

Generalization of an inequality involving maximum moduli of a polynomial and its polar derivative

by
V.K. JAIN

Abstract

For a polynomial $p(z)$ of degree n , having all its zeros in $|z| \leq 1$ Turán had shown that

$$\max_{|z|=1} |p'(z)| \geq (n/2) \max_{|z|=1} |p(z)|,$$

which was refined by Aziz and Dawood [2] in the form

$$\max_{|z|=1} |p'(z)| \geq (n/2) \{ \max_{|z|=1} |p(z)| + \min_{|z|=1} |p(z)| \}.$$

And this refinement was generalized to polar derivative of a polynomial, by Aziz and Ahmed [1] in the form

$$\max_{|z|=1} |D_{\alpha} p(z)| \geq (n/2) \{ (|\alpha| - 1) \max_{|z|=1} |p(z)| + (|\alpha| + 1) \min_{|z|=1} |p(z)| \}, |\alpha| \geq 1.$$

We have obtained a generalization of this inequality involving polar derivative in the form

$$\begin{aligned} \max_{|z|=1} |D_{\alpha_t} \dots D_{\alpha_2} D_{\alpha_1} p(z)| &\geq \{ n(n-1) \dots (n-t+1) / 2^t \} \\ &\left[\{ (|\alpha_1| - 1) \dots (|\alpha_t| - 1) \} \max_{|z|=1} |p(z)| \right. \\ &\left. + \{ 2^t |\alpha_1| |\alpha_2| \dots |\alpha_t| - ((|\alpha_1| - 1) \dots (|\alpha_t| - 1)) \} \min_{|z|=1} |p(z)| \right]; \\ &|\alpha_1| \geq 1, |\alpha_2| \geq 1, \dots, |\alpha_t| \geq 1 \ \& \ 1 \leq t < n. \end{aligned}$$

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1 Introduction and statement of results

If $p(z)$ is a polynomial of degree n then concerning the estimate of $|p'(z)|$ on the unit disc we have the following well known theorem of Bernstein [5].

Theorem A. *If $p(z)$ is a polynomial of degree n then*

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|. \quad (1.1)$$

The result is best possible and equality holds in (1.1) for $p(z) = \lambda z^n$.

For polynomials having no zeros in $|z| < 1$ Erdős conjectured a refinement and Lax [4] proved it by obtaining the result

Theorem B. *If $p(z)$ is a polynomial of degree n , having no zeros in $|z| < 1$ then*

$$\max_{|z|=1} |p'(z)| \leq (n/2) \max_{|z|=1} |p(z)|. \quad (1.2)$$

The result is best possible and equality holds in (1.2) for $p(z) = \lambda + \mu z^n$, with $|\lambda| = |\mu|$.

On the other hand for polynomials having all their zeros in $|z| \leq 1$, Turán [7] obtained

Theorem C. *If $p(z)$ is a polynomial of degree n , having all its zeros in $|z| \leq 1$ then*

$$\max_{|z|=1} |p'(z)| \geq (n/2) \max_{|z|=1} |p(z)|. \quad (1.3)$$

The result is best possible and equality holds in (1.3) for $p(z) = \lambda + \mu z^n$, with $|\lambda| = |\mu|$.

Aziz and Dawood [2] obtained the following refinement of Theorem C.

Theorem D. *If $p(z)$ is a polynomial of degree n , having all its zeros in $|z| \leq 1$ then*

$$\max_{|z|=1} |p'(z)| \geq (n/2) \{ \max_{|z|=1} |p(z)| + \min_{|z|=1} |p(z)| \}. \quad (1.4)$$

The result is best possible and equality holds in (1.4) for $p(z) = \lambda + \mu z^n$ with $|\lambda| = |\mu|$.

Shah [6], on the other hand, obtained a generalization of Theorem C to polar derivative of a polynomial by proving the result

Theorem E. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| < 1$ then*

$$\max_{|z|=1} |D_\alpha p(z)| \geq (n/2)(|\alpha| - 1) \max_{|z|=1} |p(z)|, \quad |\alpha| \geq 1, \quad (1.5)$$

The result is best possible and equality holds in (1.5) for $p(z) = \{(z-1)/2\}^n$.

Aziz and Ahmed [1] obtained similar type of generalization of Theorem D and proved

Theorem F. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$ then*

$$\max_{|z|=1} |D_{\alpha} p(z)| \geq (n/2) \{ (|\alpha| - 1) \max_{|z|=1} |p(z)| + (|\alpha| + 1) \min_{|z|=1} |p(z)| \}; |\alpha| \geq 1. \quad (1.6)$$

The result is best possible and equality holds in (1.6) for $p(z) = (z - 1)^n$ with $\alpha \geq 1$.

In this paper we have obtained a generalization of Theorem F. More precisely we have proved

Theorem 1. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$ then for $|\alpha_1| \geq 1, |\alpha_2| \geq 1, \dots, |\alpha_t| \geq 1, (t < n)$*

$$\begin{aligned} \max_{|z|=1} |D_{\alpha_t} \dots D_{\alpha_2} D_{\alpha_1} p(z)| \geq \\ \{ n(n-1) \dots (n-t+1)/2^t \} \left[\{ (|\alpha_1| - 1) \dots (|\alpha_t| - 1) \} \max_{|z|=1} |p(z)| \right. \\ \left. + \{ 2^t (|\alpha_1| |\alpha_2| \dots |\alpha_t|) - (|\alpha_1| - 1) \dots (|\alpha_t| - 1) \} \min_{|z|=1} |p(z)| \right], \quad (1.7) \end{aligned}$$

where

$$\begin{aligned} D_{\alpha_j} D_{\alpha_{j-1}} \dots D_{\alpha_1} p(z) &= p_j(z) = \\ &= (n-j+1)p_{j-1}(z) + (\alpha_j - z)p'_{j-1}(z), j = 1, 2, \dots, t, \quad (1.8) \end{aligned}$$

$$p_0(z) = p(z). \quad (1.9)$$

The result is best possible and equality holds for $p(z) = (z - 1)^n$ with $\alpha_1 \geq 1, \alpha_2 \geq 1, \dots, \alpha_t \geq 1$.

From Theorem 1 one easily gets the following generalization of Theorem E, similar to generalization of Theorem F obtained earlier

Corollary 1. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$ then for $|\alpha_1| \geq 1, |\alpha_2| \geq 1, \dots, |\alpha_t| \geq 1, (t < n)$*

$$\begin{aligned} \max_{|z|=1} |D_{\alpha_t} \dots D_{\alpha_2} D_{\alpha_1} p(z)| \geq \\ \{ n(n-1) \dots (n-t+1)/2^t \} \{ (|\alpha_1| - 1) \dots (|\alpha_t| - 1) \} \max_{|z|=1} |p(z)|. \end{aligned}$$

The result is best possible and equality holds for $p(z) = (z - 1)^n$ with $\alpha_1 \geq 1, \alpha_2 \geq 1, \dots, \alpha_t \geq 1$.

2 Lemmas.

For the proof of the theorem we require following lemmas.

Lemma 1. *If all the zeros of an n^{th} degree polynomial $p(z)$ lie in a circular region C and if none of the points $\alpha_1, \alpha_2, \dots, \alpha_t$ ($t < n$) lies in the region C then each of the polar derivatives $p_1(z), p_2(z), \dots, p_t(z)$, (defined by (1.8) & (1.9)) has all of its zeros in region C .*

This lemma is due to Laguerre [3].

Lemma 2. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$ then for $|\alpha| \geq 1$*

$$|D_\alpha p(z)| \geq n\{(|\alpha| - 1)/2\}|p(z)|, \quad |z| = 1.$$

Proof of Lemma 2. Well, Lemma follows trivially for

$$|\alpha| = 1.$$

Therefore from now onwards we shall assume that

$$|\alpha| > 1.$$

Let

$$q(z) = z^n \overline{p(1/\bar{z})}. \quad (2.1)$$

Then

$$p(z) = z^n \overline{q(1/\bar{z})}, \quad (2.2)$$

$$|q(z)| = |p(z)|, \quad |z| = 1.$$

Now an application of maximum modulus principle to the function $q(z)/p(z)$ will yield

$$|q(z)| \leq |p(z)|, \quad |z| \geq 1,$$

which implies that the polynomial

$$T(z) = q(z) - \lambda p(z), \quad |\lambda| > 1,$$

of degree n will not vanish in $|z| > 1$. Therefore by Lemma 1 the polynomial

$$\begin{aligned} D_\alpha T(z) &= (\alpha - z)q'(z) + nq(z) - \lambda D_\alpha p(z), \\ &= S(z) - \lambda D_\alpha p(z), \text{ (say), } \quad |\alpha| > 1 \end{aligned} \quad (2.3)$$

will also not vanish in $|z| > 1$ for every λ with $|\lambda| > 1$ and accordingly

$$|S(z)| \leq |D_\alpha p(z)|, \quad |z| > 1,$$

which implies

$$|S(z)| \leq |D_\alpha p(z)|, \quad |z| \geq 1. \quad (2.4)$$

Now for $|z| = 1$

$$\begin{aligned}
|D_{\alpha}p(z)| + |S(z)| &= |np(z) + (\alpha - z)p'(z)| + |nq(z) + (\alpha - z)q'(z)|, \text{ (by (2.3))}, \\
&\geq |\alpha p'(z)| - |np(z) - zp'(z)| + |\alpha q'(z)| - |nq(z) - zq'(z)|, \\
&= |\alpha||p'(z)| - |np(z) - zp'(z)| + |\alpha||nq(z) - zq'(z)| - |p'(z)|, \\
&\hspace{15em} \text{(by (2.1) and (2.2))}, \\
&= (|\alpha| - 1)(|p'(z)| + |np(z) - zp'(z)|), \\
&\geq (|\alpha| - 1)n|p(z)|
\end{aligned}$$

and Lemma follows by using (2.4).

Lemma 3. *If $p(z)$ is a polynomial of degree n , having all its zeros in $|z| \leq 1$ then for $|\alpha_1| \geq 1, |\alpha_2| \geq 1, \dots, |\alpha_t| \geq 1, (t < n)$*

$$\begin{aligned}
|D_{\alpha_t}D_{\alpha_{t-1}} \dots D_{\alpha_2}D_{\alpha_1}p(z)| &\geq \\
\{n(n-1) \dots (n-t+1)(|\alpha_1| - 1) \dots (|\alpha_t| - 1)/2^t\} |p(z)|, & |z| = 1. \quad (2.5)
\end{aligned}$$

Proof of Lemma 3: Well, Lemma follows trivially if

$$|\alpha_j| = 1 \text{ for at least one } j, 1 \leq j \leq t.$$

Therefore from now onwards we will assume that

$$|\alpha_1| > 1, |\alpha_2| > 1, \dots, |\alpha_t| > 1. \quad (2.6)$$

We will prove Lemma by mathematical induction. Lemma is true for $t = 1$, by Lemma 2. We assume that Lemma is true for

$$t = q,$$

i.e.

$$\begin{aligned}
|D_{\alpha_q} \dots D_{\alpha_1}p(z)| &\geq \{n(n-1) \dots (n-q+1)(|\alpha_1| - 1) \dots (|\alpha_q| - 1)/2^q\} |p(z)|; \\
|z| = 1 \& |\alpha_1| > 1, \dots, |\alpha_q| > 1. \quad (2.7)
\end{aligned}$$

and we will prove that Lemma is true for

$$t = q + 1, (< n).$$

Except for one value α'_1 , (say), of α_1 , $D_{\alpha_1}p(z)$ will be a polynomial of degree $(n-1)$. Let us take any $\alpha_1 (\neq \alpha'_1 \text{ if } |\alpha'_1| > 1)$, with $|\alpha_1| > 1$ and fix it up. Thus $D_{\alpha_1}p(z)$ is a polynomial of degree $(n-1)$. Now $D_{\alpha_2}D_{\alpha_1}p(z)$ will be a polynomial of degree $(n-2)$ for every α_2 , except for one value α'_2 , (say), of α_2 . Let us take any $\alpha_2 (\neq \alpha'_2 \text{ if } |\alpha'_2| > 1)$, with $|\alpha_2| > 1$ and fix it up. Thus $D_{\alpha_2}D_{\alpha_1}p(z)$ is a polynomial of degree $(n-2)$. One can continue and say that $D_{\alpha_q}D_{\alpha_{q-1}} \dots D_{\alpha_1}p(z)$ will be a polynomial of degree $(n-q)$ for every α_q , except for one value α'_q , (say), of α_q . Let us take any $\alpha_q (\neq \alpha'_q \text{ if } |\alpha'_q| > 1)$, with $|\alpha_q| > 1$

and fix it up. Thus $D_{\alpha_q} D_{\alpha_{q-1}} \dots D_{\alpha_1} p(z)$ is a polynomial of degree $(n - q)$ with fixed α_q ($|\alpha_q| > 1$), fixed α_{q-1} ($|\alpha_{q-1}| > 1$), ..., fixed α_1 ($|\alpha_1| > 1$) and has all its zeros in $|z| \leq 1$. Therefore on applying Lemma 2 to $D_{\alpha_q} \dots D_{\alpha_1} p(z)$ with

$$\alpha = \alpha_{q+1} \quad (|\alpha_{q+1}| > 1),$$

we get

$$|D_{\alpha_{q+1}} D_{\alpha_q} \dots D_{\alpha_1} p(z)| \geq (n - q) \{ (|\alpha_{q+1}| - 1) / 2 \} |D_{\alpha_q} \dots D_{\alpha_1} p(z)|, \quad |z| = 1,$$

which, on being combined with (2.7), implies

$$\begin{aligned} |D_{\alpha_{q+1}} D_{\alpha_q} \dots D_{\alpha_1} p(z)| &\geq \\ \{n(n-1) \dots (n-q) ((|\alpha_1| - 1) \dots (|\alpha_{q+1}| - 1)) / 2^{q+1} \} |p(z)|; \\ |z| = 1 \ \&\ |\alpha_{q+1}| > 1. \end{aligned} \quad (2.8)$$

Further α_q ($\neq \alpha'_q$ if $|\alpha'_q| > 1$) with $|\alpha_q| > 1$ was a fixed element but was chosen arbitrarily. Accordingly (2.8) is true for every α_{q+1} ($|\alpha_{q+1}| > 1$) and every α_q ($\neq \alpha'_q$ if $|\alpha'_q| > 1$) with $|\alpha_q| > 1$, i.e.

$$\begin{aligned} |D_{\alpha_{q+1}} D_{\alpha_q} \dots D_{\alpha_1} p(z)| &\geq \\ \{n(n-1) \dots (n-q) ((|\alpha_1| - 1) \dots (|\alpha_{q+1}| - 1)) / 2^{q+1} \} |p(z)|; \\ |z| = 1, \ |\alpha_{q+1}| > 1 \ \&\ |\alpha_q| > 1 \ (\alpha_q \neq \alpha'_q \ \text{if} \ |\alpha'_q| > 1). \end{aligned} \quad (2.9)$$

Now if $|\alpha'_q| > 1$ then for a fixed α_{q+1} ($|\alpha_{q+1}| > 1$), $D_{\alpha_{q+1}} D_{\alpha_q} \dots D_{\alpha_1} p(z)$ is a continuous function of α_q and therefore by continuity we can say that (2.9) will be true for a fixed α_{q+1} and α'_q and accordingly (2.9) will be true for every α_{q+1} ($|\alpha_{q+1}| > 1$) and α'_q . Thus (2.9) is true for every α_{q+1} ($|\alpha_{q+1}| > 1$) and every α_q ($|\alpha_q| > 1$) i.e.

$$\begin{aligned} |D_{\alpha_{q+1}} D_{\alpha_q} \dots D_{\alpha_1} p(z)| &\geq \\ \{n(n-1) \dots (n-q) ((|\alpha_1| - 1) \dots (|\alpha_{q+1}| - 1)) / 2^{q+1} \} |p(z)|; \\ |z| = 1, \ |\alpha_{q+1}| > 1 \ \&\ |\alpha_q| > 1. \end{aligned} \quad (2.10)$$

As argued for α_q & α'_q , we can argue for α_{q-1} & α'_{q-1} and say that

$$\begin{aligned} |D_{\alpha_{q+1}} D_{\alpha_q} D_{\alpha_{q-1}} \dots D_{\alpha_1} p(z)| &\geq \\ \{n(n-1) \dots (n-q) ((|\alpha_1| - 1) \dots (|\alpha_{q+1}| - 1)) / 2^{q+1} \} |p(z)|; \\ |z| = 1, \ |\alpha_{q+1}| > 1, \ |\alpha_q| > 1 \ \&\ |\alpha_{q-1}| > 1. \end{aligned}$$

One can continue similarly and obtain

$$\begin{aligned} |D_{\alpha_{q+1}} D_{\alpha_q} \dots D_{\alpha_2} D_{\alpha_1} p(z)| &\geq \\ \{n(n-1) \dots (n-q) ((|\alpha_1| - 1) \dots (|\alpha_{q+1}| - 1)) / 2^{q+1} \} |p(z)|; \\ |z| = 1, \ |\alpha_{q+1}| > 1, \ |\alpha_q| > 1, \ \dots, \ |\alpha_2| > 1, \ |\alpha_1| > 1, \end{aligned}$$

thereby proving that Lemma is true for

$$t = q + 1.$$

This completes the proof of Lemma.

3 Proof of Theorem 1

Let

$$m = \min_{|z|=1} |p(z)|. \quad (3.1)$$

If $p(z)$ has a zero on $|z| = 1$ then the result follows from Lemma 3. Therefore from now onwards we will assume that $p(z)$ has all its zeros in $|z| < 1$. By Rouché's theorem the polynomial

$$F(z) = p(z) - m\beta z^n, \quad |\beta| < 1$$

will also have all its n zeros in $|z| < 1$, thereby implying by Lemma 3 for $|\alpha_1| \geq 1$, $|\alpha_2| \geq 1, \dots, |\alpha_t| \geq 1$

$$|D_{\alpha_t} \dots D_{\alpha_2} D_{\alpha_1} F(z)| \geq \{n(n-1) \dots (n-t+1)/2^t\} \{(|\alpha_1| - 1) \dots (|\alpha_t| - 1)\} |F(z)|, \quad |z| = 1,$$

i.e.

$$\begin{aligned} |D_{\alpha_t} \dots D_{\alpha_2} D_{\alpha_1} p(z) - \beta m \{n(n-1) \dots (n-t+1) \alpha_1 \alpha_2 \dots \alpha_t\} z^{n-t}| \geq \\ \{n(n-1) \dots (n-t+1)/2^t\} \{(|\alpha_1| - 1) \dots (|\alpha_t| - 1)\} (|p(z)| - m|\beta|), \\ |z| = 1. \end{aligned} \quad (3.2)$$

But by Lemma 1 the polynomial

$$\begin{aligned} D_{\alpha_t} \dots D_{\alpha_2} D_{\alpha_1} F(z) &= D_{\alpha_t} \dots D_{\alpha_2} D_{\alpha_1} p(z) - \\ &- m\beta \{n(n-1) \dots (n-t+1) \alpha_1 \alpha_2 \dots \alpha_t\} z^{n-t}; \\ &|\alpha_1| \geq 1, \dots, |\alpha_t| \geq 1 \ \& \ |\beta| < 1 \end{aligned}$$

has all its zeros in $|z| < 1$ and therefore

$$|D_{\alpha_t} \dots D_{\alpha_2} D_{\alpha_1} p(z)| \geq mn(n-1) \dots (n-t+1) |\alpha_1| |\alpha_2| \dots |\alpha_t| |z|^{n-t}, \quad |z| \geq 1,$$

which by (3.2) implies

$$\begin{aligned} |D_{\alpha_t} \dots D_{\alpha_2} D_{\alpha_1} p(z)| - m|\beta| n(n-1) \dots (n-t+1) |\alpha_1| |\alpha_2| \dots |\alpha_t| \geq \\ \{n(n-1) \dots (n-t+1)/2^t\} \{(|\alpha_1| - 1) \dots (|\alpha_t| - 1)\} (|p(z)| - m|\beta|), \\ |\beta| < 1 \ \& \ |z| = 1. \end{aligned}$$

Now letting

$$|\beta| \rightarrow 1$$

Theorem 1 follows by using (3.1).

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Mathematics Department,
I.I.T. Kharagpur - 721302,
INDIA
E-mail: vkj@maths.iitkgp.ernet.in