_1, x_2\}^{d+d_3-3}, \dots, x_3^{d_3-2} \{x_1, x_2\}^d\}.$$

We prove by induction on $1 \leq j \leq d$ that $J_{d+d_3-2+j} = \text{Shad}(J_{d+d_3-3+j}) \cup x_3^{d_3+2j-2}\{x_1, x_2\}^{d-j}$. If $j = 1$ then $|J_{d+d_3-1}| - |\text{Shad}(J_{d+d_3-2})| = d$ so we must add d generators, which are precisely the elements of the set $x_3^{d_3}\{x_1, x_2\}^{d-1}$. Indeed, if we have a generator which does not belong to the set it is divisible by $x_3^{d_3+1}$ and therefore the map $(S/J)_{d-2} \xrightarrow{\cdot x_3^{d_3+1}} (S/J)_{d_3+d-1}$ is not injective, which is a contradiction with x_3 is a strong Lefschetz element for S/J (the map has to be bijective). The induction step is similar and finally we obtain that $J_{d_3+2d-2} = S_{d_3+2d-2}$ and thus we cannot add new minimal generators of J in degree $> d_3 + 2d - 2$.

In order to complete the proof we must consider now $d = 2$. The hypothesis implies $d_3 \geq 3$. We already seen that $J_2 = \{x_1^2, x_1x_2\}$ and $\text{Shad}(J_2) = \{x_1\{x_1, x_2\}^2, x_1x_3\{x_1, x_2\}\}$.

Using the formulae from Corollary 2.2 we have $|J_3| - |\text{Shad}(J_2)| = 1$, so there is only one generator to add to the set $\text{Shad}(J_2)$ in order to obtain J_{d+1} .

Since J is strongly stable, we have only two possibilities: x_2^3 or $x_1x_3^2$. We can not have $x_1x_3^2$, since otherwise the application

$$(S/J)_1 \xrightarrow{\cdot x_3^2} (S/J)_3,$$

with $|(S/J)_1| < |(S/J)_3|$ (see Proposition 2.1) would not be injective ($0 \neq x_1 \in (S/J)_1$ and is mapped to 0), which is a contradiction to the fact that x_3 is a strong Lefschetz element for S/J . Hence:

$$J_3 = \{\{x_1, x_2\}^3, x_1x_3\{x_1, x_2\}\}.$$

Assume now $d_3 \geq 4$. One can easily show, using induction on $1 \leq j \leq d_3 - 4$, if case, that $|J_{3+j}| = |\text{Shad}(J_{2+j})|$ and J_{3+j} is the set

$$\{\{x_1, x_2\}^{3+j}, \dots, x_3^j\{x_1, x_2\}^3, x_3^{j+1}x_1\{x_1, x_2\}\}.$$

In particular, we obtain that

$$J_{d_3-1} = \{\{x_1, x_2\}^{d_3-1}, \dots, x_3^{d_3-4}\{x_1, x_2\}^3, x_3^{d_3-3}x_1\{x_1, x_2\}\}.$$

Since $|J_{d_3}| - |\text{Shad}(J_{d_3-1})| = 1$, the generator which has to be add to $\text{Shad}(J_{d_3-1})$ is exactly $x_2^2x_3^{d_3-2}$ such that J is strongly stable. Hence, we get

$$J_{d_3} = \{\{x_1, x_2\}^{d_3}, \dots, x_3^{d_3-3}\{x_1, x_2\}^3, x_1x_3^{d_3-2}\{x_1, x_2\}\},$$

and one can check that is the same formula as in the case $d_3 = 3$.

Since $|J_{d_3+1}| - |\text{Shad}(J_{d_3})| = 1$, there is only one generator to add to the set $\text{Shad}(J_{d_3})$ in order to obtain J_{d_3+1} . Since J is strongly stable, we have only two possibilities: $x_2^2x_3^{d_3-1}$ or $x_1x_3^{d_3}$. We can not have $x_1x_3^{d_3}$, since otherwise the application

$$(S/J)_1 \xrightarrow{\cdot x_3^{d_3}} (S/J)_{d_3+1},$$

with $|(S/J)_1| = |(S/J)_3|$ (see Proposition 2.1) would not be injective ($0 \neq x_1 \in (S/J)_1$ and is mapped to 0), which is a contradiction to the fact that x_3 is a strong Lefschetz element for S/J . Hence:

$$J_{d_3+1} = \{\{x_1, x_2\}^{d_3+1}, \dots, x_3^{d_3-1}\{x_1, x_2\}^2\}.$$

Since $|J_{d_3+2}| - |\text{Shad}(J_{d_3+1})| = 2$ and J is strongly stable, we must add $x_1x_3^{d_3+1}$ and $x_2x_3^{d_3+1}$ at $\text{Shad}(J_{d_3+1})$ in order to obtain J_{d_3+2} . Hence $J_{d_3+2} = S_{d_3+2} \setminus \{x_3^{d_3+2}\}$. Finally, since $J_{d_3+3} = S_{d_3+3}$ we add $x_3^{d_3+3}$ at $\text{Shad}(J_{d_3+2})$ and thus we cannot add new minimal generators of J in degree $> d_3 + 2$. \square

Corollary 2.4. *In the conditions of the above proposition, the number of minimal generators of J is $d^2 + d + 1$.*

Example 2.5. *Let $d_1 = d_2 = 3$ and $d_3 = 9$. Proposition 2.3 implies:*

$$J = (x_1^3, x_1^2x_2, x_1x_2^3, x_2^5, x_2^4x_3^5, x_1x_2^2x_3^7, x_2^3x_3^7, x_3^9\{x_1, x_2\}^2, x_3^{11}\{x_1, x_2\}, x_3^{13}).$$

- Subcase $d_1 < d_2 < d_3$.

Proposition 2.6. Let $2 \leq d_1 < d_2 < d_3$ be positive integers such that $d_1 + d_2 \leq d_3 + 1$. The Hilbert function of the standard graded complete intersection $A = K[x_1, x_2, x_3]/I$, where I is the ideal generated by f_1, f_2, f_3 , with f_i homogeneous polynomials of degree d_i , for all i , with $1 \leq i \leq 3$, has the form:

1. $H(A, k) = \binom{k+2}{k}$, for $k \leq d_1 - 1$.
2. $H(A, k) = \binom{d_1+1}{2} + jd_1$, for $k = j + d_1 - 1$, where $0 \leq j \leq d_2 - d_1$.
3. $H(A, k) = \binom{d_1+1}{2} + d_1(d_2 - d_1) + \sum_{i=1}^j (d_1 - i)$, for $k = j + d_2 - 1$, where $0 \leq j \leq d_1 - 1$.
4. $H(A, k) = d_1d_2$, for $d_1 + d_2 - 2 \leq k \leq d_3 - 1$.
5. $H(A, k) = H(A, d_1 + d_2 + d_3 - 3 - k)$ for $k \geq d_3$.

Proof: It follows from [10, Lemma 2.9(a)]. \square

Corollary 2.7. In the conditions of Proposition 2.6, let $J = \text{Gin}(I)$ be the generic initial ideal of I with respect to the reverse lexicographic order. If we denote by J_k the set of monomials of J of degree k , then:

1. $|J_k| = 0$, for $k \leq d - 1$.

2. $|J_k| = j(j+1)/2$, for $k = j + d_1 - 1$, where $0 \leq j \leq d_2 - d_1$.
3. $|J_k| = \frac{(d_2-d_1)((d_2-d_1-1))}{2} + j(d_2 - d_1) + j(j+1)$, for $k = j + d_2 - 1$, where $0 \leq j \leq d_1 - 1$.
4. $|J_k| = \frac{d_1^2+d_2^2-d_1-d_2}{2} + j(d_1 + d_2) + \frac{j(j-1)}{2}$, for $k = j + d_1 + d_2 - 2$, where $0 \leq j \leq d_3 - d_1 - d_2 + 1$.
5. $|J_k| = \frac{d_3^2+d_3-2d_1d_2}{2} + jd_3 + j(j+1)$, for $k = j + d_3 - 1$ where $0 \leq j \leq d_1 - 1$.
6. $|J_k| = \frac{(d_1+d_3)(d_1+d_3-1)+d_1^2-d_1-2d_1d_2}{2} + j(d_3 + 2d_1) + \frac{j(j-1)}{2}$, for $k = j + d_1 + d_3 - 2$, where $0 \leq j \leq d_2 - d_1$.
7. $|J_k| = \frac{(d_2+d_3)(d_2+d_3-1)+d_1(d_1-1)}{2} + j(d_1 + d_2 + d_3)$, for $k = d_2 + d_3 - 2$, where $0 \leq j \leq d_1 - 1$.
8. $J_k = S_k$, for $k \geq 3d - 2$.

Proposition 2.8. Let $2 \leq d_1 < d_2 < d_3$ be positive integers such that $d_1 + d_2 \leq d_3 + 1$. Let $f_1, f_2, f_3 \in K[x_1, x_2, x_3]$ be a regular sequence of homogeneous polynomials of degrees d_1, d_2, d_3 . If $I = (f_1, f_2, f_3)$, $J = \text{Gin}(I)$, the generic initial ideal with respect to the reverse lexicographic order, and S/I has (SLP), then:

$$J = (x_1^{d_1}, x_1^{d_1-j} x_2^{d_2-d_1+2j-1} \text{ for } 1 \leq j \leq d_1 - 1, x_2^{d_1+d_2-1}, x_2^{d_1+d_2-2} x_3^{d_3-d_1-d_2+2}, \\ x_3^{d_3-d_1-d_2+2j+2} x_2^{d_1+d_2-2j-2} \{x_1, x_2\}^j \text{ for } 1 \leq j \leq d_1 - 2, \\ x_3^{d_3+d_1-d_2-2+2j} x_2^{d_2-d_1+1-j} \{x_1, x_2\}^{d_1-1} \text{ for } 1 \leq j \leq d_2 - d_1, \\ x_3^{d_3+d_2-d_1+2j-2} \{x_1, x_2\}^{d_1-j} \text{ for } 1 \leq j \leq d_1).$$

Proof: We have $|J_{d_1}| = 1$, hence $J_{d_1} = \{x_1^{d_1}\}$, since J is a strongly stable ideal. Therefore:

$$\text{Shad}(J_{d_1}) = \{x_1^{d_1} \{x_1, x_2\}, x_1^{d_1} x_3\}.$$

Assume $d_2 > d_1 + 1$. Since $|J_{d_1+1}| = |\text{Shad}(J_{d_1})|$ from the formulae of 2.7, it follows $J_{d_1+1} = \text{Shad}(J_{d_1})$. We prove by induction on $1 \leq j \leq d_2 - d_1 - 1$ that

$$J_{d_1+j} = \text{Shad}(J_{d_1+j-1}) = \{x_1^{d_1} \{x_1, x_2\}^j, x_3 x_1^{d_1} \{x_1, x_2\}^{j-1}, \dots, x_3^j x_1^{d_1}\}.$$

Indeed, the case $j = 1$ is already proved. Suppose the assertion is true for some $j < d_2 - d_1 - 1$. Since $|J_{d_1+j+1}| - |\text{Shad}(J_{d_1+j})| = 0$ it follows that

$$J_{d_1+j+1} = \text{Shad}(J_{d_1+j}) = \{x_1^{d_1} \{x_1, x_2\}^{j+1}, x_3 x_1^{d_1} \{x_1, x_2\}^j, \dots, x_3^{j+1} x_1^{d_1}\}$$

thus we are done. In particular, we get

$$J_{d_2-1} = \{x_1^{d_1} \{x_1, x_2\}^{d_2-d_1-1}, x_3 x_1^{d_1} \{x_1, x_2\}^{d_2-d_1-2}, \dots, x_3^{d_2-d_1-1} x_1^{d_1}\}$$

which is the same formula as in the case $d_2 = d_1 + 1$.

We have $|J_{d_2}| - |\text{Shad}(J_{d_2-1})| = 1$ so we must add a new generator to $\text{Shad}(J_{d_2-1})$ to obtain J_{d_2} . Since J is strongly stable and x_3 is a strong Lefschetz element for S/J this new generator is $x_1^{d_1-1}x_2^{d_2-d_1+1}$, therefore

$$J_{d_2} = \{x_1^{d_1-1}\{x_1, x_2\}^{d_2-d_1+1}, x_3x_1^{d_1}\{x_1, x_2\}^{d_2-d_1-1}, \dots, x_3^{d_2-d_1}x_1^{d_1}\}.$$

Assume $d_1 > 2$. We prove by induction on $1 \leq j \leq d_1 - 1$ that

$$J_{d_2-1+j} = \text{Shad}(J_{d_2-2+j}) \cup \{x_1^{d_1-j}x_2^{d_2-d_1+2j-1}\} = \{x_1^{d_1-j}\{x_1, x_2\}^{d_2-d_1+2j-1}, \\ x_3x_1^{d_1-j+1}\{x_1, x_2\}^{d_2-d_1+2j-3}, \dots, x_3^jx_1^{d_1}\{x_1, x_2\}^{d_2-d_1-1}, \dots, x_3^{d_2-d_1+j-1}x_1^{d_1}\}.$$

The assertion was proved for $j = 1$. Suppose $1 \leq j < d_1 - 1$ and the assertion is true for j . We have $|J_{d_2+j}| - |\text{Shad}(J_{d_2-1+j})| = 1$, thus we must add a new generator to $\text{Shad}(J_{d_2-1+j})$ in order to obtain J_{d_2+j} and since J is strongly stable and x_3 is a strong Lefschetz element for S/J , this is $x_1^{d_1-j-1}x_2^{d_2-d_1+2j+1}$ and we are done. In particular, we obtain:

$$J_{d_1+d_2-2} = \{x_1\{x_1, x_2\}^{d_1+d_2-3}, x_3x_1^2\{x_1, x_2\}^{d_1+d_2-5}, \dots, \\ x_3^{d_1-1}x_1^{d_1}\{x_1, x_2\}^{d_2-d_1-1}, x_3^{d_1}x_1^{d_1}\{x_1, x_2\}^{d_2-d_1-2}, \dots, x_3^{d_2-2}x_1^{d_1}\}$$

and one can check that is the same expression as in the case $d_1 = 2$.

In order to compute J_k , with $2d - 2 \leq k \leq d_3$, we must consider two possibilities.

- $1.d_3 = d_1 + d_2 - 1$.

Since $|J_{d_1+d_2-1}| - |\text{Shad}(J_{d_1+d_2-2})| = 2$, there are two generators to add to $\text{Shad}(J_{d_1+d_2-2})$ to get $J_{d_1+d_2-1}$, but on the other hand J is strongly stable and x_3 is a strong Lefschetz element for S/J so these generators must be $x_2^{d_1+d_2-1}, x_2^{d_1+d_2-2}x_3$. Therefore:

$$J_{d_1+d_2-1} = J_{d_3} = \{\{x_1, x_2\}^{d_1+d_2-1}, x_3\{x_1, x_2\}^{d_1+d_2-2}, \dots, x_1^{d_1}x_3^{d_2-1}\}.$$

- $2.d_3 > d_1 + d_2 - 1$.

Since $|J_{d_1+d_2-1}| - |\text{Shad}(J_{d_1+d_2-2})| = 1$, there is only one generator to add to $\text{Shad}(J_{d_1+d_2-2})$, which is precisely $x_2^{d_1+d_2-1}$ since J is strongly stable and x_3 is a strong Lefschetz element for S/J . Therefore

$$J_{d_1+d_2-1} = \{\{x_1, x_2\}^{d_1+d_2-1}, x_1x_3\{x_1, x_2\}^{d_1+d_2-3}, \dots, x_1^{d_1}x_3^{d_2-1}\}.$$

One can easily show, using induction on $1 \leq j \leq d_3 - d_1 - d_2$, if case, that $|J_{d_1+d_2-1+j}| = |\text{Shad}(J_{d_1+d_2-2+j})|$ and $J_{d_1+d_2-1+j}$ is the set

$$\{\{x_1, x_2\}^{j+d_1+d_2-1}, x_3\{x_1, x_2\}^{j+d_1+d_2-2}, \dots, x_3^j\{x_1, x_2\}^{d_1+d_2-1},$$

$$x_3^{j+1}x_1\{x_1, x_2\}^{d_1+d_2-3}, \dots, x_3^{d_1+j}x_1^{d_1}\{x_1, x_2\}^{d_2-d_1-1}, \dots, x_3^{j+d_2-1}x_1^{d_1}\}$$

$$\text{So } J_{d_3-1} = \{\{x_1, x_2\}^{d_3-1}, x_3\{x_1, x_2\}^{d_3-2}, \dots, x_3^{d_3-d_1-d_2}\{x_1, x_2\}^{d_1+d_2-1}, \\ x_3^{d_3-d_1-d_2+1}x_1\{x_1, x_2\}^{d_1+d_2-3}, \dots, x_3^{d_3-d_2}x_1^{d_1}\{x_1, x_2\}^{d_2-d_1-1}, \dots, x_3^{d_3-d_1-1}x_1^{d_1}\}$$

Since $|J_{d_3}| - |\text{Shad}(J_{d_3-1})| = 1$, J is strongly stable and x_3 is a strong Lefschetz element for S/J , the generator which has to be added to $\text{Shad}(J_{d_3-1})$ is $x_2^{d_1+d_2-2}x_3^{d_3-d_1-d_2+2}$. Hence, we get

$$J_{d_3} = \{\{x_1, x_2\}^{d_3}, x_3\{x_1, x_2\}^{d_3-1}, \dots, x_3^{d_3-d_1-d_2+2}\{x_1, x_2\}^{d_1+d_2-2}, \\ x_3^{d_3-d_1-d_2+3}x_1^2\{x_1, x_2\}^{d_1+d_2-5}, \dots, x_3^{d_3-d_2+1}x_1^{d_1}\{x_1, x_2\}^{d_2-d_1-1}, \dots, x_3^{d_3-d_1}x_1^{d_1}\},$$

and one can check that is the same formula as in 1. ($d_3 = d_1 + d_2 - 1$).

Assume now $d_1 > 2$. We show by induction on $1 \leq j \leq d_1 - 2$ that

$$J_{d_3+j} = \text{Shad}(J_{d_3-1+j}) \cup \\ \cup \{x_1^j x_3^{d_3-d_1-d_2+2j+2} x_2^{d_1+d_2-2j-2}, \dots, x_3^{d_3-d_1-d_2+2j+2} x_2^{d_1+d_2-2-j}\}.$$

Indeed, for $j = 1$, $|J_{d_3+1}| - |\text{Shad}(J_{d_3})| = 2$ so we must add two generators to $\text{Shad}(J_{d_3})$ in order to obtain J_{d_3+1} . Since J is strongly stable and x_3 is a strong Lefschetz element for S/J , these new generators are $x_1 x_3^{d_3-d_1-d_2+4} x_2^{d_1+d_2-2j-4}$ and $x_3^{d_3-d_1-d_2+4} x_2^{d_1+d_2-2j-3}$. Assume now that we proved the assertion for some $j < d_1 - 2$. Then $|J_{d_3+j+1}| - |\text{Shad}(J_{d_3+j})| = j + 2$ and since J is strongly stable and x_3 is a strong Lefschetz element for S/J , the new generators are $x_3^{d_3-d_1-d_2+2j+4} x_2^{d_1+d_2-2j-4} \{x_1, x_2\}^{j+1}$ as required. Hence, we get

$$J_{d_1+d_3-2} = \{\{x_1, x_2\}^{d_1+d_3-2}, \dots, x_3^{d_3+d_1-d_2-2}\{x_1, x_2\}^{d_2}, \\ x_1^{d_1} x_3^{d_3+d_1-d_2-1}\{x_1, x_2\}^{d_2}, \dots, x_1^{d_1} x_3^{d_3-2}\},$$

and one can check that is the same formula as in the case $d_1 = 2$.

We have $|J_{d_1+d_3-1}| - |\text{Shad}(J_{d_1+d_3-2})| = d_1$ so we must add d_1 new generators to $\text{Shad}(J_{d_1+d_3-2})$ and since J is strongly stable and x_3 is a strong Lefschetz element for S/J , they are $x_3^{d_3+d_1-d_2} x_2^{d_2-d_1} \{x_1, x_2\}^{d_1-1}$. Therefore $J_{d_1+d_3-1}$ is the set

$$\{\{x_1, x_2\}^{d_1+d_3-1}, \dots, x_3^{d_3+d_1-d_2}\{x_1, x_2\}^{d_2-1}, \\ x_1^{d_1} x_3^{d_3+d_1-d_2+1}\{x_1, x_2\}^{d_2-d_1-2}, \dots, x_1^{d_1} x_3^{d_3-1}\}.$$

Suppose $d_1 > 2$. We prove by induction on $1 \leq j \leq d_2 - d_1$ that:

$$J_{d_1+d_3-2+j} = \text{Shad}(J_{d_1+d_3-3+j}) \cup \{x_3^{d_3+d_1-d_2-2+2j} x_2^{d_2-d_1+1-j} \{x_1, x_2\}^{d_1-1}\} = \\ = \{\{x_1, x_2\}^{d_1+d_3-2+j}, \dots, x_3^{d_3+d_1-d_2-2+2j} \{x_1, x_2\}^{d_2-j}, \\ x_1^{d_1} x_3^{d_3+d_1-d_2-1+2j} \{x_1, x_2\}^{d_2-d_1-j-1}, \dots, x_1^{d_1} x_3^{d_3-2+j}\}.$$

We already proved this for $j = 1$. Suppose the assertion is true for some $j < d_2 - d_1$. Since $|J_{d_1+d_3-1+j}| - |\text{Shad}(J_{d_1+d_3-2+j})| = d_1$ we must add d_1 new generators to $\text{Shad}(J_{d_1+d_3-2+j})$ and these new generators are

$$x_3^{d_3+d_1-d_2-2+2j} x_2^{d_2-d_1+1-j} \{x_1, x_2\}^{d_1-1}$$

because J is strongly stable and x_3 is a strong Lefschetz element for S/J . In particular, we get:

$$J_{d_2+d_3-2} = \{\{x_1, x_2\}^{d_2+d_3-2}, x_3\{x_1, x_2\}^{d_2+d_3-3}, \dots, x_3^{d_3+d_2-d_1-2}\{x_1, x_2\}^{d_1}\},$$

which is the same formula as in the case $d_1 = 2$.

We prove by induction on $1 \leq j \leq d_1$ that

$$J_{d_2+d_3-2+j} = \text{Shad}(J_{d_2+d_3-3+j}) \cup \{x_3^{d_3+d_2-d_1+2j-2}\{x_1, x_2\}^{d_1-j}\}.$$

If $j = 1$ then $|J_{d_2+d_3-1}| - |\text{Shad}(J_{d_2+d_3-2})| = d_1$ so we must add d_1 generators, which are precisely the elements of the set $x_3^{d_3}\{x_1, x_2\}^{d_1-1}$ since J is strongly stable and x_3 is a strong Lefschetz element for S/J . The induction step is similar and finally we obtain that $J_{d_3+2d-2} = S_{d_3+2d-2}$ and thus we cannot add new minimal generators of J in degrees $> d_3 + 2d - 2$. \square

Corollary 2.9. *In the conditions of the above proposition, the number of minimal generators of J is $1 + d_1 + d_1 d_2$.*

Example 2.10. *Let $d_1 = 3$, $d_2 = 4$ and $d_3 = 9$. Then*

$$J = (x_1^3, x_1^2 x_2^2, x_1 x_2^4, x_2^6, x_2^5 x_3, x_3^6 x_2^3 \{x_1, x_2\}, \\ x_3^8 x_2 \{x_1, x_2\}^2, x_3^{10} \{x_1, x_2\}^2, x_3^{12} \{x_1, x_2\}, x_3^{14}).$$

3 Case $d_1 + d_2 > d_3 + 1$.

- Subcase $d_1 = d_2 = d_3$.

Proposition 3.1. Let $2 \leq d := d_1 = d_2 = d_3$ be positive integers. The Hilbert function of the standard graded complete intersection $A = K[x_1, x_2, x_3]/I$, where I is the ideal generated by f_1, f_2, f_3 , with f_i homogeneous polynomials of degree d_i , for all i , with $1 \leq i \leq 3$, has the form:

1. $H(A, k) = \binom{k+2}{k}$, for $k \leq d - 1$.
2. $H(A, k) = \binom{k+2}{2} - \frac{3j(j+1)}{2}$, for $k = j + d - 1$, where $0 \leq j \leq \lfloor \frac{d-1}{2} \rfloor$.
3. $H(A, k) = H(A, 3d - k - 3)$, for $k \geq \lceil \frac{3d-3}{2} \rceil$.

Proof: It follows from [10, Lemma 2.9(b)]. \square

Corollary 3.2. In the conditions of Proposition 3.1, let $J = \text{Gin}(I)$ be the generic initial ideal of I with respect to the reverse lexicographic order. If we denote by J_k the set of monomials of J of degree k , then:

1. $|J_k| = 0$, for $k \leq d - 1$.
2. $|J_k| = \frac{3j(j+1)}{2}$, for $k = d - 1 + j$, where $0 \leq j \leq \lfloor \frac{3d-1}{2} \rfloor$.
3. If d is even, then $|J_k| = \frac{3d^2+3d(4j+2)}{8} + \frac{3j(j+1)}{2}$, for $k = j + \frac{3d-2}{2}$, where $0 \leq j \leq \frac{d-2}{2}$.
If d is odd, then $|J_k| = \frac{3(d^2-1)+12jd}{8} + \frac{3j^2}{2}$, for $k = j + \frac{3d-3}{2}$, where $0 \leq j \leq \frac{d-1}{2}$.
4. $|J_k| = \frac{3d(d-1)}{2} + 3jd$, for $k = j + 2d - 2$, where $0 \leq j \leq d - 1$.
5. $J_k = S_k$, for $k \geq 3d - 2$.

Proposition 3.3. Let $2 \leq d := d_1 = d_2 = d_3$ be positive integers. Let $f_1, f_2, f_3 \in K[x_1, x_2, x_3]$ be a regular sequence of homogeneous polynomials of degrees d_1, d_2, d_3 . If $I = (f_1, f_2, f_3)$, $J = \text{Gin}(I)$, the generic initial ideal with respect to the reverse lexicographic order, and S/I has (SLP), then:

$$J = (x_1^{d-2}\{x_1, x_2\}^2, x_1^{d-2j-1}x_2^{3j+1}, x_1^{d-2j-2}x_2^{3j+2} \text{ for } 1 \leq j \leq \frac{d-3}{2}, \\ x_2^{\frac{3d-1}{2}}, x_3x_2^{\frac{3d-3}{3}}, x_3^{2j+1}x_1^{2j}x_2^{\frac{3d-3}{2}-3j}, \dots, x_3^{2j+1}x_2^{\frac{3d-3}{2}-j}, \\ 1 \leq j \leq \frac{d-3}{2}, x_3^{d-2+2j}\{x_1, x_2\}^{d-j}, 1 \leq j \leq d)$$

if d is odd, or

$$J = (x_1^{d-2}\{x_1, x_2\}^2, x_1^{d-2j-1}x_2^{3j+1}, x_1^{d-2j-2}x_2^{3j+2} \text{ for } 1 \leq j \leq \frac{d-4}{2}, \\ x_1x_2^{\frac{3d-4}{2}}, x_2^{\frac{3d-2}{2}}, x_3^{2j}x_1^{2j-1}x_2^{\frac{3d}{2}-3j}, \dots, x_3^{2j}x_2^{\frac{3d-2}{2}-j}, \\ 1 \leq j \leq \frac{d-2}{2}, x_3^{d-2+2j}\{x_1, x_2\}^{d-j}, 1 \leq j \leq d)$$

if d is even.

Proof: We have $|J_d| = 3$, hence $J_d = \{x_1^{d-2}\{x_1, x_2\}^2\}$, since J is strongly stable and x_3 is strong Lefschetz for S/J . Therefore:

$$\text{Shad}(J_d) = \{x_1^{d-2}\{x_1, x_2\}^3, x_1^{d-2}x_3\{x_1, x_2\}^2\}.$$

Now we have four possibilities to analyze: $d = 2$, $d = 3$, $d = 4$ and $d \geq 5$.

d = 2. Using the formulae from Corrolary 3.2 we have $|J_3| - |\text{Shad}(J_2)| = 2$ so there are two generators to add to $\text{Shad}(J_2)$ to obtain J_3 . Since J is strongly stable and x_3 is a strong Lefschetz element for S/J these new generators are $x_3^2x_1$ and $x_3^2x_2$. Therefore

$$J_3 = \{\{x_1, x_2\}^3, x_3\{x_1, x_2\}^2, x_3^2\{x_1, x_2\}\}.$$

Since $|J_4| - |\text{Shad}(J_3)| = 1$ there is only one generator to add to $\text{Shad}(J_3)$ and this is precisely x_3^4 . It follows $J_4 = S_4$ and thus we cannot add new minimal generators of J in degree > 5 .

d = 3. We have $|J_4| - |\text{Shad}(J_3)| = 2$ so there are two generators to add to $\text{Shad}(J_3)$ to obtain J_4 . Since J is strongly stable and x_3 is a strong Lefschetz element for S/J these new generators are x_2^4 and $x_3x_2^3$. Therefore

$$J_4 = \{\{x_1, x_2\}^4, x_3\{x_1, x_2\}^3\}.$$

Since $|J_5| - |\text{Shad}(J_4)| = 3$ there are three new monomials to add to $\text{Shad}(J_4)$ in order to obtain J_5 . Since J is strongly stable and x_3 is a strong Lefschetz element for S/J these new generators are $x_3^3\{x_1, x_2\}^2$. Analogously, we must add two new monomial to $\text{Shad}(J_5)$ in order to obtain J_6 and these are $x_3^5\{x_1, x_2\}$. Finally, we will add x_3^7 and thus we cannot add new minimal generators of J in degree > 7 .

d = 4. We have $|J_4| - |\text{Shad}(J_3)| = 2$ so there are two generators to add to $\text{Shad}(J_4)$ to obtain J_5 . Since J is strongly stable and x_3 is a strong Lefschetz element for S/J these new generators are $x_1x_2^4$ and x_2^5 . Therefore

$$J_5 = \{\{x_1, x_2\}^5, x_1^2x_3\{x_1, x_2\}^2\}.$$

Since $|J_6| - |\text{Shad}(J_5)| = 2$ there are two new monomials to add to $\text{Shad}(J_5)$ in order to obtain J_6 and using the usual argument these new monomials are $x_3^2x_1x_2^2, x_3^2x_2^4$. It follows

$$J_6 = \{\{x_1, x_2\}^6, x_3\{x_1, x_2\}^5, x_3^2\{x_1, x_2\}^4\}.$$

Finally, we will add consequently $x_3^4\{x_1, x_2\}^3, x_3^6\{x_1, x_2\}^2, x_3^8\{x_1, x_2\}$ and x_3^{10} .

Suppose now $d \geq 5$. We have $|J_{d+1}| - |\text{Shad}(J_d)| = 2$ so there are two generators to add to $\text{Shad}(J_d)$ to obtain J_{d+1} . Since J is strongly stable and x_3 is a strong Lefschetz element for S/J these new generators are $x_1^{d-3}x_2^4, x_1^{d-4}x_2^5$. It follows

$$J_{d+1} = \{x_1^{d-4}\{x_1, x_2\}^5, x_1^{d-2}x_3\{x_1, x_2\}^3, x_1^{d-2}x_3^2\{x_1, x_2\}^2\}.$$

We prove by induction on j , with $1 \leq j \leq \lfloor \frac{d-3}{2} \rfloor$ that

$$\begin{aligned} J_{d+j} &= \text{Shad}(J_{d+j-1}) \cup \{x_1^{d-2j-1}x_2^{3j+1}, x_1^{d-2j-2}x_2^{3j+2}\} = \\ &= \{x_1^{d-2j-2}\{x_1, x_2\}^{3j+2}, x_1^{d-2j+1}x_3\{x_1, x_2\}^{3j-1}, \dots, x_1^{d-2}x_3^j\{x_1, x_2\}^2\}, \end{aligned}$$

the assertion being checked for $j = 1$. Assume now that the statement is true for some $j \leq \lfloor \frac{d-3}{2} \rfloor$. Then

$$\begin{aligned} \text{Shad}(J_{d+j}) &= \\ &= \{x_1^{d-2j-2}\{x_1, x_2\}^{3j+3}, x_1^{d-2j-2}x_3\{x_1, x_2\}^{3j+2}, \dots, x_1^{d-2}x_3^{j+1}\{x_1, x_2\}^2\}. \end{aligned}$$

Since $|J_{d+j+1}| - |\text{Shad}(J_{d+j})| = 2$ we must add two generators to $\text{Shad}(J_{d+j})$ to obtain J_{d+j+1} . Using the fact that J is strongly stable and x_3 is a strong Lefschetz element for S/J it follows that these new generators are

$$x_1^{d-2j-3}x_2^{3j+4}, x_1^{d-2j-4}x_2^{3j+5},$$

so the induction step is fulfilled.

We must consider now two possibilities.

1. d is odd. We obtain

$$J_{\frac{3d-3}{2}} = \{x_1\{x_1, x_2\}^{\frac{3d-5}{2}}, x_1^3x_3\{x_1, x_2\}^{\frac{3d-11}{2}}, \dots, x_1^{d-2}x_3^{\frac{d-3}{2}}\{x_1, x_2\}^2\}.$$

Since $|J_{\frac{3d-1}{2}}| - |\text{Shad}(J_{\frac{3d-3}{2}})| = 2$ there are two generators to add to $\text{Shad}(J_{\frac{3d-3}{2}})$ to obtain $J_{\frac{3d-1}{2}}$, and they must be $x_2^{\frac{3d-1}{2}}$, $x_3x_2^{\frac{3d-3}{2}}$ using the usual argument. Therefore,

$$J_{\frac{3d-1}{2}} = \{\{x_1, x_2\}^{\frac{3d-1}{2}}, x_3\{x_1, x_2\}^{\frac{3d-3}{2}}, \dots, x_1^{d-2}x_3^{\frac{d-1}{2}}\{x_1, x_2\}^2\}.$$

Since $|J_{\frac{3d+1}{2}}| - |\text{Shad}(J_{\frac{3d-1}{2}})| = 1$ we must add a 3 new generators to $\text{Shad}(J_{\frac{3d-1}{2}})$ to obtain $J_{\frac{3d+1}{2}}$ and since J is strongly stable and x_3 is a strong Lefschetz element for S/J , they are $x_1^2x_3^3x_2^{\frac{3d-9}{2}}$, $x_1x_3^3x_2^{\frac{3d-7}{2}}$, $x_3^3x_2^{\frac{3d-5}{2}}$.

We prove by induction on j , with $1 \leq j \leq \frac{d-3}{2}$ that

$$\begin{aligned} J_{\frac{3d-1}{2}+j} &= \{\text{Shad}(J_{\frac{3d-3}{2}+j})\} \cup \{x_3^{2j+1}x_1^{2j}x_2^{\frac{3d-3}{2}-3j}, \dots, x_3^{2j+1}x_2^{\frac{3d-3}{2}-j}\} = \\ &= \{\{x_1, x_2\}^{\frac{3d-1}{2}+j}, x_3\{x_1, x_2\}^{\frac{3d-3}{2}+j}, \dots, x_3^{2j+1}\{x_1, x_2\}^{\frac{3d-3}{2}-j}, \\ &\quad x_3^{2j+2}x_1^{2j+3}\{x_1, x_2\}^{\frac{3d-11}{2}-3j}, \dots, x_1^{d-2}x_3^{\frac{d-1}{2}+j}\{x_1, x_2\}^2\}. \end{aligned}$$

This assertion is proved for $j = 1$. Assume the assertion is true for some $j < \frac{d-3}{2}$. Since $|J_{\frac{3d+1}{2}+j}| - |\text{Shad}(J_{\frac{3d-1}{2}+j})| = 2j+3$ we must add $2j+3$ new generators to $\text{Shad}(J_{\frac{3d-1}{2}+j})$ in order to obtain $J_{\frac{3d+1}{2}+j}$. The usual argument implies that those new generators are $x_3^{2j+3}x_1^{2j+2}x_2^{\frac{3d-3}{2}-3j-3}, \dots, x_3^{2j+3}x_2^{\frac{3d-3}{2}-j-1}$, which conclude the induction.

2. d is even. We obtain

$$J_{\frac{3d-4}{2}} = \{x_1^2\{x_1, x_2\}^{\frac{3d-8}{2}}, x_1^4x_3\{x_1, x_2\}^{\frac{3d-14}{2}}, \dots, x_1^{d-2}x_3^{\frac{d-4}{2}}\{x_1, x_2\}^2\}.$$

We have $|J_{\frac{3d-2}{2}}| - |\text{Shad}(J_{\frac{3d-4}{2}})| = 2$ so we must add two new generators to $\text{Shad}(J_{\frac{3d-4}{2}})$ to obtain $J_{\frac{3d-2}{2}}$. Since J is strongly stable and x_3 is a strong Lefschetz element for S/J , they are $x_1x_2^{\frac{3d-4}{2}}$ and $x_2^{\frac{3d-2}{2}}$, therefore

$$J_{\frac{3d-2}{2}} = \left\{ \{x_1, x_2\}^{\frac{3d-2}{2}}, x_1^2x_3\{x_1, x_2\}^{\frac{3d-8}{2}}, \dots, x_1^{d-2}x_3^{\frac{d-2}{2}}\{x_1, x_2\}^2 \right\}.$$

Since $|J_{\frac{3d}{2}}| - |\text{Shad}(J_{\frac{3d-2}{2}})| = 2$ we must add two new generators to $\text{Shad}(J_{\frac{3d-2}{2}})$ in order to obtain $J_{\frac{3d}{2}}$ and since J is strongly stable and x_3 is a strong Lefschetz element for S/J , they are $x_1x_3^2x_2^{\frac{3d-6}{2}}$ and $x_3^2x_2^{\frac{3d-4}{2}}$.

We prove by induction on j , with $1 \leq j \leq \frac{d-2}{2}$ that

$$J_{\frac{3d-2}{2}+j} = \text{Shad}(J_{\frac{3d-4}{2}+j}) \cup \left\{ x_3^{2j}x_1^{2j-1}x_2^{\frac{3d}{2}-3j}, \dots, x_3^{2j}x_2^{\frac{3d-2}{2}-j} \right\}.$$

The assertion has been proved for $j = 1$ and suppose it is true for some $j < \frac{d-2}{2}$. Since $|J_{\frac{3d}{2}+j}| - |\text{Shad}(J_{\frac{3d-2}{2}+j})| = 2j + 2$ we must add $2j + 2$ generators to $\text{Shad}(J_{\frac{3d-2}{2}+j})$ in order to obtain $J_{\frac{3d}{2}+j}$ and since J is strongly stable and x_3 is a strong Lefschetz element for S/J they must be

$$x_3^{2j+2}x_1^{2j+1}x_2^{\frac{3d}{2}-3j-3}, \dots, x_3^{2j+2}x_2^{\frac{3d-4}{2}-j},$$

which conclude the induction.

Either if d is even, either if d is odd, we obtain

$$\begin{aligned} J_{2d-2} &= \left\{ \{x_1, x_2\}^{2d-2}, x_3\{x_1, x_2\}^{2d-2}, \dots, x_3^{d-2}\{x_1, x_2\}^d \right\} = \\ &= \left\{ \{x_1, x_2\}^d \{x_1, x_2, x_3\}^{d-2} \right\}. \end{aligned}$$

Since $|J_{2d-1}| - |\text{Shad}(J_{2d-2})| = d$ we must add d new generators to $\text{Shad}(J_{2d-2})$ to obtain J_{2d-1} . But J is strongly stable and x_3 is a strong Lefschetz element for S/J , so we must add $x_3^d\{x_1, x_2\}^{d-1}$, therefore $J_{2d-1} = \left\{ \{x_1, x_2\}^{d-1}\{x_1, x_2, x_3\}^d \right\}$. Using induction on $j \leq d$, we prove that

$$J_{2d-2+j} = \text{Shad}(J_{2d-3+j}) \cup \left\{ x_3^{d-2+2j}\{x_1, x_2\}^{d-j} \right\} = \left\{ x_1, x_2 \right\}^{d-j} \left\{ x_1, x_2, x_3 \right\}^{d-2+2j}.$$

For $j = 1$ we already proved. Suppose the assertion is true for $j < d$. We have $|J_{2d-1+j}| - |\text{Shad}(J_{2d-2+j})| = d - j$ so we must add $d - j$ new monomials to $\text{Shad}(J_{2d-2+j})$ to obtain J_{2d-1+j} and from the usual argument, these new monomials are $x_3^{d+2j}\{x_1, x_2\}^{d-j-1}$. Finally, since $J_{3d-2} = S_{3d-2}$ we cannot add new minimal generators of J in degree $> 3d - 2$. \square

Corollary 3.4. *In the conditions above, the number of minimal generators of J is $1 + \frac{d(d+1)}{2} + \frac{(d+1)^2}{4}$ when d is odd, or $1 + \frac{d(d+1)}{2} + \frac{d(d+2)}{4}$ when d is even.*

Example 3.5. 1. Let $d_1 = d_2 = d_3 = 5$. Then

$$J = (x_1^3\{x_1, x_2\}^2, x_1^2x_2^4, x_1x_2^5, x_2^7, x_3x_2^6, x_2^3x_3^3\{x_1, x_2\}^2, \\ x_3^5\{x_1, x_2\}^4, x_3^7\{x_1, x_2\}^3, x_3^9\{x_1, x_2\}^2, x_3^{11}\{x_1, x_2\}, x_3^{13}).$$

2. Let $d_1 = d_2 = d_3 = 6$. Then

$$J = (x_1^4\{x_1, x_2\}^2, x_1^3x_2^4, x_1^2x_2^5, x_1x_2^7, x_2^8, x_2^6x_3^2\{x_1, x_2\}, x_2^3x_3^4\{x_1, x_2\}^3, \\ x_3^6\{x_1, x_2\}^5, x_3^8\{x_1, x_2\}^4, x_3^{10}\{x_1, x_2\}^3, x_{12}^6\{x_1, x_2\}^2, x_{14}^6\{x_1, x_2\}, x_{16}^6)$$

- Subcase $d_1 = d_2 < d_3$.

Proposition 3.6. Let $2 \leq d := d_1 = d_2 < d_3$ be positive integers such that $d_1 + d_2 > d_3 + 1$. The Hilbert function of the standard graded complete intersection $A = K[x_1, x_2, x_3]/I$, where I is the ideal generated by f_1, f_2, f_3 , with f_i homogeneous polynomials of degree d_i , for all i , with $1 \leq i \leq 3$, has the form:

1. $H(A, k) = \binom{k+2}{k}$, for $k \leq d - 1$.
2. $H(A, k) = \binom{d+1}{2} + \sum_{i=1}^j (d - i)$, for $k = j + d - 1$, where $0 \leq j \leq d_3 - d$.
3. $H(A, k) = \binom{d+1}{2} + \sum_{i=1}^{d_3-d} (d - i) + \sum_{i=1}^j (2d - d_3 - 2i)$, for $k = j + d_3 - 1$, where $0 \leq j \leq \lfloor \frac{2d-d_3-1}{2} \rfloor$.
4. $H(A, k) = H(A, d_3 + 2d - 3 - k)$, for $k \geq \lceil \frac{d_3+2d-3}{2} \rceil$.

Proof: It follows from [10, Lemma 2.9(b)]. \square

Corollary 3.7. In the conditions of Proposition 3.6, let $J = \text{Gin}(I)$ be the generic initial ideal of I with respect to the reverse lexicographic order. If we denote by J_k the set of monomials of J of degree k , then:

1. $|J_k| = 0$, for $k \leq d - 1$.
2. $|J_k| = j(j + 1)$, for $k = j + d - 1$ where $0 \leq j \leq d_3 - d$.
3. $|J_k| = d_3^2 + d_3 - d^2 - d - 2dd_3 + j(2d_3 - d) + \frac{3j(j+1)}{2}$, for $k = j + d_3 - 1$, where $0 \leq j \leq \lfloor \frac{2d-d_3-1}{2} \rfloor$.
4. If d_3 is even then $|J_k| = \frac{4d^2+3d_3^2-4dd_3+4d}{8} + \frac{j(2d+d_3)}{2} + \frac{3j(j+1)}{2}$, for $k = j + \frac{2d+d_3-2}{2}$, where $0 \leq j \leq \frac{2d-d_3-2}{2}$.
If d_3 is odd then $|J_k| = \frac{3d_3^2+4d^2-4dd_3-3}{2} + \frac{j(2d+d_3-3)}{2} + \frac{3j(j+1)}{2}$, for $k = j + \frac{2d+d_3-3}{2}$, where $0 \leq j \leq \frac{2d-d_3-1}{2}$.

5. $|J_k| = 3d^2 - 2d + \frac{d_3(d_3+1)}{2} - 2dd_3 + (4d - d_3)j + j(j-1)$, for $k = j + 2d - 2$, where $0 \leq j \leq d_3 - d$.
6. $|J_k| = \frac{(d+d_3)(d+d_3-1)}{2} - \frac{d(d+1)}{2} + j(2d + d_3)$, for $k = j + d_3 + d - 2$, where $0 \leq j \leq d - 1$.
7. $J_k = S_k$, for $k \geq 3d - 2$.

Proposition 3.8. Let $2 \leq d := d_1 = d_2 < d_3$ be positive integers such that $2d > d_3 + 1$. Let $f_1, f_2, f_3 \in K[x_1, x_2, x_3]$ be a regular sequence of homogeneous polynomials of degrees d_1, d_2, d_3 . If $I = (f_1, f_2, f_3)$, $J = \text{Gin}(I)$, the generic initial ideal with respect to the reverse lexicographic order and S/I has (SLP), then if d_3 is even, we have:

$$\begin{aligned}
J &= (x_1^d, x_1^{d-1}x_2, x_1^{d-j-1}x_2^{2j+1}) \text{ for } 1 \leq j \leq d_3 - d - 1, \\
&x_1^{2d-d_3-2j+1}x_2^{2d_3-2d+3j-2}, x_1^{2d-d_3-2j}x_2^{2d_3-2d+3j-1} \text{ for } 1 \leq j \leq \frac{2d-d_3}{2}, \\
&x_3^{2j}x_1^{2j-1}x_2^{\frac{2d+d_3-2}{2}-3j}, \dots, x_3^{2j}x_2^{\frac{2d+d_3-4}{2}-j} \text{ for } 1 \leq j \leq \frac{2d-d_3-2}{2}, \\
&x_3^{2d-d_3-2+2j}x_2^{2d_3-2d+2-2j}\{x_1, x_2\}^{2d-d_3+j-2}, 1 \leq j \leq d_3 - d, \\
&x_3^{d_3-2+2j}\{x_1, x_2\}^{d-j}, 1 \leq j \leq d.
\end{aligned}$$

Otherwise, if d_3 is odd, we have

$$\begin{aligned}
J &= (x_1^d, x_1^{d-1}x_2, x_1^{d-j-1}x_2^{2j+1}) \text{ for } 1 \leq j \leq d_3 - d - 1, \\
&x_1^{2d-d_3-2j+1}x_2^{2d_3-2d+3j-2}, x_1^{2d-d_3-2j}x_2^{2d_3-2d+3j-1} \text{ for } 1 \leq j \leq \frac{2d-d_3-1}{2}, \\
&x_2^{\frac{2d+d_3-1}{2}}, x_3x_2^{\frac{2d+d_3-3}{2}}, x_3^{2j+1}x_1^{2j}x_2^{\frac{2d+d_3-3}{2}-3j}, \dots, x_3^{2j+1}x_2^{\frac{2d+d_3-3}{2}-j}, \\
1 \leq j \leq \frac{2d-d_3-3}{2}, &x_3^{2d-d_3-2+2j}x_2^{2d_3-2d+2-2j}\{x_1, x_2\}^{2d-d_3+j-2}, \\
1 \leq j \leq d_3 - d, &x_3^{d_3-2+2j}\{x_1, x_2\}^{d-j}, 1 \leq j \leq d.
\end{aligned}$$

Proof: We note first that $d \geq 3$. Indeed, if $d = 2$ then the condition $2d > d_3 + 1$ implies $d_3 = 2$ which is a contradiction. We have $|J_d| = 2$, hence $J_d = x_1^{d-1}\{x_1, x_2\}$, since J is strongly stable. Therefore:

$$\text{Shad}(J_d) = \{x_1^{d-1}\{x_1, x_2\}^2, x_1^{d-1}x_3\{x_1, x_2\}\}.$$

Assume $d_3 > d + 1$. Since $|J_{d+1}| - |\text{Shad}(J_d)| = 1$ we must add a new generator to $\text{Shad}(J_d)$ in order to obtain J_{d+1} . On the other hand, J is strongly stable and x_3 is a strong Lefschetz element for S/J so this new generator is $x_1^{d-2}x_3^3$, therefore

$$J_{d+1} = \{x_1^{d-2}\{x_1, x_2\}^3, x_1^{d-1}x_3\{x_1, x_2\}\}.$$

We prove by induction on $1 \leq j \leq d_3 - d - 1$ that

$$J_{d+j} = \text{Shad}(J_{d-1+j}) \cup \{x_1^{d-j-1}x_2^{2j+1}\} = \\ \{x_1^{d-j-1}\{x_1, x_2\}^{2j+1}, \dots, x_1^{d-1}x_3^j\{x_1, x_2\}\}.$$

The case $j = 1$ was done. Suppose the assertion is true for some $j < d_3 - d - 1$. Then, since $|J_{d+j+1}| - |\text{Shad}(J_{d+j})| = 1$ it follows that we must add one generator to $\text{Shad}(J_{d+j})$ in order to obtain J_{d+j+1} . Since J is strongly stable and x_3 is a strong Lefschetz element for S/J , this new generator must be $x_1^{d-j-2}x_2^{2j+3}$. In particular,

$$J_{d_3-1} = \{x_1^{2d-d_3}\{x_1, x_2\}^{2d_3-2d-1}, x_1^{2d-d_3+1}x_3\{x_1, x_2\}^{2d_3-2d-3}, \\ \dots, x_3^{d_3-d-1}x_1^{d-1}\{x_1, x_2\}\},$$

which is the same formula as in the case $d_3 = d + 1$.

We need to consider several possibilities. First, suppose $d_3 = 2d - 2$. We have $|J_{d_3}| - |\text{Shad}(J_{d_3-1})| = 2$ so we must add two generators to $\text{Shad}(J_{d_3-1})$ in order to obtain $J_{d_3} = J_{2d-2}$. But since J is strongly stable and x_3 is a strong Lefschetz element for S/J , these new generators are $x_1x_2^{2d-3}$ and x_2^{2d-2} .

Suppose now $d_3 < 2d - 2$. Since $|J_{d_3}| - |\text{Shad}(J_{d_3-1})| = 2$ we must add two generators to $\text{Shad}(J_{d_3-1})$ in order to obtain J_{d_3} . J strongly stable and x_3 is a strong Lefschetz element for S/J force us to choose

$$x_1^{2d-d_3-1}x_2^{2d_3-2d+1}, x_1^{2d-d_3-2}x_2^{2d_3-2d+2},$$

so

$$J_{d_3} = \{x_1^{2d-d_3-2}\{x_1, x_2\}^{2d_3-2d}, x_1^{2d-d_3}x_3\{x_1, x_2\}^{2d_3-2d-1}, \\ x_1^{2d-d_3+1}x_3^2\{x_1, x_2\}^{2d_3-2d-3}\}.$$

We show by induction on $1 \leq j \leq \lfloor \frac{2d-d_3+1}{2} \rfloor$ that

$$J_{d_3-1+j} = \text{Shad}(J_{d_3-2+j}) \cup \{x_1^{2d-d_3-2j+1}x_2^{2d_3-2d+3j-2}, x_1^{2d-d_3-2j}x_2^{2d_3-2d+3j-1}\} = \\ = \{x_1^{2d-d_3-2j}\{x_1, x_2\}^{2d_3-2d+3j-1}, x_3x_1^{2d-d_3-2j+2}\{x_1, x_2\}^{2d_3-2d+3j-4}, \dots, \\ x_3^jx_1^{2d-d_3}\{x_1, x_2\}^{2d_3-2d-1}, x_3^{j+1}x_1^{2d-d_3+1}\{x_1, x_2\}^{2d_3-2d-3}, \\ \dots, x_3^{d_3-d-1+j}x_1^{d-1}\{x_1, x_2\}\}.$$

We already done the case $j = 1$. Suppose the assertion is true for some $j < \lfloor \frac{2d-d_3+1}{2} \rfloor$. We have $|J_{d_3+j}| - |\text{Shad}(J_{d_3+j-1})| = 2$ so we add two generators to $\text{Shad}(J_{d_3+j-1})$ and they must be

$$x_1^{2d-d_3-2j-1}x_2^{2d_3-2d+3j+1}, x_1^{2d-d_3-2j-2}x_2^{2d_3-2d+3j+2}$$

from the usual argument. In the following, we distinguish between two possibilities: d is even or d is odd. If d is even, we get

$$J_{\frac{2d+d_3-4}{2}} = \{x_1^2\{x_1, x_2\}^{\frac{2d+d_3-8}{2}}, x_3x_1^4\{x_1, x_2\}^{\frac{2d+d_3-14}{2}}, \dots, \\ x_3^{\frac{2d-d_3-2}{2}}x_1^{2d-d_3}\{x_1, x_2\}^{2d_3-2d-1}, \dots, x_3^{\frac{d_3-3}{2}}x_1^{d-1}\{x_2, x_2\}\}.$$

We have $|J_{\frac{2d+d_3-2}{2}}| - |\text{Shad}(J_{\frac{2d+d_3-4}{2}})| = 2$ so we add two generators to $\text{Shad}(J_{\frac{2d+d_3-4}{2}})$ and they must be $x_1x_2^{\frac{2d+d_3-4}{2}}, x_2^{\frac{2d+d_3-2}{2}}$, so:

$$J_{\frac{2d+d_3-2}{2}} = \{\{x_1, x_2\}^{\frac{2d+d_3-2}{2}}, x_3x_1^2\{x_1, x_2\}^{\frac{2d+d_3-8}{2}}, \dots, \\ x_3^{\frac{2d-d_3}{2}}x_1^{2d-d_3}\{x_1, x_2\}^{2d_3-2d-1}, \dots, x_3^{\frac{d_3-2}{2}}x_1^{d-1}\{x_2, x_2\}\}.$$

One can easily show by induction on $1 \leq j \leq \frac{2d-d_3-2}{2}$, if case, that

$$J_{\frac{2d+d_3-2}{2}+j} = \text{Shad}(J_{\frac{2d+d_3-4}{2}+j}) \cup \{x_3^{2j}x_1^{2j-1}x_2^{\frac{2d+d_3-2}{2}-3j}, \dots, x_3^{2j}x_2^{\frac{2d+d_3-4}{2}-j}\} = \\ = \{\{x_1, x_2\}^{\frac{2d+d_3-2}{2}+j}, \dots, x_3^{2j}\{x_1, x_2\}^{\frac{2d+d_3-2}{2}-j}, x_3^{2j+1}x_1^{2j+2}\{x_1, x_2\}^{\frac{2d+d_3-8}{2}-3j}, \\ \dots, x_3^{\frac{2d-d_3}{2}+j}x_1^{2d-d_3}\{x_1, x_2\}^{2d_3-2d-1}, \dots, x_3^{\frac{d_3-2}{2}+j}x_1^{d-1}\{x_1, x_2\}\}.$$

Indeed, if $j = 1$, we have $|J_{\frac{2d+d_3}{2}}| - |\text{Shad}(J_{\frac{2d+d_3-2}{2}+j})| = 2$ so we must add two monomials to $\text{Shad}(J_{\frac{2d+d_3-2}{2}+j})$ in order to obtain $J_{\frac{2d+d_3}{2}+j}$. Since J is strongly stable and x_3 is a strong Lefschetz element for S/J , these new monomials are $x_3^2x_1x_2^{\frac{2d+d_3-2}{2}-3}$ and $x_3^2x_2^{\frac{2d+d_3-2}{2}-2}$. The induction step is similar.

If d is odd, we have $J_{\frac{2d+d_3-3}{2}} = \{x_1\{x_1, x_2\}^{\frac{2d+d_3-5}{2}}, x_3x_1^3\{x_1, x_2\}^{\frac{2d+d_3-11}{2}}, \dots, \\ x_3^{\frac{2d-d_3-1}{2}}x_1^{2d-d_3}\{x_1, x_2\}^{2d_3-2d-1}, \dots, x_3^{\frac{d_3-3}{2}}x_1^{d-1}\{x_2, x_2\}\}.$

Since $|J_{\frac{2d+d_3-1}{2}}| - |\text{Shad}(J_{\frac{2d+d_3-3}{2}})| = 2$ we add two generators to $\text{Shad}(J_{\frac{2d+d_3-3}{2}})$ in order to obtain $J_{\frac{2d+d_3-1}{2}}$. Since J is strongly stable and x_3 is a strong Lefschetz element for S/J , these new monomials are $x_2^{\frac{2d+d_3-1}{2}}, x_3x_2^{\frac{2d+d_3-3}{2}}$, therefore:

$$J_{\frac{2d+d_3-1}{2}} = \{\{x_1, x_2\}^{\frac{2d+d_3-1}{2}}, x_3\{x_1, x_2\}^{\frac{2d+d_3-3}{2}}, x_3^2x_1^3\{x_1, x_2\}^{\frac{2d+d_3-11}{2}}, \dots, \\ x_3^{\frac{2d-d_3+1}{2}}x_1^{2d-d_3}\{x_1, x_2\}^{2d_3-2d-1}, \dots, x_3^{\frac{d_3-1}{2}}x_1^{d-1}\{x_2, x_2\}\}.$$

Assume $d_3 < 2d-3$ (otherwise $\frac{2d+d_3-1}{2} = 2d-2$). $|J_{\frac{2d+d_3+1}{2}}| - |\text{Shad}(J_{\frac{2d+d_3-3}{2}})| = 3$ so we must add 3 generators to $\text{Shad}(J_{\frac{2d+d_3-3}{2}})$ to obtain $J_{\frac{2d+d_3+1}{2}}$. The usual argument implies that they are $x_1^2x_3^3x_2^{\frac{2d+d_3-9}{2}}, x_1x_3^3x_2^{\frac{2d+d_3-7}{2}}, x_3^3x_2^{\frac{2d+d_3-5}{2}}$ and thus

$$J_{\frac{2d+d_3+1}{2}} = \{\{x_1, x_2\}^{\frac{2d+d_3+1}{2}}, x_3\{x_1, x_2\}^{\frac{2d+d_3-1}{2}}, x_3^2\{x_1, x_2\}^{\frac{2d+d_3-3}{2}}, \\ x_1^2x_3^3x_2^{\frac{2d+d_3-9}{2}}, x_1x_3^3x_2^{\frac{2d+d_3-7}{2}}, x_3^3x_2^{\frac{2d+d_3-5}{2}}\}.$$

$$x_3^3 \{x_1, x_2\}^{\frac{2d+d_3-5}{2}}, \dots, x_3^{\frac{2d-d_3+3}{2}} x_1^{2d-d_3} \{x_1, x_2\}^{2d_3-2d-1}, \dots, x_3^{\frac{d_3+1}{2}} x_1^{d-1} \{x_2, x_2\}\}.$$

One can easily prove by induction on $1 \leq j \leq \frac{2d-d_3-3}{2}$ that

$$\begin{aligned} J_{\frac{2d+d_3-1}{2}+j} &= \text{Shad}(J_{\frac{2d+d_3-3}{2}+j}) \cup \{x_3^{2j+1} x_1^{2j} x_2^{\frac{2d+d_3-3}{2}-3j}, \dots, x_3^{2j+1} x_2^{\frac{2d+d_3-3}{2}-j}\} = \\ &= \{\{x_1, x_2\}^{\frac{2d+d_3-1}{2}+j}, \dots, x_3^{2j+1} \{x_1, x_2\}^{\frac{2d+d_3-3}{2}-j}, x_3^{2j+2} x_1^{2j+3} \{x_1, x_2\}^{\frac{2d+d_3-11}{2}-3j}, \\ &\quad x_3^{\frac{2d-d_3+1}{2}+j} x_1^{2d-d_3} \{x_1, x_2\}^{2d_3-2d-1}, \dots, x_3^{\frac{d_3-1}{2}+j} x_1^{d-1} \{x_1, x_2\}\}. \end{aligned}$$

In all cases above, we get:

$$\begin{aligned} J_{2d-2} &= \{\{x_1, x_2\}^{2d-2}, x_3 \{x_1, x_2\}^{2d-3}, \dots, \\ &\quad x_3^{2d-d_3-2} \{x_1, x_2\}^{d_3}, x_3^{2d-d_3-1} x_1^{2d-d_3} \{x_1, x_2\}^{2d_3-2d-1}, \dots, x_3^{d-2} x_1^{d-1} \{x_1, x_2\}\}. \end{aligned}$$

We have $|J_{2d-1}| - |\text{Shad}(J_{2d-2})| = 2d - d_3$ so we must add $2d - d_3$ new generators to $\text{Shad}(J_{2d-2})$ to obtain J_{2d-1} . We get

$$\begin{aligned} J_{2d-1} &= \{\{x_1, x_2\}^{2d-1}, x_3 \{x_1, x_2\}^{2d-2}, \dots, \\ &\quad x_3^{2d-d_3} \{x_1, x_2\}^{d_3-1}, x_3^{2d-d_3+1} x_1^{2d-d_3+1} \{x_1, x_2\}^{2d_3-2d-3}, \dots, x_3^{d-1} x_1^{d-1} \{x_1, x_2\}\}. \end{aligned}$$

One can easily show by induction of $1 \leq j \leq d_3 - d$ that

$$\begin{aligned} J_{2d-2+j} &= \text{Shad}(J_{2d-3+j}) \cup \{x_3^{2d-d_3-2+2j} x_2^{2d_3-2d+2-2j} \{x_1, x_2\}^{2d-d_3+j-2}\} = \\ &= \{\{x_1, x_2\}^{2d-2+j}, x_3 \{x_1, x_2\}^{2d-3+j}, \dots, x_3^{2d-d_3-2+2j} \{x_1, x_2\}^{d_3-j}, \\ &\quad x_3^{2d-d_3-1+2j} x_1^{2d-d_3+j} \{x_1, x_2\}^{2d_3-2d-1-2j}, \dots, x_3^{d-2+j} x_1^{d-1} \{x_1, x_2\}\}. \end{aligned}$$

In particular, we get:

$$J_{d+d_3-2} = \{\{x_1, x_2\}^{d+d_3-2}, x_3 \{x_1, x_2\}^{d+d_3-3}, \dots, x_3^{d_3-2} \{x_1, x_2\}^d\}.$$

We have $|J_{d+d_3-1}| - |\text{Shad}(J_{d+d_3-2})| = d$ so we must add d new generators to $\text{Shad}(J_{d+d_3-2})$ in order to obtain J_{d+d_3-1} . Since J is strongly stable and x_3 is a strong Lefschetz element for S/J , these new generators are $x_3^{d_3} \{x_1, x_2\}^{d-1}$ so

$$J_{d+d_3-1} = \{\{x_1, x_2\}^{d+d_3-1}, x_3 \{x_1, x_2\}^{d+d_3-2}, \dots, x_3^{d_3} \{x_1, x_2\}^{d-1}\}.$$

Now, one can easily prove by induction on $1 \leq j \leq d-1$ that J_{d+d_3-2+j} is the set

$$\begin{aligned} &\text{Shad}(J_{d+d_3-3+j}) \cup \{x_3^{d_3-2+2j} \{x_1, x_2\}^{d-j}\} = \\ &= \{\{x_1, x_2\}^{d+d_3-2+j}, \dots, x_3^{d_3-2+2j} \{x_1, x_2\}^{d-j}\}. \end{aligned}$$

Finally we obtain that $J_{d_3+2d-2} = S_{d_3+2d-2}$ and thus we cannot add new minimal generators of J in degree $> d_3 + 2d - 2$. \square

Corollary 3.9. *In the conditions of the above proposition, the number of minimal generators of J is $d(d+1) - \left(\frac{2d-d_3}{2}\right)^2 + 1$ if d_3 is even or $d(d+1) - \frac{(2d-d_3)^2-1}{4} + 1$ if d_3 is odd.*

Example 3.10. 1. Let $d_1 = d_2 = 4$ and $d_3 = 6$. We have

$$J = (x_1^4, x_1^3x_2, x_1^2x_2^3, x_1x_2^5, x_2^6, x_3^2x_2^4\{x_1, x_2\}, x_3^4x_2^2\{x_1, x_2\}^2, \\ x_3^6\{x_1, x_2\}^3, x_3^8\{x_1, x_2\}^2, x_3^{10}\{x_1, x_2\}, x_3^{12}).$$

2. Let $d_1 = d_2 = 4$ and $d_3 = 5$. We have:

$$J = (x_1^4, x_1^3x_2, x_1^2x_2^3, x_1x_2^4, x_2^6, x_2^5x_3, x_3^3x_2^2\{x_1, x_2\}^2, \\ x_3^5\{x_1, x_2\}^3, x_3^7\{x_1, x_2\}^2, x_3^9\{x_1, x_2\}, x_3^{11}).$$

- Subcase $d_1 < d_2 = d_3$.

Proposition 3.11. Let $2 \leq d_1 < d_2 = d_3 =: d$ be positive integers. The Hilbert function of the standard graded complete intersection $A = K[x_1, x_2, x_3]/I$, where I is the ideal generated by f_1, f_2, f_3 , with f_i homogeneous polynomials of degree d_i , for all i , with $1 \leq i \leq 3$, has the form:

1. $H(A, k) = \binom{k+2}{k}$, for $k \leq d_1 - 2$.
2. $H(A, k) = \binom{d_1+1}{2} + jd_1$, for $k = j + d_1 -$, where $0 \leq j \leq d - d_1$.
3. $H(A, k) = \binom{d_1+1}{2} + d_1(d - d_1) + \sum_{i=1}^j (d_1 - 2i)$, for $k = j + d - 1$, where $0 \leq j \leq \lfloor \frac{d_1-1}{2} \rfloor$.
4. $H(A, k) = H(A, d_1 + 2d - 3 - k)$, for $k \geq \lceil \frac{d_1+2d-3}{2} \rceil$.

Proof: It follows from [10, Lemma 2.9(b)]. □

Corollary 3.12. In the conditions of Proposition 3.11, let $J = \text{Gin}(I)$ be the generic initial ideal of I with respect to the reverse lexicographic order. If we denote by J_k the set of monomials of J of degree k , then:

1. $|J_k| = 0$, for $k \leq d_1 - 1$.
2. $|J_k| = j(j+1)/2$, for $k = j + d_1 - 1$, where $0 \leq j \leq d - d_1$.
3. $|J_k| = \frac{(d-d_1)(d-d_1-1)}{2} + j(d-d_1) + \frac{3j(j+1)}{2}$, for $k = j + d - 1$, where $0 \leq j \leq \lfloor \frac{d_1-1}{2} \rfloor$.

4. If d_1 is even then $|J_k| = \frac{3d_1^2+2d_1+4d^2+4d-4dd_1}{8} + \frac{j(2d+d_1)}{2} + \frac{3j(j+1)}{2}$, for $k = j + \frac{2d+d_1-2}{2}$, where $0 \leq j \leq \frac{d_1-2}{2}$.
- If d_1 is odd then $|J_k| = \frac{3d_1^2+4d^2-4dd_1-3}{2} + \frac{j(2d+d_1)}{2} + \frac{3j^2}{2}$ for $k = j + \frac{2d+d_1-3}{2}$, where $0 \leq j \leq \frac{d_1-1}{2}$.
5. $|J_k| = \frac{d(d-1)}{2} + d_1(d_1-1) + j(2d_1+d) + \frac{j(j-1)}{2}$, for $k = j + d_1 + d - 2$, where $0 \leq j \leq d - d_1$.
6. $|J_k| = \frac{2d(2d-1)-d_1(d_1-1)}{2} + j(2d+d_1)$, for $k = j+2d-2$, where $0 \leq j \leq d_1-1$.
7. $J_k = S_k$, for $k \geq 3d - 2$.

Proposition 3.13. Let $2 \leq d_1 < d_2 = d_3 =: d$ be positive integers. Let $f_1, f_2, f_3 \in K[x_1, x_2, x_3]$ be a regular sequence of homogeneous polynomials of degrees d_1, d_2, d_3 . Let $I = (f_1, f_2, f_3)$, $J = \text{Gin}(I)$, the generic initial ideal with respect to the reverse lexicographic order and S/I has (SLP), then if d_1 is even we have:

$$J = (x_1^{d_1}, x_1^{d_1-2j+1} x_2^{d-d_1-2+3j}, x_1^{d_1-2j} x_2^{d-d_1-1+3j} \text{ for } 1 \leq j \leq \frac{d_1-2}{2},$$

$$x_1 x_2^{\frac{d_1+2d-4}{2}}, x_2^{\frac{d_1+2d-4}{2}}, x_3^{2j} x_1^{2j-1} x_2^{\frac{d_1+2d}{2}-3j}, \dots, x_3^{2j} x_2^{\frac{d_1+2d-2}{2}-j} \text{ for } 1 \leq j \leq \frac{d_1-4}{2},$$

$$x_3^{d_1-2+2j} x_2^{d-d_1-j+1} \{x_1, x_2\}^{d_1-1} \text{ for } 1 \leq j \leq d - d_1,$$

$$x_3^{2d-d_1-2+2j} \{x_1, x_2\}^{d_1-j} \text{ for } 1 \leq j \leq d_1).$$

Otherwise, if d_1 is odd, then:

$$J = (x_1^{d_1}, x_1^{d_1-2j+1} x_2^{d-d_1-2+3j}, x_1^{d_1-2j} x_2^{d-d_1-1+3j} \text{ for } 1 \leq j \leq \frac{d_1-1}{2},$$

$$x_2^{\frac{d_1+2d-1}{2}}, x_3 x_2^{\frac{d_1+2d-3}{2}}, x_3^{2j+1} x_1^{2j} x_2^{\frac{d_1+2d-3}{2}-3j}, \dots, x_3^{2j+1} x_2^{\frac{d_1+2d-3}{2}-3j}$$

$$\text{for } 1 \leq j \leq \frac{d_1-3}{2}, x_3^{d_1-2+2j} x_2^{d-d_1-j+1} \{x_1, x_2\}^{d_1-1} \text{ for } 1 \leq j \leq d - d_1,$$

$$x_3^{2d-d_1-2+2j} \{x_1, x_2\}^{d_1-j} \text{ for } 1 \leq j \leq d_1).$$

Proof: We have $|J_{d_1}| = 1$, hence $J_{d_1} = \{x_1^{d_1}\}$, since J is a strongly stable. Therefore:

$$\text{Shad}(J_{d_1}) = \{x_1^{d_1} \{x_1, x_2\}, x_1^{d_1} x_3\}.$$

Assume $d > d_1 + 1$. Using the formulae from 3.12 we get $|J_{d_1+1}| - |\text{Shad}(J_{d_1})| = 0$ so $J_{d_1+1} = \text{Shad}(J_{d_1})$. We show by induction on $1 \leq j \leq d - d_1$ that

$$J_{d_1+j} = \text{Shad}(J_{d_1-1+j}) = \{x_1^{d_1} \{x_1, x_2\}^j, x_1^{d_1} x_3 \{x_1, x_2\}^{j-1}, \dots, x_1^{d_1} x_3^j\}.$$

We already prove this for $j = 1$. Suppose the assertion is true for some $j < d - d_1$. Since $|J_{d_1+j}| = |Shad(J_{d_1-1+j})|$ we have $J_{d_1+j} = Shad(J_{d_1-1+j})$ thus the induction step is done. In particular, we get:

$$J_{d-1} = \{x_1^{d_1} \{x_1, x_2\}^{d-d_1-1}, x_1^{d_1} x_3 \{x_1, x_2\}^{d-d_1-2}, \dots, x_1^{d_1} x_3^{d-d_1-1}\}.$$

which is the same expression as in the case $d = d_1 + 1$.

In the following, we consider two possibilities. First, suppose $d_1 = 2$. We have $|J_d| - |Shad(J_{d-1})| = 2$ so we must add two new generators to $Shad(J_{d-1})$ in order to obtain J_d . Since J is strongly stable and x_3 is a strong Lefschetz element for S/J these new generators are $x_1 x_2^{d-1}$ and x_2^d .

Suppose now $d_1 > 2$. We have $|J_d| - |Shad(J_{d-1})| = 2$ so we must add two new generators to $Shad(J_{d-1})$ in order to obtain J_d . Since J is strongly stable and x_3 is a strong Lefschetz element for S/J these new generators are $x_1^{d_1-1} x_2^{d-d_1+1}, x_1^{d_1-2} x_2^{d-d_1+2}$. Therefore

$$J_d = \{x_1^{d_1-2} \{x_1, x_2\}^{d-d_1}, x_1^{d_1} x_3 \{x_1, x_2\}^{d-d_1-1}, \dots, x_1^{d_1} x_3^{d-d_1}\}.$$

We prove by induction on $1 \leq j \leq \lfloor \frac{d_1-1}{2} \rfloor$ that:

$$\begin{aligned} J_{d-1+j} &= Shad(J_{d-2+j}) \cup \{x_1^{d_1-2j+1} x_2^{d-d_1-2+3j}, x_1^{d_1-2j} x_2^{d-d_1-1+3j}\} = \\ &= \{x_1^{d_1-2j} \{x_1, x_2\}^{d-d_1-1-3j}, x_1^{d_1-2j+2} x_3 \{x_1, x_2\}^{d-d_1-4-3j}, \dots, \\ & x_1^{d_1} x_3^j \{x_1, x_2\}^{d-d_1-1}, x_1^{d_1} x_3^{j+1} \{x_1, x_2\}^{d-d_1-2}, \dots, x_1^{d_1} x_3^{d-d_1+j-1}\}. \end{aligned}$$

We already done the case $j = 1$. Suppose the assertion is true for some $j < \lfloor \frac{d_1-1}{2} \rfloor$. Since $|J_{d+j}| - |Shad(J_{d-1+j})| = 2$ we must add two generators to $Shad(J_{d-1+j})$ in order to obtain J_{d+j} and they are

$$x_1^{d_1-2j-1} x_2^{d-d_1+3j+1}, x_1^{d_1-2j-2} x_2^{d-d_1+3j+2}$$

because J is strongly stable and x_3 is a strong Lefschetz element for S/J . Therefore, the induction step is done.

In the following, we consider two possibilities: d_1 is even or d_1 is odd. First, suppose d_1 is even. We have

$$\begin{aligned} J_{\frac{d_1+2d-4}{2}} &= \{x_1^2 \{x_1, x_2\}^{\frac{d_1+2d-8}{2}}, x_1^4 x_3 \{x_1, x_2\}^{\frac{d_1+2d-14}{2}}, \dots, x_1^{d_1} x_3^{\frac{d_1-2}{2}} \{x_1, x_2\}^{d-d_1-1}, \\ & x_1^{d_1} x_3^{\frac{d_1}{2}} \{x_1, x_2\}^{d-d_1-2}, \dots, x_1^{d_1} x_3^{\frac{2d-d_1-4}{2}}\}. \end{aligned}$$

Since $|J_{\frac{d_1+2d-2}{2}}| - |Shad(J_{\frac{d_1+2d-4}{2}})| = 2$ we need to add two new monomials to $Shad(J_{\frac{d_1+2d-4}{2}})$ and since J is strongly stable and x_3 is a strong Lefschetz element

for S/J , they are $x_1 x_2^{\frac{d_1+2d-4}{2}}, x_2^{\frac{d_1+2d-4}{2}}$, thus:

$$J_{\frac{d_1+2d-2}{2}} = \{\{x_1, x_2\}^{\frac{d_1+2d-2}{2}}, x_3 x_1^2 \{x_1, x_2\}^{\frac{d_1+2d-8}{2}}, x_1^4 x_3^2 \{x_1, x_2\}^{\frac{d_1+2d-14}{2}}, \dots,$$

$$x_1^{d_1} x_3^{\frac{d_1}{2}} \{x_1, x_2\}^{d-d_1-1}, x_1^{d_1} x_3^{\frac{d_1+2}{2}} \{x_1, x_2\}^{d-d_1-2}, \dots, x_1^{d_1} x_3^{\frac{2d-d_1-2}{2}} \}.$$

One can easily show by induction on $1 \leq j \leq \frac{d_1-2}{2}$ that

$$\begin{aligned} J_{\frac{d_1+2d-2}{2}+j} &= \text{Shad}(J_{\frac{d_1+2d-4}{2}+j}) \cup \{x_3^{2j} x_1^{2j-1} x_2^{\frac{d_1+2d}{2}-3j}, \dots, x_3^{2j} x_2^{\frac{d_1+2d-2}{2}-j}\} = \\ &= \{\{x_1, x_2\}^{\frac{d_1+2d-2}{2}+j}, x_3 \{x_1, x_2\}^{\frac{d_1+2d-4}{2}+j}, \dots, x_3^{2j} \{x_1, x_2\}^{\frac{d_1+2d-2}{2}-j}, \\ &x_3^{2j+1} x_1^{2j+2} \{x_1, x_2\}^{\frac{d_1+2d-8}{2}-3j}, \dots, x_3^{\frac{d_1}{2}} x_1^{d_1} \{x_1, x_2\}^{d-d_1-1}, \dots, x_1^{d_1} x_3^{\frac{2d-d_1-2}{2}+j}\}. \end{aligned}$$

The assertion was already done for $j = 1$ and the induction step is similar.

If d_1 is odd, we get

$$\begin{aligned} J_{\frac{d_1+2d-3}{2}} &= \{x_1 \{x_1, x_2\}^{\frac{d_1+2d-5}{2}}, x_1^3 x_3 \{x_1, x_2\}^{\frac{d_1+2d-11}{2}}, \dots, x_1^{d_1} x_3^{\frac{d_1-1}{2}} \{x_1, x_2\}^{d-d_1-1}, \\ &x_1^{d_1} x_3^{\frac{d_1+1}{2}} \{x_1, x_2\}^{d-d_1-2}, \dots, x_1^{d_1} x_3^{\frac{2d-d_1-3}{2}} \}. \end{aligned}$$

Since $|J_{\frac{d_1+2d-1}{2}}| - |\text{Shad}(J_{\frac{d_1+2d-3}{2}})| = 2$ we add two generators to

$\text{Shad}(J_{\frac{d_1+2d-3}{2}})$ in order to obtain $J_{\frac{d_1+2d-1}{2}}$ and they must be $x_2^{\frac{d_1+2d-1}{2}}, x_3 x_2^{\frac{d_1+2d-3}{2}}$, therefore:

$$\begin{aligned} J_{\frac{d_1+2d-1}{2}} &= \{\{x_1, x_2\}^{\frac{d_1+2d-1}{2}}, x_3 \{x_1, x_2\}^{\frac{d_1+2d-3}{2}}, x_1^3 x_3^2 \{x_1, x_2\}^{\frac{d_1+2d-11}{2}}, \dots, \\ &x_1^{d_1} x_3^{\frac{d_1+1}{2}} \{x_1, x_2\}^{d-d_1-1}, x_1^{d_1} x_3^{\frac{d_1+3}{2}} \{x_1, x_2\}^{d-d_1-2}, \dots, x_1^{d_1} x_3^{\frac{2d-d_1-1}{2}} \}. \end{aligned}$$

Since $|J_{\frac{d_1+2d+1}{2}}| - |\text{Shad}(J_{\frac{d_1+2d-1}{2}})| = 3$, we add 3 new generators to

$\text{Shad}(J_{\frac{d_1+2d-1}{2}})$ in order to obtain $J_{\frac{d_1+2d+1}{2}}$ and they are

$$x_1^2 x_3^3 x_2^{\frac{d_1+2d-9}{2}}, x_1 x_3^3 x_2^{\frac{d_1+2d-7}{2}}, x_3^3 x_2^{\frac{d_1+2d-5}{2}},$$

therefore

$$\begin{aligned} J_{\frac{d_1+2d+1}{2}} &= \{\{x_1, x_2\}^{\frac{d_1+2d+1}{2}}, x_3 \{x_1, x_2\}^{\frac{d_1+2d-1}{2}}, x_3^2 \{x_1, x_2\}^{\frac{d_1+2d-3}{2}}, \\ &x_3^3 \{x_1, x_2\}^{\frac{d_1+2d-5}{2}}, \dots, x_1^{d_1} x_3^{\frac{d_1+3}{2}} \{x_1, x_2\}^{d-d_1-1}, \\ &x_1^{d_1} x_3^{\frac{d_1+5}{2}} \{x_1, x_2\}^{d-d_1-2}, \dots, x_1^{d_1} x_3^{\frac{2d-d_1+1}{2}} \}. \end{aligned}$$

One can easily prove by induction on $1 \leq j \leq \frac{d_1-3}{2}$ that

$$\begin{aligned} J_{\frac{d_1+2d-1}{2}+j} &= \text{Shad}(J_{\frac{d_1+2d-3}{2}+j}) \cup \{x_3^{2j+1} x_1^{2j} x_2^{\frac{d_1+2d-3}{2}-3j}, \dots, x_3^{2j+1} x_2^{\frac{d_1+2d-3}{2}-3j}\} = \\ &= \{\{x_1, x_2\}^{\frac{d_1+2d-1}{2}+j}, x_3 \{x_1, x_2\}^{\frac{d_1+2d-3}{2}+j}, \dots, x_3^{2j+2} \{x_1, x_2\}^{\frac{d_1+2d-3}{2}-j}, \end{aligned}$$

$$x_3^{2j+2} x_1^{2j+3} \{x_1, x_2\}^{\frac{d_1+2d-11}{2}-3j}, \dots, x_3^{\frac{d_1+1}{2}} x_1^{d_1} \{x_1, x_2\}^{d-d_1-1}, \dots, x_1^{d_1} x_3^{\frac{2d-d_1-1}{2}+j} \}.$$

The assertion was already proved for $j = 1$ and the induction step is similar.

In all cases, we obtain

$$J_{d_1+d-2} = \{ \{x_1, x_2\}^{d_1+d-2}, x_3 \{x_1, x_2\}^{d_1+d-3}, \dots, x_3^{d_1-2} \{x_1, x_2\}^d, \\ x_1^{d_1} x_3^{d_1-1} \{x_1, x_2\}^{d-d_1-1}, \dots, x_1^{d_1} x_3^{d-2} \}.$$

We have $|J_{d_1+d-1}| - |\text{Shad}(J_{d_1+d-2})| = d_1$ so we must add d_1 new monomials to $\text{Shad}(J_{d_1+d-2})$ in order to obtain J_{d_1+d-1} . Since J is strongly stable and x_3 is a strong Lefschetz element for S/J , these new monomials are $x_3^{d_1} x_2^{d-d_1} \{x_1, x_2\}^{d_1-1}$, therefore

$$J_{d_1+d-1} = (\{x_1, x_2\}^{d_1+d-1}, x_3 \{x_1, x_2\}^{d_1+d-2}, \dots, x_3^{d_1} \{x_1, x_2\}^{d-1}, \\ x_1^{d_1} x_3^{d_1+1} \{x_1, x_2\}^{d-d_1-2}, \dots, x_1^{d_1} x_3^{d-1}).$$

One can easily prove by induction on $1 \leq j \leq d - d_1$ that

$$J_{d_1+d-1+j} = \text{Shad}(J_{d_1+d-2+j}) \cup \{x_3^{d_1-2+2j} x_2^{d-d_1-j+1} \{x_1, x_2\}^{d_1-1}\} = \\ = \{ \{x_1, x_2\}^{d_1+d-2+j}, x_3 \{x_1, x_2\}^{d_1+d-3+j}, \dots, x_3^{d_1-2+2j} \{x_1, x_2\}^{d-j}, \\ x_1^{d_1} x_3^{d_1-1+2j} \{x_1, x_2\}^{d-d_1-j-1}, \dots, x_1^{d_1} x_3^{d-2+j} \}$$

the case $j = 1$ being already done and then, the induction step being similar. In particular, $J_{2d-2} = \{ \{x_1, x_2\}^{2d-2}, x_3 \{x_1, x_2\}^{2d-3}, \dots, x_3^{2d-d_1-2} \{x_1, x_2\}^{d_1} \}.$

We have $|J_{2d-1}| - |\text{Shad}(J_{2d-2})| = d_1$ so we must add d_1 new generators to $\text{Shad}(J_{2d-2})$ in order to obtain J_{2d-1} . Since J is strongly stable and x_3 is a strong Lefschetz element for S/J , these new monomials are $x_3^{2d-d_1} \{x_1, x_2\}^{d_1-1}$ so

$$J_{2d-1} = \{ \{x_1, x_2\}^{2d-1}, x_3 \{x_1, x_2\}^{2d-2}, \dots, x_3^{2d-d_1} \{x_1, x_2\}^{d_1-1} \}.$$

One can easily show by induction on $1 \leq j \leq d_1$ that

$$J_{2d-2+j} = \text{Shad}(J_{2d-3+j}) \cup \{x_3^{2d-d_1-2+2j} \{x_1, x_2\}^{d_1-j}\} = \\ = \{ \{x_1, x_2\}^{2d-2+j}, \dots, x_3^{2d-d_1-2+2j} \{x_1, x_2\}^{d_1-j} \}.$$

Finally, we obtain $J_{d_1+2d-2} = S_{d_1+2d-2}$ and therefore we cannot add new minimal generators of J in degrees $> d_1 + 2d - 2$. \square

Corollary 3.14. *In the conditions of the above proposition, the number of minimal generators of J is $d_1(d+1) - \binom{d_1}{2} + 1$ if d is even; $d_1(d+1) - \frac{d_1-1}{4} + 1$ if d is odd.*

Example 3.15. If $d_1 = 4$, $d_2 = d_3 = 6$, then $J = (x_1^4, x_1^3x_2^3, x_1^2x_2^4, x_2^7, x_3x_2^6, x_3^2x_1x_2^5, x_3^2x_2^6, x_3^4x_2^2\{x_1, x_2\}^3, x_3^6x_2\{x_1, x_2\}^3, x_3^8\{x_1, x_2\}^3, x_3^{10}\{x_1, x_2\}^2, x_3^{12}\{x_1, x_2\}, x_3^{14})$.

If $d_1 = 3$ and $d_2 = d_3 = 6$, then: $J = (x_1^3, x_1^2x_2^4, x_1x_2^5, x_1x_2^6, x_2^7, x_3^2x_2^3\{x_1, x_2\}^2, x_3^5x_2^2\{x_1, x_2\}^2, x_3^7x_2\{x_1, x_2\}^2, x_3^9\{x_1, x_2\}^2, x_3^{11}\{x_1, x_2\}, x_3^{11})$.

- Subcase $d_1 < d_2 < d_3$.

Proposition 3.16. Let $2 \leq d_1 < d_2 < d_3$ be positive integers such that $d_1 + d_2 > d_3 + 1$. The Hilbert function of the standard graded complete intersection $A = K[x_1, x_2, x_3]/I$, where I is the ideal generated by f_1, f_2, f_3 , with f_i homogeneous polynomials of degree d_i , for all i , with $1 \leq i \leq 3$, has the form:

1. $H(A, k) = \binom{k+2}{k}$, for $k \leq d_1 - 2$.
2. $H(A, k) = \binom{d_1+1}{2} + jd_1$, for $k = j + d_1 - 1$, where $0 \leq j \leq d_2 - d_1$.
3. $H(A, k) = \binom{d_1+1}{2} + d_1(d_2 - d_1) + \sum_{i=1}^j (d_1 - i)$, for $k = j + d_2 - 1$, where $0 \leq k \leq d_3 - d_2$.
4. $H(A, k) = \binom{d_1+1}{2} + d_1(d_2 - d_1) + \sum_{i=1}^{d_3-1} (d_1 - j) + \sum_{i=1}^j (d_1 + d_2 - d_3 - 2i)$, for $k = j + d_3 - 1$, where $0 \leq j \leq \lfloor \frac{d_1+d_2-d_3-1}{2} \rfloor$.
5. $H(A, k) = H(A, d_1 + d_2 + d_3 - 3 - k)$ for $k > d_3 - 1 + \lfloor \frac{d_1+d_2-d_3-1}{2} \rfloor$.

Proof: It follows from [10, Lemma 2.9(b)]. \square

Corollary 3.17. In the conditions of Proposition 3.16, let $J = \text{Gin}(I)$ be the generic initial ideal of I with respect to the reverse lexicographic order. If we denote by J_k the set of monomials of J of degree k , then:

1. $|J_k| = 0$, for $k \leq d_1 - 1$.
2. $|J_k| = j(j+1)/2$, for $k = j + d_1 - 1$, where $0 \leq j \leq d_2 - d_1$.
3. $|J_k| = d_2(d_2 - 1) + j(d_2 - d_1) + j(j+1)$, for $k = j + d_2 - 1$, where $0 \leq j \leq d_3 - d_2$.
4. $|J_k| = |J_{d_3-1}| + j(2d_3 - d_1 - d_2) + \frac{3j(j+1)}{2}$, for $k = j + d_3 - 1$, where $0 \leq j \leq \lfloor \frac{d_1-1}{2} \rfloor$.
5. If $d_1 + d_2 + d_3$ is even then $|J_k| = |J_{\frac{d_1+d_2+d_3-4}{2}}| + \frac{(j+1)(d_1+d_2+d_3)}{2} + \frac{3j(j+1)}{2}$, for $k = j + \frac{d_1+d_2+d_3-2}{2}$, where $0 \leq j \leq \frac{d_1+d_2-d_3-2}{2}$.
If $d_1 + d_2 + d_3$ is odd then $|J_k| = |J_{\frac{d_1+d_2+d_3-3}{2}}| + \frac{j(d_1+d_2+d_3)}{2} + \frac{3j^2}{2}$, for $k = j + \frac{d_1+d_2+d_3-3}{2}$, where $0 \leq j \leq \frac{d_1+d_2-d_3-1}{2}$.

6. $|J_k| = |J_{d_1+d_2-2}| + j(2d_1 + 2d_2 - d_3 - 1) + j^2$, for $k = j + d_1 + d_2 - 2$, where $0 \leq d_3 - d_2$.
7. $|J_k| = |J_{d_1+d_3-2}| + j(2d_1 + d_3 - 1) + \frac{j(j+1)}{2}$, for $k = j + d_1 + d_3 - 2$, where $0 \leq j \leq d_2 - d_1$.
8. $|J_k| = |J_{d_2+d_3-1}| + j(d_1 + d_2 + d_3)$, for $k = j + d_1 + d_3 - 2$, where $0 \leq j \leq d_1 - 1$.
9. $J_k = S_k$, for $k \geq 3d - 2$.

Proposition 3.18. Let $2 \leq d_1 < d_2 < d_3$ be positive integers such that $d_1 + d_2 > d_3 + 1$. Let $f_1, f_2, f_3 \in K[x_1, x_2, x_3]$ be a regular sequence of homogeneous polynomials of degrees d_1, d_2, d_3 . Let $\alpha = d_1 + d_2 - d_3$. Let $I = (f_1, f_2, f_3)$, $J = \text{Gin}(I)$, the generic initial ideal with respect to the reverse lexicographic order, and suppose S/I has (SLP). If α is even, then:

$$\begin{aligned}
J = & (x_1^{d_1}, x_1^{d_1-1} x_2^{d_2-d_1+1}, x_1^{d_1-2} x_2^{d_2-d_1+3}, \dots, x_1^{d_1+d_2-d_3} x_2^{2d_3-d_1-d_2-1}, \\
& x_1^{d_1+d_2-d_3-2j} x_2^{2d_3-d_1-d_2+3j-1}, \\
& x_1^{d_1+d_2-d_3-2j+1} x_2^{2d_3-d_1-d_2+3j-2} \text{ for } j = 1, \dots, \frac{\alpha-2}{2}, \\
& x_2^{\frac{d_1+d_2+d_3-2}{2}}, x_1 x_2^{\frac{d_1+d_2+d_3-4}{2}}, x_3^{2j} x_2^{\frac{d_1+d_2+d_3}{2}-3j} \{x_1, x_2\}^{2j-1} \text{ for } j = 1, \dots, \frac{\alpha-2}{2}, \\
& x_1^{d_1+d_2-d_3+j-2} x_2^{2d_3-d_1-d_2-2j+2} x_3^{d_1+d_2-d_3+2j-2}, \\
& \dots, x_2^{d_3-j} x_3^{d_1+d_2-d_3+2j-2} \text{ for } j = 1, \dots, d_3 - d_2, \\
& x_1^{d_1-1} x_3^{d_1+d_3-d_2+2j-2} x_2^{d_2-d_1-j+1}, \dots, x_3^{d_1+d_3-d_2+2j-2} x_2^{d_2-j} \text{ for } j = 1, \dots, d_2 - d_1, \\
& \{x_1, x_2\}^{d_1-j} x_3^{d_2+d_3-d_1-2+2j} \text{ for } j = 1, \dots, d_1).
\end{aligned}$$

Otherwise, if α is odd, then:

$$\begin{aligned}
J = & (x_1^{d_1}, x_1^{d_1-1} x_2^{d_2-d_1+1}, x_1^{d_1-2} x_2^{d_2-d_1+3}, \dots, x_1^{d_1+d_2-d_3} x_2^{2d_3-d_1-d_2-1}, \\
& x_1^{d_1+d_2-d_3-2j} x_2^{2d_3-d_1-d_2+3j-1}, x_1^{d_1+d_2-d_3-2j+1} x_2^{2d_3-d_1-d_2+3j-2}, j = 1, \dots, \frac{\alpha-1}{2}, \\
& x_2^{\frac{d_1+d_2+d_3-1}{2}}, x_3 x_2^{\frac{d_1+d_2+d_3-3}{2}}, x_1^{2j} x_3^{2j+1} x_2^{\frac{d_1+d_2+d_3-3}{2}-3j}, \\
& \dots, x_3^{2j+1} x_2^{\frac{d_1+d_2+d_3-3}{2}-j}, j = 1, \dots, \frac{\alpha-3}{2}, \\
& x_1^{d_1+d_2-d_3+j-2} x_2^{2d_3-d_1-d_2-2j+2} x_3^{d_1+d_2-d_3+2j-2}, \\
& \dots, x_2^{d_3-j} x_3^{d_1+d_2-d_3+2j-2} \text{ for } j = 1, \dots, d_3 - d_2, \\
& x_1^{d_1-1} x_3^{d_1+d_3-d_2+2j-2} x_2^{d_2-d_1-j+1}, \dots, x_3^{d_1+d_3-d_2+2j-2} x_2^{d_2-j} \text{ for } j = 1, \dots, d_2 - d_1, \\
& \{x_1, x_2\}^{d_1-j} x_3^{d_2+d_3-d_1-2+2j} \text{ for } j = 1, \dots, d_1).
\end{aligned}$$

Proof: We have $|J_{d_1}| = 1$, hence $J_{d_1} = \{x_1^{d_1}\}$, since J is strongly stable. Therefore:

$$\text{Shad}(J_{d_1}) = \{x_1^{d_1}\{x_1, x_2\}, x_1^{d_1}x_3\}.$$

Assume $d_2 > d_1 + 1$. Using the formulae from 3.17 we get $|J_{d_1+1}| - |\text{Shad}(J_{d_1})| = 0$, therefore

$$J_{d_1+1} = \text{Shad}(J_{d_1}) = \{x_1^{d_1}\{x_1, x_2\}, x_1^{d_1}x_3\}.$$

Using induction on $1 \leq j \leq d_2 - d_1 - 1$ we prove that

$$J_{d_1+j} = \text{Shad}(J_{d_1+j-1}) = \{x_1^{d_1}\{x_1, x_2\}^j, \dots, x_3^j x_1^{d_1}\}.$$

Indeed, this assertion was already proved for $j = 1$, and if we suppose that is true for some $j < d_2 - d_1 - 1$ we get $|J_{d_1+j+1}| = |\text{Shad}(J_{d_1+j})|$ so we are done. In particular, we obtain

$$J_{d_2-1} = \{x_1^{d_1}\{x_1, x_2\}^{d_2-d_1-1}, x_3x_1^{d_1}\{x_1, x_2\}^{d_2-d_1-2}, \dots, x_3^{d_2-d_1-1}x_1^{d_1}\}.$$

We have $|J_{d_2}| - |\text{Shad}(J_{d_2-1})| = 1$ so we must add a new generator to $\text{Shad}(J_{d_2-1})$ in order to obtain J_{d_2} . But since J is strongly stable and x_3 is a strong Lefschetz element for S/J , this new generator is $x_1^{d_1-1}x_2^{d_2-d_1+1}$ and therefore $J_{d_2} = \{x_1^{d_1-1}\{x_1, x_2\}^{d_2-d_1+1}, x_3x_1^{d_1}\{x_1, x_2\}^{d_2-d_1-1}, \dots, x_3^{d_2-d_1}x_1^{d_1}\}$.

We show by induction on $1 \leq j \leq d_3 - d_2$ that

$$\begin{aligned} J_{d_2-1+j} &= \text{Shad}(J_{d_2-2+j}) \cup \{x_1^{d_1-j}x_2^{d_2-d_1+2j-1}\} = \\ &= \{x_1^{d_1-j}\{x_1, x_2\}^{d_2-d_1+2j-1}, \dots, x_3^j x_1^{d_1}\{x_1, x_2\}^{d_2-d_1-1}, \dots, x_3^{d_2-d_1+j-1}x_1^{d_1}\}. \end{aligned}$$

The first step of induction was already done. Suppose the assertion is true for some $j < d_3 - d_2$. Since $|J_{d_2+j}| - |\text{Shad}(J_{d_2-1+j})| = 1$ and J is strongly stable and x_3 is a strong Lefschetz element for S/J , we add $x_1^{d_1-j-1}x_2^{d_2-d_1+2j+1}$ to $\text{Shad}(J_{d_2-1+j})$ in order to obtain J_{d_2+j} . Thus, we are done. In particular, we get

$$\begin{aligned} J_{d_3-1} &= \{x_1^{d_1+d_2-d_3}\{x_1, x_2\}^{2d_3-d_1-d_2-1}, x_3x_1^{d_1+d_2-d_3+1}\{x_1, x_2\}^{2d_3-d_1-d_2-3}, \dots, \\ &\quad x_3^{d_3-d_2}x_1^{d_1}\{x_1, x_2\}^{d_2-d_1-1}, \dots, x_3^{d_3-d_1-1}x_1^{d_1}\}. \end{aligned}$$

We consider first $\alpha = 2$, i.e. $d_3 = d_1 + d_2 - 2$. In this case, since $|J_{d_3}| - |\text{Shad}(J_{d_3-1})| = 2$ we add two new generators to $\text{Shad}(J_{d_3-1})$ in order to obtain J_{d_3} . Since J is strongly stable and x_3 is a strong Lefschetz element for S/J , these new generators are $x_2^{d_3}$ and $x_1x_2^{d_3-1}$.

Suppose now $\alpha \geq 2$. We have $|J_{d_3}| - |\text{Shad}(J_{d_3-1})| = 2$ so we must add two new generators to $\text{Shad}(J_{d_3-1})$ to obtain J_{d_3} . Since J is strongly stable and x_3 is a strong Lefschetz element for S/J , these new generators are $x_1^{d_1+d_2-d_3-2}x_2^{2d_3-d_1-d_2+2}$ and $x_1^{d_1+d_2-d_3-1}x_2^{2d_3-d_1-d_2+1}$, therefore

$$J_{d_3} = \{x_1^{d_1+d_2-d_3-2}\{x_1, x_2\}^{2d_3-d_1-d_2+2}, x_3x_1^{d_1+d_2-d_3}\{x_1, x_2\}^{2d_3-d_1-d_2-1},$$

$$x_3^2 x_1^{d_1+d_2-d_3+1} \{x_1, x_2\}^{2d_3-d_1-d_2-3}, \dots, x_3^{d_3-d_2} x_1^{d_1} \{x_1, x_2\}^{d_2-d_1}, \dots, x_3^{d_3-d_1} x_1^{d_1} \}.$$

One can prove by induction on $1 \leq j \leq \lfloor \frac{\alpha-1}{2} \rfloor$ that J_{d_3-1+j} is the set

$$\begin{aligned} & \text{Shad}(J_{d_3-2+j}) \cup \\ & \cup \{x_1^{d_1+d_2-d_3-2j} x_2^{2d_3-d_1-d_2+3j-1}, x_1^{d_1+d_2-d_3-2j+1} x_2^{2d_3-d_1-d_2+3j-2}\} = \\ & \quad \{x_1^{d_1+d_2-d_3-2j} \{x_1, x_2\}^{2d_3-d_1-d_2+3j-1}, \\ & \quad \dots, x_1^{d_1+d_2-d_3-2} x_3^{j-1} \{x_1, x_2\}^{2d_3-d_1-d_2+2}, x_3^j J_{d_3-1}\}. \end{aligned}$$

Indeed, the assertion was proved for $j = 1$ and the induction step is similar. In the following, we must consider two possibilities: α is even or α is odd. Suppose first α is even. We obtain that $J_{\frac{d_1+d_2+d_3-4}{2}}$ is the set

$$\begin{aligned} & \{x_1^2 \{x_1, x_2\}^{\frac{d_1+d_2+d_3-8}{2}}, \dots, \\ & x_1^{d_1+d_2-d_3-2} x_3^{\frac{d_1+d_2-d_3-4}{2}} \{x_1, x_2\}^{2d_3-d_1-d_2+2}, x_3^{\frac{d_1+d_2-d_3-2}{2}} J_{d_3-1}\}. \end{aligned}$$

We have $|J_{\frac{d_1+d_2+d_3-2}{2}}| - |\text{Shad}(J_{\frac{d_1+d_2+d_3-4}{2}})| = 2$ so we must add two generators to $\text{Shad}(J_{\frac{d_1+d_2+d_3-4}{2}})$ and, since J is strongly stable and x_3 is a strong Lefschetz element for S/J , these new generators are $x_2^{\frac{d_1+d_2+d_3-2}{2}}$ and $x_1 x_2^{\frac{d_1+d_2+d_3-4}{2}}$. Therefore

$$\begin{aligned} J_{\frac{d_1+d_2+d_3-2}{2}} &= \{ \{x_1, x_2\}^{\frac{d_1+d_2+d_3-2}{2}}, x_1^2 x_3 \{x_1, x_2\}^{\frac{d_1+d_2+d_3-8}{2}}, \dots, \\ & x_1^{d_1+d_2-d_3-2} x_3^{\frac{d_1+d_2-d_3-2}{2}} \{x_1, x_2\}^{2d_3-d_1-d_2+2}, x_3^{\frac{d_1+d_2-d_3}{2}} J_{d_3-1} \}. \end{aligned}$$

One can easily prove by induction on $1 \leq j \leq \frac{\alpha-4}{2}$ that $J_{\frac{d_1+d_2+d_3-2}{2}+j}$ is the set

$$\begin{aligned} &= \text{Shad}(J_{\frac{d_1+d_2+d_3-4}{2}+j}) \cup \{x_3^{2j} x_1^{2j-1} x_2^{\frac{d_1+d_2+d_3-2}{2}-3j}, \dots, x_3^{2j} x_2^{\frac{d_1+d_2+d_3-2}{2}-j}\} = \\ & \quad \{ \{x_1, x_2\}^{\frac{d_1+d_2+d_3-2}{2}+j}, \dots, x_3^{2j} \{x_1, x_2\}^{\frac{d_1+d_2+d_3-2}{2}-j}, \\ & \quad x_1^{2j+2} x_3^{2j+1} \{x_1, x_2\}^{\frac{d_1+d_2+d_3-8}{2}-3j}, \dots, \\ & \quad x_1^{d_1+d_2-d_3-2} x_3^{\frac{d_1+d_2-d_3-2}{2}+j} \{x_1, x_2\}^{2d_3-d_1-d_2+2}, x_3^{\frac{d_1+d_2-d_3}{2}+j} J_{d_3-1} \}. \end{aligned}$$

Suppose now α is odd. We have that $J_{\frac{d_1+d_2+d_3-3}{2}}$ is the set

$$\begin{aligned} & \{x_1 \{x_1, x_2\}^{\frac{d_1+d_2+d_3-5}{2}}, \dots, \\ & x_1^{d_1+d_2-d_3-2} x_3^{\frac{d_1+d_2-d_3-3}{2}} \{x_1, x_2\}^{2d_3-d_1-d_2+2}, x_3^{\frac{d_1+d_2-d_3-1}{2}} J_{d_3-1}\}. \end{aligned}$$

We have $|J_{\frac{d_1+d_2+d_3-1}{2}}| - |\text{Shad}(J_{\frac{d_1+d_2+d_3-3}{2}})| = 2$ so we must add two new generators to $\text{Shad}(J_{\frac{d_1+d_2+d_3-3}{2}})$ in order to obtain $J_{\frac{d_1+d_2+d_3-1}{2}}$. Since J is strongly

stable and x_3 is a strong Lefschetz element for S/J , these new generators are $x_2^{\frac{d_1+d_2+d_3-1}{2}}$ and $x_3x_2^{\frac{d_1+d_2+d_3-3}{2}}$. Therefore

$$\begin{aligned} J_{\frac{d_1+d_2+d_3-1}{2}} &= \{ \{x_1, x_2\}^{\frac{d_1+d_2+d_3-1}{2}}, x_3 \{x_1, x_2\}^{\frac{d_1+d_2+d_3-3}{2}}, \\ &x_1^3 x_3^2 \{x_1, x_2\}^{\frac{d_1+d_2+d_3-11}{2}}, \dots, x_1^{d_1+d_2-d_3-2} x_3^{\frac{d_1+d_2-d_3-1}{2}} \{x_1, x_2\}^{2d_3-d_1-d_2+2}, \\ &x_3^{\frac{d_1+d_2-d_3+1}{2}} J_{d_3-1} \}. \end{aligned}$$

One can easily prove by induction on $1 \leq j \leq \frac{\alpha-3}{2}$ that

$$\begin{aligned} J_{\frac{d_1+d_2+d_3-1}{2}+j} &= \text{Shad}(J_{\frac{d_1+d_2+d_3-3}{2}+j}) \cup \\ &\cup \{ x_1^{2j} x_3^{2j+1} x_2^{\frac{d_1+d_2+d_3-3}{2}-3j}, \dots, x_3^{2j+1} x_2^{\frac{d_1+d_2+d_3-3}{2}-j} \}. \end{aligned}$$

For $j = 1$, we notice that $|J_{\frac{d_1+d_2+d_3+1}{2}}| - |\text{Shad}(J_{\frac{d_1+d_2+d_3-1}{2}})| = 3$ so we must add 3 new monomials to $\text{Shad}(J_{\frac{d_1+d_2+d_3-1}{2}})$ in order to obtain $J_{\frac{d_1+d_2+d_3+1}{2}}$. But, since J is strongly stable and x_3 is a strong Lefschetz element for S/J , they are exactly $x_1^2 x_3^3 x_2^{\frac{d_1+d_2+d_3-3}{2}-3}$, $x_1 x_3^3 x_2^{\frac{d_1+d_2+d_3-3}{2}-2}$ and $x_3^3 x_2^{\frac{d_1+d_2+d_3-3}{2}-1}$, as required. The induction step is similar.

In all cases, we obtain that $J_{d_1+d_2-2}$ is the set

$$\begin{aligned} &\{ \{x_1, x_2\}^{d_1+d_2-2}, x_3 \{x_1, x_2\}^{d_1+d_2-3}, \dots, \\ &x_3^{d_1+d_2-d_3-2} \{x_1, x_2\}^{d_3}, x_3^{d_1+d_2-d_3-1} J_{d_3-1} \}. \end{aligned}$$

We have $|J_{d_1+d_2-1}| - |\text{Shad}(J_{d_1+d_2-2})| = \alpha$, so we must add α new generators to obtain $J_{d_1+d_2-1}$. Since J is strongly stable and x_3 is a strong Lefschetz element for S/J , they are

$$x_1^{d_1+d_2-d_3-1} x_2^{2d_3-d_1-d_2} x_3^{d_1+d_2-d_3}, \dots, x_2^{d_3-1} x_3^{d_1+d_2-d_3},$$

therefore $J_{d_1+d_2-1}$ is the set

$$\begin{aligned} &\{ \{x_1, x_2\}^{d_1+d_2-1}, \dots, x_3^{d_1+d_2-d_3} \{x_1, x_2\}^{d_3-1}, \\ &x_3^{d_1+d_2-d_3+1} x_1^{d_1+d_2-d_3+1} \{x_1, x_2\}^{2d_3-d_1-d_2-3}, \\ &\dots, x_3^{d_1} x_1^{d_1} \{x_1, x_2\}^{d_2-d_1-1}, x_1^{d_1} x_3^{d_1+1} \{x_1, x_2\}^{d_2-d_1-2}, \dots, x_1^{d_1} x_3^{d_2-1} \}. \end{aligned}$$

One can easily prove by induction on $1 \leq j \leq d_3 - d_2$ that $J_{d_1+d_2-1+j}$ is the set

$$\begin{aligned} &\text{Shad}(J_{d_1+d_2-2+j}) \cup \\ &\cup \{ x_1^{d_1+d_2-d_3+j-1} x_2^{2d_3-d_1-d_2-2j} x_3^{d_1+d_2-d_3+2j}, \dots, x_2^{d_3-1-j} x_3^{d_1+d_2-d_3+2j} \} \end{aligned}$$

Indeed, the case $j = 1$ was already done and the induction step is similar. In particular, we get

$$J_{d_1+d_3-2} = \{\{x_1, x_2\}^{d_1+d_3-2}, x_3\{x_1, x_2\}^{d_1+d_3-3}, \dots, x_3^{d_1+d_3-d_2-2}\{x_1, x_2\}^{d_2}, \\ x_1^{d_1}x_3^{d_1+d_3-d_2-1}\{x_1, x_2\}^{d_2-d_1-1}, \dots, x_1^{d_1}x_3^{d_3-2}\}.$$

Since $|J_{d_1+d_3-1}| - |\text{Shad}(J_{d_1+d_3-2})| = d_1$ we must add d_1 generators to $\text{Shad}(J_{d_1+d_3-2})$ in order to obtain $J_{d_1+d_3-1}$. Since J is strongly stable and x_3 is a strong Lefschetz element for S/J , these new generators are

$$x_1^{d_1-1}x_3^{d_1+d_3-d_2}x_2^{d_2-d_1}, \dots, x_3^{d_1+d_3-d_2}x_2^{d_2-1},$$

so $J_{d_1+d_3-1}$ is the set

$$\{\{x_1, x_2\}^{d_1+d_3-1}, \dots, x_3^{d_1+d_3-d_2}\{x_1, x_2\}^{d_2-1}, \\ x_1^{d_1}x_3^{d_1+d_3-d_2+1}\{x_1, x_2\}^{d_2-d_1-2}, \dots, x_1^{d_1}x_3^{d_3-1}\}.$$

We prove by induction on $1 \leq j \leq d_2 - d_1$ that $J_{d_1+d_3-2+j}$ is the set

$$\text{Shad}(J_{d_1+d_3-3+j}) \cup \{x_1^{d_1-1}x_3^{d_1+d_3-d_2+2j-2}x_2^{d_2-d_1-j+1}, \dots, x_3^{d_1+d_3-d_2+2j-2}x_2^{d_2-j}\}.$$

Indeed, we already proved this for $j = 1$ and the induction step is similar. We get

$$J_{d_2+d_3-2} = \{\{x_1, x_2\}^{d_1+d_3-2}, x_3\{x_1, x_2\}^{d_1+d_3-3}, \dots, x_3^{d_3+d_2-d_1-2}\{x_1, x_2\}^{d_1}\}.$$

One can easily prove by induction on $1 \leq j \leq d_1$ that

$$J_{d_2+d_3-2+j} = \text{Shad}(J_{d_2+d_3-3+j}) \cup \{x_3^{d_2+d_3-d_1-2+2j}\{x_1, x_2\}^{d_1-j}\} = \\ = \{\{x_1, x_2\}^{d_2+d_3-2+j}, \dots, x_3^{d_2+d_3-d_1-2+2j}\{x_1, x_2\}^{d_1-j}\}.$$

Finally, we obtain $J_{d_1+d_2+d_3-2} = S_{d_1+d_2+d_3-2}$ and therefore we cannot add new minimal generators of J in degrees $> d_1 + d_2 + d_3 - 2$. \square

Corollary 3.19. *In the above conditions of the above proposition, the number of minimal generators of J is $d_1(d_2+1) - \left(\frac{\alpha}{2}\right)^2 + 1$ if α is even or $d_1(d_2+1) - \frac{\alpha^2-1}{4} + 1$ if α is odd.*

Example 3.20. 1. Let $d_1 = 3$, $d_2 = 5$ and $d_3 = 6$. Then

$$J = (x_1^3, x_1^2x_2^3, x_1x_2^5, x_2^6, x_2^4x_3^2\{x_1, x_2\}, x_2^2x_3^4\{x_1, x_2\}^2, x_2x_3^6\{x_1, x_2\}^2, \\ x_3^8\{x_1, x_2\}^2, x_3^{10}\{x_1, x_2\}, x_3^{12}).$$

2. Let $d_1 = 4$, $d_2 = 5$ and $d_3 = 6$. Then $J = (x_1^4, x_1^3x_2^2, x_1^2x_2^4, x_1x_2^5, x_2^7, x_3x_2^6, \\ x_2^3x_3^3\{x_1, x_2\}^2, x_2x_3^5\{x_1, x_2\}^3, x_3^7\{x_1, x_2\}^3, x_3^9\{x_1, x_2\}^2, x_3^{11}\{x_1, x_2\}, x_3^{13}).$

Remark 3.21. *If $f_1, f_2, f_3 \in S = K[x_1, x_2, x_3]$ is a regular sequence of homogeneous polynomials of given degrees d_1, d_2, d_3 such that $S/(f_1, f_2, f_3)$ has (SLP), then the number of minimal generators of $J = \text{Gin}((f_1, f_2, f_3))$, $\mu(J) \leq d_1(d_2 + 1) + 1$. This follows immediately from 2.4, 2.9, 3.4, 3.9, 3.14 and 3.19.*

References

- [1] D. ANICK, "Thin algebras of embedding dimension three", J.Algebra 100 (1986), 235-259.
- [2] W. BRUNS, J. HERZOG, "*Cohen-Macaulay rings*", Revised Edition, Cambridge, 1996.
- [3] R. FRÖBERG, "An inequality for Hilbert series of graded algebras", *Mathematica Scandinavica* 56, 1985, 117-144.
- [4] T. HARIMA, J. MIGLIORE, U. NAGEL, J. WATANABE, "The weak and strong Lefschetz properties for artinian K -algebras", J. Algebra 262 (2003), no. 1, 99–126.
- [5] T. HARIMA, J. WATANABE, "The finite free extension of K -algebras with the strong Lefschetz property", *Rend. Sem. Mat. Univ. Padova*, 110 (2003), 119-146.
- [6] J. HERZOG, D. POPESCU, "The strong Lefschetz property and simple extensions", Preprint.
- [7] G. MORENO, "These", Ecole Polytechnique, 1991.
- [8] K. PARDUE, "Generic polynomials", Preprint 1999.
- [9] D. POPESCU, "The strong Lefschetz property and certain complete intersection extensions", *Bull.Math Soc.Sc.Math.Roumanie*, 48(96),no 4,(2005),421-431.
- [10] D. POPESCU, M. VLADOIU, "Strong Lefschetz property on algebras of embedding dimension three", *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)* 49(97) (2006), no. 1, 75–86.
- [11] A. WIEBE, "The Lefschetz property for componentwise linear ideals and Gotzmann ideals", *Comm. Algebra* 32 (2004), no. 12, 4601–4611.

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